Probabilistic Satisfiability with Imprecise Probabilities

Pierre Hansen, Fabien Chauny
GERAD and École des Hautes Études Commerciales, Montréal, Canada
e-mail: {pierreh, fabien}@crt.umontreal.ca
Marcus Poggi de Aragão
e-mail: poggi@inf.puc-rio.br
Brigitte Jaumard
GERAD and École Polytechnique de Montréal, Canada
e-mail: brigitt@crt.umontreal.ca
Sylvain Perron
École Polytechnique de Montréal, Canada
e-mail: sylvain@crt.umontreal.ca

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Abstract: *Treatment of imprecise probabilities within the probabilistic satisfiability approach to uncertainty in knowledge-based systems is surveyed and discussed. Both probability intervals and qualitative probabilities are considered. Analytical and numerical methods to test coherence and bound the probability of a conclusion are reviewed. They use polyhedral combinatorics and advanced methods of linear programming.*

Keywords: *Satisfiability, Probability intervals, Qualitative probabilities, Polyhedra, Linear programming, Column generation.*

Resumo: Este trabalho apresenta uma revisão do tratamento das probabilidades imprecisas dentro da abordagem da satisfatibilidade probabilística para a incerteza em sistemas de bases de conhecimento. São considerados intervalos de probabilidade e probabilidades qualitativas. Métodos analíticos e numéricos para testar a coerência de uma base de conhecimento e limitar a probabilidade que pode ser atribuída a uma conclusão sobre esta base são revistos. Estes utilizam combinatória poliédrica e métodos avançados de programação linear.

Palavras-chave: Satisfatibilidade, intervalos de probabilidade, probabilidades qualitativas, poliedros, programação linear, geração de colunas.

1 Introduction

Probabilistic Satisfiability and its extensions is a well developed approach to the treatment of uncertainty in knowledge-based systems. Given logical sentences and their probabilities of being true, it studies coherence of these probabilities (or, in Boole's words, if they satisfy the "conditions of possible experience") and provides analytical or numerical bounds on the probability of being true for an additional sentence. Probabilistic Satisfiability has the following five characteristics: (i) it is the oldest approach, its roots going back to Boole's famous book of 1854 on "The Laws of Thought" [2]; (ii) it agrees with the classical theories of logic and probability; (iii) it is powerful, i.e., available algorithms allow solution of problems with several hundred logical sentences; (iv) it is versatile, as conditional probabilities, additional linear constraints, operations on logical sentences and ways to restore coherence can be addressed in simple extensions of the basic model; (v) it is applicable in many fields such as expert systems, automated theorem proving, testing of combinational circuits, reliability and physics. Some main papers and books on this approach are [2], [7], [8], [14], [10], [6], [15], [12], [13]. A recent survey is given in [11].

In this paper, we present a survey, with some new results, on the treatment of imprecise probabilities within the probabilistic satisfiability approach. This can be done in two ways. First, one may consider probability intervals instead of point values as in the work of Boole [2], a proposal already made by Hailperin [7]. Analytical solution is possible, based on Boole's version of Fourier-Motzkin elimination or on enumeration of extreme points and rays of polyhedra [7][13]. Numerical solution of large instances is obtained through the column generation technique of linear programming and nonlinear 0-1 optimization. Problems with conditional probabilities can be treated in a similar way [15]. Second, one may consider qualitative probabilities, i.e., inequalities between probabilities whose values are unknown. Coherence and inequalities between an additional sentence and the initial ones can again be studied, this time by parametric linear programming.

2 Problems Statement

The *probabilistic satisfiability problem in decision form* may be defined as follows: Consider m logical sentences $S_1, S_2, \ldots, S_m$ defined on n logical variables $x_1, x_2, \ldots, x_n$ with the usual Boolean operators $\vee$ (logical sum), $\wedge$ (logical product) and $^-$ (negation, or complementation). Assume probabilities $\pi_1, \pi_2, \ldots, \pi_m$ for these sentences to be true are given. Are these probabilities consistent?

There are 2^n complete products w_j, for $j = 1, 2, \ldots, 2^n$, of the variables $x_1, x_2, \ldots, x_n$ in direct or complemented form. These products may be called, following Leibniz, *possible worlds*. In each possible world w_j any sentence S_i is true or false. The probabilistic satisfiability problem may then be reformulated as follows: is there a probability distribution $p_1, p_2, \ldots, p_{2^n}$ on the set of possible worlds such that the sum of the probabilities of the possible worlds in which sentence S_i is true is equal to its probability π_i of being true, for $i = 1, 2, \ldots, m$.

Defining the $m \times 2^n$ matrix $A = (a_{ij})$ by

$$a_{ij} = \begin{cases} 1 & \text{if } S_i \text{ is true in possible world } w_j \\ 0 & \text{otherwise} \end{cases}$$

the decision form of probabilistic satisfiability may be written:

$$\begin{array}{rcl} \mathbb{1}p & = & 1 \\ Ap & = & \pi \\ p & \geq & 0 \end{array} \tag{1}$$

where $\mathbb{1}$ is a 2^n unit row vector, p and π are the column vectors $(p_1, p_2, \ldots, p_{2^n})^T$ and $(\pi_1, \pi_2, \ldots, \pi_m)^T$ respectively. The answer is yes if there is a vector p satisfying (1) and no otherwise. Note that not all columns of A need be different. Moreover, not all 2^m possible different column vectors of A need, or in most cases will, be present. This is due to the fact that some subset of sentences being true will force other sentences to be true or prohibit them from being so.

Taking imprecision on the sentences' probabilities into account implies a slight change in the above model. It amounts to allow probabilities π_i to belong to an interval, say $[\underline{\pi}_i, \overline{\pi}_i]$. This was first suggested by Hailperin [7] and adds marginal complexity to the original problem. The problem can be rewritten as:

$$\begin{array}{rcl} \mathbb{1}p & = & 1 \\ \underline{\pi} \leq Ap & \leq & \overline{\pi} \\ p & \geq & 0. \end{array} \tag{2}$$

Considering one more sentence S_{m+1}, with an unknown probability π_{m+1} leads to the *optimization form of probabilistic satisfiability.* Usually the constraints (1) or (2) do not impose a unique value for the probability π_{m+1} of S_{m+1}, but some bounds. The satisfiability problem in optimization form is to find the best possible such bounds. It can be written

$$\begin{array}{rl} \min / \max & A_{m+1}p \\ \text{subject to:} & \\ & \mathbb{1}p = 1 \\ & \underline{\pi} \leq Ap \leq \overline{\pi} \\ & p \geq 0. \end{array} \tag{3}$$

Fagin, Halpern and Megiddo [5] note that if some of the π_i are not fixed they may be subject to $v \geq 1$ further linear equalities or inequalities. This leads to another extension:

$$\begin{array}{rl} \min / \max & A_{m+1}p \\ \text{subject to:} & \mathbb{1}p = 1 \\ & Ap = \pi \\ & B\pi = b \\ & p \geq 0 \end{array} \tag{4}$$

where B and b are a $(v \times m)$-matrix and a v-column vector of real numbers. This includes, after addition of slack or surplus variables, problem (3) and the problem of coherence of qualitative probabilities studied by, among others, Coletti [3] where only order relations between probabilities are given (with an arbitrarily small approximation if some or all of the inequalities are strict), i.e., $B\pi = b$ can be written $\pi_i \leq \pi_j$ $(i,j) \in Q$, where Q denotes a given set of pairs of indices.

Another important extension is to consider conditional probabilities instead of, or in addition to, unconditional ones. This was already discussed by Boole [2], for particular examples, and much later by various authors [7][15][3]. Three cases arise: conditionals may be in the constraints of (4), in the objective function or in both. Several ways of representing the conditional probability $\text{prob}(S_k|S_\ell) = \frac{\text{prob}(S_k \wedge S_\ell)}{\text{prob}(S_\ell)} = \pi_{k|\ell}$ in (1) have been proposed. Introducing a variable π_ℓ for the unknown probability $\text{prob}(S_\ell)$ leads to the two constraints [15]:

$$\begin{array}{rcl} A_{k\wedge\ell}p & - & \pi_{k|\ell}\pi_\ell = 0 \\ A_\ell p & - & \pi_\ell = 0 \end{array} \tag{5}$$

where $A_{k\wedge\ell} = (a_{k\wedge\ell,j})$ with $a_{k\wedge\ell,j} = 1$ if both S_ℓ and S_k are true in possible world w_j and 0 otherwise. This way to express conditional probabilities is close to that of Boole [2] who also introduces an unknown parameter. A more compact expression is obtained by eliminating π_ℓ [8]:

$$A'_{k\wedge\ell}p = (A_{k\wedge\ell} - \pi_{k|\ell}A_\ell)p = 0 \tag{6}$$

i.e., $A'_{k\wedge\ell} = (a'_{k\wedge\ell,j})$ where $a'_{k\wedge\ell,j} = 1 - \pi_{k|\ell}$ if S_k and S_ℓ are true, $-\pi_{k|\ell}$ if S_k is false and S_ℓ true and 0 if S_ℓ is false in possible world w_j. Adding $\pi_{k|\ell}\,\mathbb{1}$ to both sides of (6) gives the equation

$$A''_{k\wedge\ell}p = \pi_{k|\ell} \tag{7}$$

where $A''_{k\wedge\ell} = (a''_{k\wedge\ell,j})$ is such that $a''_{k\wedge\ell,j} = 1$ if S_k and S_ℓ are true, 0 if S_k is false and S_ℓ true and $\pi_{k|\ell}$ if S_ℓ is false. Observe that these three values coincide with those given by de Finetti [4] in his definition of the probability of a conditional event in terms of a bet won, lost or cancelled.

Imprecise conditional probabilities can be treated similarly to imprecise probabilities. If $\underline{\pi}_{k|\ell} \leq \pi_{k|\ell} \leq \overline{\pi}_{k\ell}$ the corresponding lines in the linear program are

$$\begin{array}{rcl} A_{k\wedge\ell}p - \underline{\pi}_{k|\ell}A_\ell p & \geq & 0 \\ A_{k\wedge\ell}p - \overline{\pi}_{k|\ell}A_\ell p & \leq & 0. \end{array} \tag{8}$$

The model with unconditional (π_i) and conditional $(\pi_{k|\ell})$ probabilities, can then be written as follows:

$$\begin{array}{rcll} \mathbb{1}p & = & 1 & \\ \underline{\pi}_i \leq A_i p & \leq & \overline{\pi}_i & \forall \pi_i \\ (A_{k\wedge\ell} - \underline{\pi}_{k|\ell}A_\ell)p & \geq & 0 & \forall \pi_{k|\ell} \\ (A_{k\wedge\ell} - \overline{\pi}_{k|\ell}A_\ell)p & \leq & 0 & \forall \pi_{k|\ell} \\ p & \geq & 0. & \end{array} \tag{9}$$

When bounds on a conditional probability $\text{prob}(S_k|S_\ell)$ are sought, it appears in the objective function, and the problem becomes one of hyperbolic (or fractional) programming:

$$\begin{array}{rc} \min / \max & \frac{A_{k\wedge\ell}p}{A_\ell p} \\ \text{subject to: (9)} & \end{array} \tag{10}$$

This model handles all cases discussed above and can be numerically solved for a few hundred unconditional and/or conditional sentences.

3 Analytical Solution

Finding analytical solution for problems (1) or (2) amounts to providing all of Boole's "conditions of possible experience" for a given set of sentences, i.e., necessary and sufficient conditions for the coherence of the probabilities associated to them. If imprecise probabilities are considered, the conditions refer to lower and upper bounds on these probabilities. When an additional sentence is given, as in (3), the analytical solution determines bounds for the probability of this sentence being true as a function of the point values for, or the interval on, the probabilities associated to the sentences.

The case where there are unconditional sentences only was already solved by Boole [2] who proposes a successive elimination algorithm close to that of Fourier-Motzkin (note however that if all conditions are to be found one must replace equalities by pairs of inequalities). Hailperin [7] [9] observes that, as the equations are linear in p, this procedure extends to the case where there are conditional sentences too. Then the conditions of possible experience and bounds need not be linear in the π_i and $\pi_{i|j}$ anymore.

Example 1: (Generalization of an example of Suppes [17] and Hailperin [7][9]) Given $\text{prob}(x_1) \in [\underline{\pi}_1, \overline{\pi}_1]$, $\text{prob}(x_2|x_1) \in [\underline{\pi}_{2|1}, \overline{\pi}_{2|1}]$, let $p_1 = \text{prob}(x_1 x_2)$, $p_2 = \text{prob}(x_1 \overline{x}_2)$, $p_3 = \text{prob}(\overline{x}_1 x_2)$ and $p_4 = \text{prob}(\overline{x}_1 \overline{x}_2)$. Find best possible bounds on $\text{prob}(x_2)$ and conditions of possible experience. This problem can be expressed as:

$$
\begin{array}{ll}
\min / \max & \pi = p_1 + p_3 \\
\text{subject to:} & \underline{\pi}_1 \leq p_1 + p_2 \leq \overline{\pi}_1 \\
& (1 - \underline{\pi}_{2|1}) p_1 - \underline{\pi}_{2|1} p_2 \geq 0 \\
& (1 - \overline{\pi}_{2|1}) p_1 - \overline{\pi}_{2|1} p_2 \leq 0 \\
& p_1 + p_2 + p_3 + p_4 = 1 \\
& p_1, p_2, p_3, p_4 \geq 0.
\end{array}
$$

Eliminating successively p_4, p_3, p_2 and p_1 yields the bounds

$$\underline{\pi}_{2|1} \underline{\pi}_1 \leq \pi \leq 1 - \underline{\pi}_1 (1 - \overline{\pi}_{2|1}).$$

and the trivial conditions $0 \leq \underline{\pi}_1 \leq \overline{\pi}_1 \leq 1$, $0 \leq \underline{\pi}_{2|1} \leq \overline{\pi}_{2|1} \leq 1$. □

Other methods than Fourier-Motzkin elimination for obtaining an analytical solution of probabilistic satisfiability have been devised. They are based on the study of the dual polyhedra for (3). Let the dual of (3) be written:

$$
\begin{array}{lllllll}
\max & y_0 & + & \underline{\pi}^t . y & + & \overline{\pi}^t . y' & \\
& \mathbb{1} . y_0 & + & A^t . y & + & A^t . y' \leq A^t_{m+1} & \\
& y \geq 0, & & y' \leq 0 & & &
\end{array}
\tag{11}
$$

$$\begin{pmatrix} \min & y_0 & + & \underline{\pi}^t.y & + & \overline{\pi}^t.y' \\ & \mathbb{1}.y_0 & + & A^t.y & + & A^t.y' \geq A^t_{m+1} \\ & y \leq 0, & y' \geq 0 \end{pmatrix}.$$

Observe that the constraints of (11) are satisfied by the vector $(0,0,0)$ $\big((1,0,0)\big)$, so the corresponding polyhedra are non-empty. Then, the use of the duality theorem of linear programming yields:

Theorem 1 *(Slight generalization of Hailperin, [7])*
The best lower (upper) bound for $\underline{\pi}_{m+1}$ $(\overline{\pi}_{m+1})$ is given by the following convex (concave) piecewise linear function of the probability assignment:

$$\begin{aligned} \underline{\pi}_{m+1}(\underline{\pi},\overline{\pi}) &= \max_{j=1,2,\ldots,k_{\max}} (1,\underline{\pi}^t,\overline{\pi}^t)y^j_{\max} \\ \Big(\overline{\pi}_{m+1}(\underline{\pi},\overline{\pi}) &= \min_{j=1,2,\ldots,k_{\min}} (1,\underline{\pi}^t,\overline{\pi}^t)y^j_{\min}\Big) \end{aligned} \tag{12}$$

where $y^j_{\max}$ $(y^j_{\min})$ for all j represent the $k_{\max}$ $(k_{\min})$ extreme points of (11)*.*

This result gives bounds on $\underline{\pi}_{m+1}$ and $\overline{\pi}_{m+1}$ but not the conditions of possible experience. To obtain these, consider the dual of the probabilistic satisfiability problem in decision form (2), with a dummy objective function, $0p$, to be minimized:

$$\begin{array}{rl} \max & y_0 + \underline{\pi}^t y + \overline{\pi}^t y' \\ \text{subject to:} & \mathbb{1} y_0 + A^t y + A^t y' \leq 0. \\ & y \geq 0, \quad y' \leq 0 \end{array} \tag{13}$$

Then using the fact that any point in a polyhedron can be expressed as a convex linear combination of its extreme points plus a linear combination of its extreme rays, and once again the duality theorem, the following arises:

Theorem 2 *(Slight generalization of Hansen, Jaumard and Poggi de Aragão, [13])*
The probabilistic satisfiability problem (1) *is consistent if and only if*

$$(1,\underline{\pi}^t,\overline{\pi}^t)r \leq 0 \tag{14}$$

for all extreme rays r of (13)*.*

Therefore, (14) yields all conditions of possible experience for problem (2).

Example 2. (Extension of Boole's challenge problem of 1851 [2]) Let prob$(S_1 \equiv x_1) = [\underline{\pi}_1, \overline{\pi}_1]$, prob$(S_2 \equiv x_2) = [\underline{\pi}_2, \overline{\pi}_2]$, prob$(S_3 \equiv x_1 x_3) = [\underline{\pi}_3, \overline{\pi}_3]$, prob$(S_4 \equiv x_2 x_3) = [\underline{\pi}_4, \overline{\pi}_4]$ and prob$(S_5 \equiv \overline{x}_1 \overline{x}_2 x_3) = [\underline{\pi}_5, \overline{\pi}_5]$. Find best possible bounds on the probability of $S_6 = x_3$.

Enumeration of the extreme points and rays [13] leads to:

Conditions of possible experience		**Lower bounds**	**Upper bounds**
$\underline{\pi}_i \leq \overline{\pi}_i$	$i = 1, 2, \ldots, 5$	$\underline{\pi}_3 + \underline{\pi}_5$	$(1 - \underline{\pi}_1) + \overline{\pi}_3$
$0 \leq \underline{\pi}_i$	$i = 1, 2, \ldots, 5$	$\underline{\pi}_4 + \underline{\pi}_5$	$(1 - \underline{\pi}_2) + \overline{\pi}_4$
$\overline{\pi}_i \leq 1$	$i = 1, 2, \ldots, 5$		$\overline{\pi}_3 + \overline{\pi}_4 + \overline{\pi}_5$
$\underline{\pi}_3 \leq \overline{\pi}_1$			$\overline{\pi}_1 + \overline{\pi}_4 + \overline{\pi}_5$
$\underline{\pi}_4 \leq \overline{\pi}_2$			$\overline{\pi}_2 + \overline{\pi}_3 + \overline{\pi}_5$
$\underline{\pi}_1 + \underline{\pi}_5 \leq 1$			$\overline{\pi}_1 + \overline{\pi}_2 + \overline{\pi}_5$
$\underline{\pi}_2 + \underline{\pi}_5 \leq 1$			
$\underline{\pi}_4 + \underline{\pi}_5 \leq 1$			
$\underline{\pi}_1 + \underline{\pi}_4 + \underline{\pi}_5 \leq \overline{\pi}_3 + 1$			
$\underline{\pi}_2 + \underline{\pi}_3 + \underline{\pi}_5 \leq \overline{\pi}_4 + 1$			

As this example illustrates, probabilistic satisfiability can be viewed as a technique for automated theorem-proving in the theory of probabilities.

4 Numerical Solution

Methods for deciding whether a given set of (imprecise) probabilities defines a coherent knowledge-base when assigned to a set of sentences, or to bound the probability of being true of an additional sentence, are numerical. The methods next described are aimed at solving (3). Solution (9) or (10), which comprise unconditional and conditional sentences, is similar, possibly after replacing a fractional objective function by a linear one through a standard change of variables [11].

Solving (3) presents two major difficulties: (i) The enormous number of columns in the linear program. The number of columns of (3)is bounded by $\min\{2^n, 2^m\}$, and is much too large just to write them down explicitly. However, the linear program (3) can be solved exactly by a column generation technique, where only a small number of columns are considered at a time. The resulting algorithm solves a "small" linear program at each iteration and checks if the solution obtained is optimal by solving a compact pricing subproblem which considers the whole set of columns implicitly; (ii) The fact that at each iteration of the column generation algorithm, deciding whether the algorithm should stop is NP-complete.

The subproblem is to compute the reduced cost, i.e., solve

$$\min_{j \in N} A_{m+1} - y_0 - y^t A^j - y'^t A^j \tag{15}$$

where N is the set of nonbasic columns, A^j the j^{th} column in N and y_0, y, y' the current dual variables, without considering nonbasic columns one at a time. This is done by a specific algorithm in which the coefficients in the columns A^j are the variables.

Observe that

$$\min_{j \in N} A_{m+1} - y_0 - y^t A^j - y'^t A^j = S_{m+1} - y_0 - \sum_{i=1}^{m} y_i S_i - \sum_{i=1}^{m} y'_i S_i \tag{16}$$

where the values True and False for the S_i, $i = 1, \ldots, m+1$ are identified with the numbers 1 and 0. Then (16) is transformed into an algebraic expression involving the logical variables $x_1, \ldots, x_n$ appearing in the S_i, with values true and false also associated with 1 and 0. This is done by eliminating the usual boolean connectives $\vee, \wedge$ and $^-$ using relations

$$\begin{aligned} x_i \vee x_j &\equiv x_i + x_j - x_i \times x_j \\ x_i \wedge x_j &\equiv x_i \times x_j \\ \overline{x}_i &\equiv 1 - x_i. \end{aligned} \tag{17}$$

The resulting expression is a nonlinear (or multilinear) real-valued function in 0–1 variables, or *nonlinear* 0–1 *function*, or *pseudo-boolean function.*

There are various techniques to solve the subproblem exactly, e.g., use an algebraic method (called the basic algorithm) or linearize the nonlinear 0–1 function and solve the resulting linear 0–1 program with e.g. CPLEX MIP. A quicker way is to use a heuristic, e.g. of Tabu Search type, which may be applied as long as it gives a reduced cost of the desired sign. Use of such a heuristic is important in the solution of large instances. For optimization problem and for infeasible decision problems an exact algorithm must be applied to the subproblem at least once. It turns out not to be too time consuming as in practice most dual variables are equal to 0 at the optimum and hence the nonlinear 0–1 function to be optimized contains very few terms.

Computational experience on a SUN ULTRA 2 computer, using CPLEX as the linear programming solver, shows that problems with up to 500 unconditional sentences can be solved. In earlier experiments [15], problems with 200 sentences, 20 of which are conditional ones were solved in reasonable time (30 min of CPU time on a SUN Sparc 4), using XMP as linear programming solver.

5 Qualitative Probabilities

Coletti [3] and others members of the Italian school of subjective probability, have extensively studied qualitative probabilistic satisfiability from a theoretical point of view. They provide formulations for the case of unconditional sentences equivalent to linear programming and for the case of conditional sentences also equivalent to linear programming if the conditioning sentence is the same for both conditionals and to quadratic programming otherwise (this last case is out of the scope of the present paper). Coherence conditions, in terms of bets are provided, but no algorithm likely to solve large instances is presented. Inequalities between sentences may be large or strict. The later case does require introduction of an additional variable to fit into the linear programming framework [16]:

$$\begin{array}{rll}
\max & t & \\
\text{subject to:} & \mathbb{1}p = 1 & \\
& Ap = \pi & \\
& \pi_i - \pi_j \leq 0 \quad \forall (i,j) \in Q_1 & \\
& \pi_i - \pi_j + t \leq 0 \quad \forall (i,j) \in Q_2 & \\
& \underline{\pi} \leq \pi \leq \overline{\pi} & \\
& p \geq 0 &
\end{array} \tag{18}$$

where Q_1(resp. Q_2) denotes the set of index pairs for inequalities (resp. strict inequalities) between pairs of sentences. The sentences and constraints between probabilities are coherent if and only if $t > 0$ in the optimal solution of (18). If an additional sentence is considered, a difficulty arises due to the two objectives $A_{m+1}p$ and t to be optimized. The solution is recourse to parametric programming. One considers the program.

$$\begin{array}{rll}
\min / \max & A_{m+1}p & \\
\text{subject to:} & \mathbb{1}p = 1 & \\
& Ap = \pi & \\
& \pi_i - \pi_j \leq 0 \quad \forall (i,j) \in Q_1 & \\
& \pi_i - \pi_j + t \leq 0 \quad \forall (i,j) \in Q_2 & \\
& t \geq b & \\
& \underline{\pi} \leq \pi \leq \overline{\pi} & \\
& p \geq 0 &
\end{array} \tag{19}$$

where b is a parameter. Then, solve first problem (18) to find if the problem is coherent and what is the maximum value of t. Taking $t = b = 0$ in (19) provides another solution, possibly not satisfying the $\pi_i \leq \pi_j \ \forall (i,j) \in Q_2$ all as strict inequalities. If both solutions give the same value to $A_{m+1}p$, stop, as t does not influence the value of the solution. Otherwise solve (19) parametrically, i.e., reduce progressively b from the largest value of t, checking if the values of the solution at $t = 0$ and of the current one agree, each time a new basis is considered. The method can be extended to find all inequalities which hold between prob(S_{m+1}) and the prob(S_i) for $i = 1, 2, \ldots, m$ instead of numerical bounds on prob(S_{m+1}). Computational results [16] show that problems with up to 100 logical variables, 200 sentences and 200 relations can be solved in reasonable time (i.e., about 1000 seconds of CPU time on a SUN-Ultra 2 with 300 Mhz and 384 Mb of RAM, linear programming computations being done with CPLEX).

References

[1] Boole, G., Proposed Question in the Theory of Probabilities, *The Cambridge and Dublin Mathematical Journal* 6 (1851) 186.

[2] Boole, G., *An Investigation of the Laws of Thought, on which are Founded the Mathematical Theories of Logic and Probabilities*, London: Walton and Maberley, 1854 (reprint New York: Dover 1958).

[3] Coletti, G., Coherent Qualitative Probability, *Journal of Mathematical Psychology* 34 (1990) 297–310.

[4] de Finetti, B., *Theory of Probability – A Critical Introductory Treatment*, Vol. 1, Wiley, New York, 1974.

[5] Fagin, R., J.Y. Halpern et N. Megiddo, A Logic for Reasoning about Probabilities, *Information and Computation* 87 (1990) 78–128.

[6] Georgakopoulos, G., D. Kavvadias, and C.H. Papadimitriou, Probabilistic Satisfiability, *Journal of Complexity* 4, 1-11, 1988.

[7] Hailperin, T., Best Possible Inequalities for the Probability of a Logical Function of Events, *American Mathematical Monthly* 72 (1965) 343–359.

[8] Hailperin, T., *Boole's Logic and Probability*, Studies in Logic and the Foundations of Mathematics 85, North Holland, Amsterdam, 2^{nd} enlarged edition, 1986.

[9] Hailperin, T., *Sentential Probability Logic*, Lehigh University Press, Bethlehem and Associated University Presses, London, 1996.

[10] Kavvadias, D., and C.H. Papadimitriou, Linear Programming Approach to Reasoning about Probabilities, *Annals of Mathematics and Artificial Intelligence*, 1, 189-205, 1990.

[11] Hansen, P. and B. Jaumard, Probabilistic Satisfiability, *Les Cahiers du GERAD*, G-96-31, 1996.

[12] Hansen, P., B. Jaumard and M. Poggi de Aragão, Mixed-Integer Column Generation Algorithms and the Probabilistic Maximum Satisfiability Problem, *Integer Programming and Combinatorial Optimization II*, E. Balas, G. Cornuejols and R. Kannan (Eds.), Pittsburgh: Carnegie Mellon University, (1992) 165–180.

[13] Hansen, P., B. Jaumard and M. Poggi de Aragão, Boole's Conditions of Possible Experience and Reasoning Under Uncertainty, *Discrete Applied Mathematics* 60 (1995) 181–193.

[14] Nilsson, N.J., Probabilistic logic, *Artificial Intelligence* 28(1) (1986) 71–87.

[15] Jaumard, B., P. Hansen and M. Poggi de Aragão, Column Generation Methods for Probabilistic Logic, *ORSA Journal on Computing* 3 (1991) 135–148.

[16] Perron, S., Le problème de la satisfaisabilité probabiliste qualitative, Travail dirigé de maîtrise, École des Hautes Études Commerciales, Montréal, Canada, 1997.

[17] Suppes, P., Probabilistic Inference and the Concept of Total Evidence, in: J. Hintikka and P. Suppes (Eds.), *Aspects of Inductive Logic*, Amsterdam: North-Holland, 1966, 49–65.