Some Remarks on the size of Boolean Functions

Vaston Gonçalves da Costa
vaston@inf.puc-rio.br

Eduardo Sany Laber
laber@inf.puc-rio.br

Edward Hermann Haeusler
hermann@inf.puc-rio.br

PUC-RioInf.MCC51/2004 December, 2004

Abstract: This report discusses some aspects regarding the size of boolean functions, their minterm and maxterm concepts and some graph properties associated to boolean functions and circuits.

Keywords: Combinatorial optimization, Boolean Functions, Lower Bound.

Resumo: Esta monografia discute alguns aspectos envolvendo o tamanho de funções booleanas, seus mintermos e maxtermos e algumas propriedades de grafos associados a funções booleanas e circuitos.

Palavras-chave: Otimização combinatória, Funções Booleanas, Cota inferior.

1 Introduction

This report discusses some aspects regarding the size of boolean functions, their minterm and maxterm concepts and some graph properties associated to boolean functions and circuits.

2 Basic Terminology

This section presentes the basic concepts for the understanding of the whole work.

Definition 2.1 *Every function* $f : \{0,1\}^n \rightarrow \{0,1\}$ *is a boolean function.*

$V(f)$ is used, to denote the set of variable of a boolean function f or simply V.
$|T|$ denotes the cardinality of set T.

Definition 2.2 *Let* f *be a boolean function with* n*-variables,* $V = \{x_1, \ldots, x_n\}$*. A minimal set* $S \subseteq V$ *is a* minterm *if and only if setting all variables in* S *to 1, forces the value of* f *to 1.*

Definition 2.3 *Let* f *be a boolean function with* n*-variables,* $V = \{x_1, \ldots, x_n\}$*. A minimal set* $S \subseteq V$ *is a* maxterm *if, and only if, setting all variables in* S *to 0, forces the value of* f *to 0.*

Let $MIN(f)$ denotes the set of all minterms and $MAX(f)$ the set of all maxterms of f.

Theorem 2.1 $\forall\ T \in MAX(f)$ *and* $\forall\ S \in MIN(f) \Rightarrow T \cap S \neq \emptyset$.

[Gur77] and [KLN$^+$93] present theorem 2.2, which connect boolean function and And-Or trees.
To understand the theorem the following definitions are necessary:

Definition 2.4 *A boolean function* $f : \{0,1\}^n \rightarrow \{0,1\}$ *is monotone if* $\vec{X} \leq \vec{Y} \Rightarrow f(\vec{X}) \leq f(\vec{Y})$.

Definition 2.5 (And-Or Tree) *Let* V *be a finite set. An* And-Or tree *is a rooted tree whose leaves are labeled with members of* V*, and whose internal nodes are labeled with the Boolean operation And(*$\wedge$*), Or(*$\vee$*) each* $x \in V$ *labels only one leaf ot a And-Or tree.*

Theorem 2.2 *A monotone Boolean function* f *that depends on all its variables has an And-Or tree representation, if and only if,*
$T \in MAX(f), S \in MIN(f) \Rightarrow |S \cap T| = 1$.

By the theorem $|k(f).l(f)| \geq n$, then $|k(f)|$ or $|l(f)|$ must be larger than $\sqrt{n}$, where $k(f)$ and $l(f)$ are used to denote the size of the largest minterm and the largest maxterm of f, respectively.

3 Estimating on the lower bound of maxterms

The main result and some useful concepts are presented in this section.

Definition 3.1 *Let f be a boolean function, $x \in V(f)$ and $H_x = \{S | S \in MIN(f) \text{ e } x \in S\}$.*

The degree of x, $Q(x)$, is

$$Q(x) = |H_x|.$$

Lemma 3.1 *Let $Q(x)$ be the maximum of f.*

If $|S| \leq k, \forall S \in MIN(f)$, then $|T| \geq \left\lceil \frac{|MIN(f)|}{Q(x)} \right\rceil$, for all $T \in MAX(f)$.

Proof.

By the theorem (2.1), each $T \in MAX(f)$ has to intercept each $S \in MIN(f)$.

Assume that, for contradiction, there is $T \in MAX(f)$ such that $|T| < \left\lceil \frac{|MIN(f)|}{Q(x)} \right\rceil$.

In this case, there is $y \in V \cap T$ such that $Q(y) \geq Q(x)$.

However, by the assumption, $Q(x)$ is maximum.

Thus, there is not $T \in MAX(f)$ such that $|T| < \left\lceil \frac{|MIN(f)|}{Q(x)} \right\rceil$.

■

Lemma 3.2 *Let $f : \{0,1\}^n \rightarrow \{0,1\}$ be such that $\forall S \subset MIN(f) \Rightarrow |S| \leq k, k \in \mathbb{N}$, where $Q(x) = 1$ for all $x \in V$. Thus, $\forall T \in MAX(f)$, $|T| \geq |MIN(f)|$.*

Proof.

By the theorem 2.1 each $T \in MAX(f)$ has to intercept $S_i \in MIN(f)$ at least once.

Since $Q(x) = 1$ for all $x \in V$, T has to intercept each $S \in MIN(f)$ at least once.

Therefore, $|T| \geq |MIN(f)|$. ■

Theorem 3.3 (Principal) *Let $f : \{0,1\}^n \rightarrow \{0,1\}$ be such that $\forall S \subset MIN(f) \Rightarrow |S| \leq k, k \in \mathbb{N}$. Thus, $\exists T \in MAX(f)$ where $|T| \geq |MIN(f)|^{\frac{1}{k}}$.*

Proof.

Assume that, there is $x_1 \in V$ such that $Q(x_1) > 1$. Otherwise, by the lemma (3.2) $|T| \geq |MIN(f)| \geq |MIN(f)|^{\frac{1}{k}}$ for all $T \in MAX(f)$.

Let $x_1 \in V$ be such that $Q(x_1) \geq Q(y) > 1 \;\; \forall y \in V$ and

$$H_1 = \{T \;\; |T \in MIN(f) \text{ e } x_1 \in T\}$$

By the lemma (3.1):

$$\forall \;\; T \in MAX(f) \Rightarrow |T| \geq \left\lceil \frac{|MIN(f)|}{Q(x_1)} \right\rceil \tag{1}$$

If $Q(x_1) < |MIN(f)|^{\frac{k-1}{k}}$, the theorem holds.

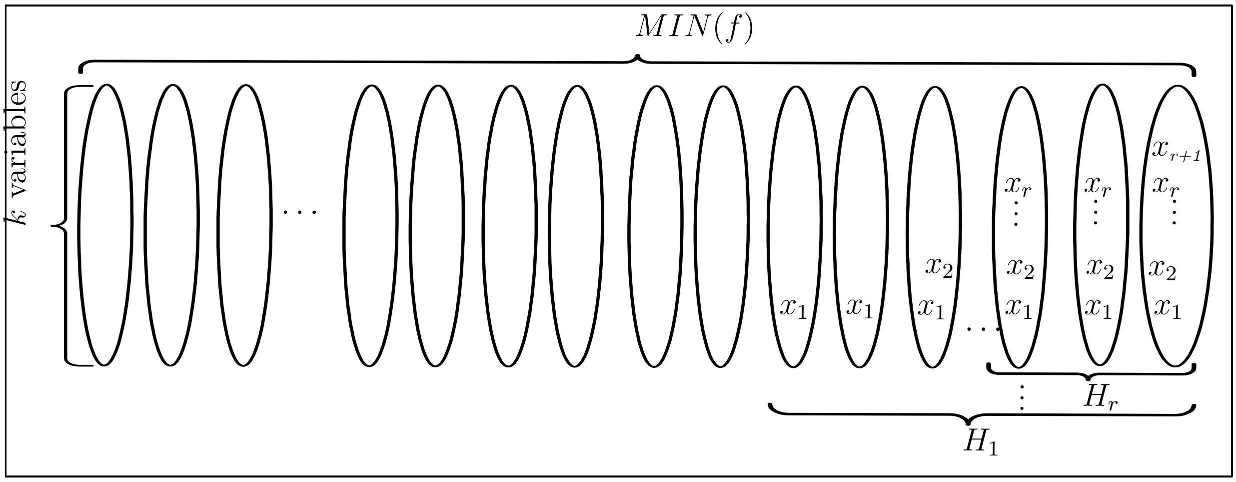

Figure 1: Representation of $MIN(f)$

Otherwise, it is necessary to prove the theorem if $Q(x_1) \geq |MIN(f)|^{\frac{k-1}{k}}$.
Let $x_2 \in H_1$, $Q(x_2) > 1$ such that $Q(x_2) \geq Q(y) \forall y \in V - x_1$.
Clearly, $Q(x_1) \geq Q(x_2)$.
Define the set:

$$H_2 = \{T \ \ |T \in H_1 \text{ e } x_2 \in T\}$$

Thus,

$$\forall \ T \in MAX(f), T \cap \{x_1\} = \emptyset \Rightarrow |T| \geq \left\lceil \frac{Q(x_1)}{Q(x_2)} \right\rceil \tag{2}$$

For x_1 may not be the only variable with Q maximum and x_2 may be presented in other minterms of f that do not belong to H_1.
By (2), if $Q(x_2) < |MIN(f)|^{\frac{k-2}{k}}$, the theorem holds.
Otherwise, it is necessary to prove the theorem if $Q(x_2) \geq |MIN(f)|^{\frac{k-2}{k}}$.
Analogally,
For $x_i \in H_{i-1}$, $Q(x_i) > 1$ maximum.

$$\forall \ T \in MAX(f), T \cap \bigcup_{i=1}^{r} \{x_i\} = \emptyset \Rightarrow |T| \geq \left\lceil \frac{Q(x_r)}{Q(x_{r+1})} \right\rceil \tag{3}$$

Where $Q(x_{r+1}) = 1$, $Q(x_r) \geq |MIN(f)|^{\frac{k-r}{k}}$ and $r + 1 \leq k$. (See figure 2).

Note that, $Q(x_r) = 1$ for some $x_r \in H_{r-1}$. Otherwise, two sets of $MIN(f)$ must be equal.
Thus, the process of picking up $x_i \in H_{i-1}$ will stop.
By the constrution $|T| \geq |MIN(f)|^{\frac{k-r}{k}} \geq |MIN(f)|^{\frac{1}{k}}$.

■

4 Final remarks

If k=2 in the theorem 3.3 we have the following application in graphs.

If we consider the set of minterms as pair of vertex (x, y) such that $(x, y) \in V$ and de maxterms as a cover of G.

Let G be a simple graph, we give a bound relating the size of the largest minnimal cover of G and its number of variables. More specifically, we prove that if the largest minimal cover in G has t vertex then G has at most $t^2 + t$ variables. Futhermore, we prove that this bound is tight.

Definition 4.1 (External neighborhood) *Let G(V,E) be a graph and let $C \subseteq V$. The* external neighborhood *of C is the set*

$$D(C) = \{y \mid y \in V - C \text{ and } (x, y) \in E \text{ for } x \in C\}$$

Definition 4.2 (External degree) *Let C be a cover of G(V,E) and let $x \in C$. The* external degree of x, *$d(x)$, is the cardinality of $D(\{x\})$.*

Definition 4.3 (Maximum External degree) *Let G(V,E) be a simple graph, $C \in V$ and $x \in C$. The external degree of x is maximum if, and only if, $d(x) \geq d(y)$ for all $y \in C$.*

Lemma 4.1 *Let G(V,E) be a simple graph and let C be a maximal cover of G. If $x \in C$ is the vertex with maximum external degree, $d(x) > 1$, then, there is $S \subseteq C - \{x\}$ such that:*

- *$|S| \geq d - 1$ and*
- *$D(S) \in D(\{x\})$.*

Proof. *Let $D(x) = \{y_1, ..., y_d\}$. As C is a cover of G, hence minimal, $C - \{x\}$ can not be a cover of G.*

Let $S_1 = C - \{x\} \cup D(x)$. Since C is a maximal cover of G, S_1 can not be a cover of G.

Indeed, at least $d - 1$ vertex of $C - \{x\}$ are edged to some vertex of $D(\{x\})$.

Let S be the set of vertex of $C - \{x\}$ that are edged to some vertex of $D(\{x\})$.

■

When considering most of the vertexes wiht maximal external degree we have that the size of G is at most $d(x).(t - d(x) + 1))$

Theorem 4.2 *Let G(V,E) be a simple graph and let C be a maximal cover of G. If $|C| = t$ and $x \in C$ is the vertex with maximal external degree, then $|V| \leq d(x).(t - d(x) + 1))$.*

If we take graphs with a even number of vertex we have this upper bound is exact.

As can be seen from the example below, (see figure 2), the upper bound is a tight one.

A maximal cover of G, C, has $|C| = 2t - 1$ end $|V| = t^2 + t$.

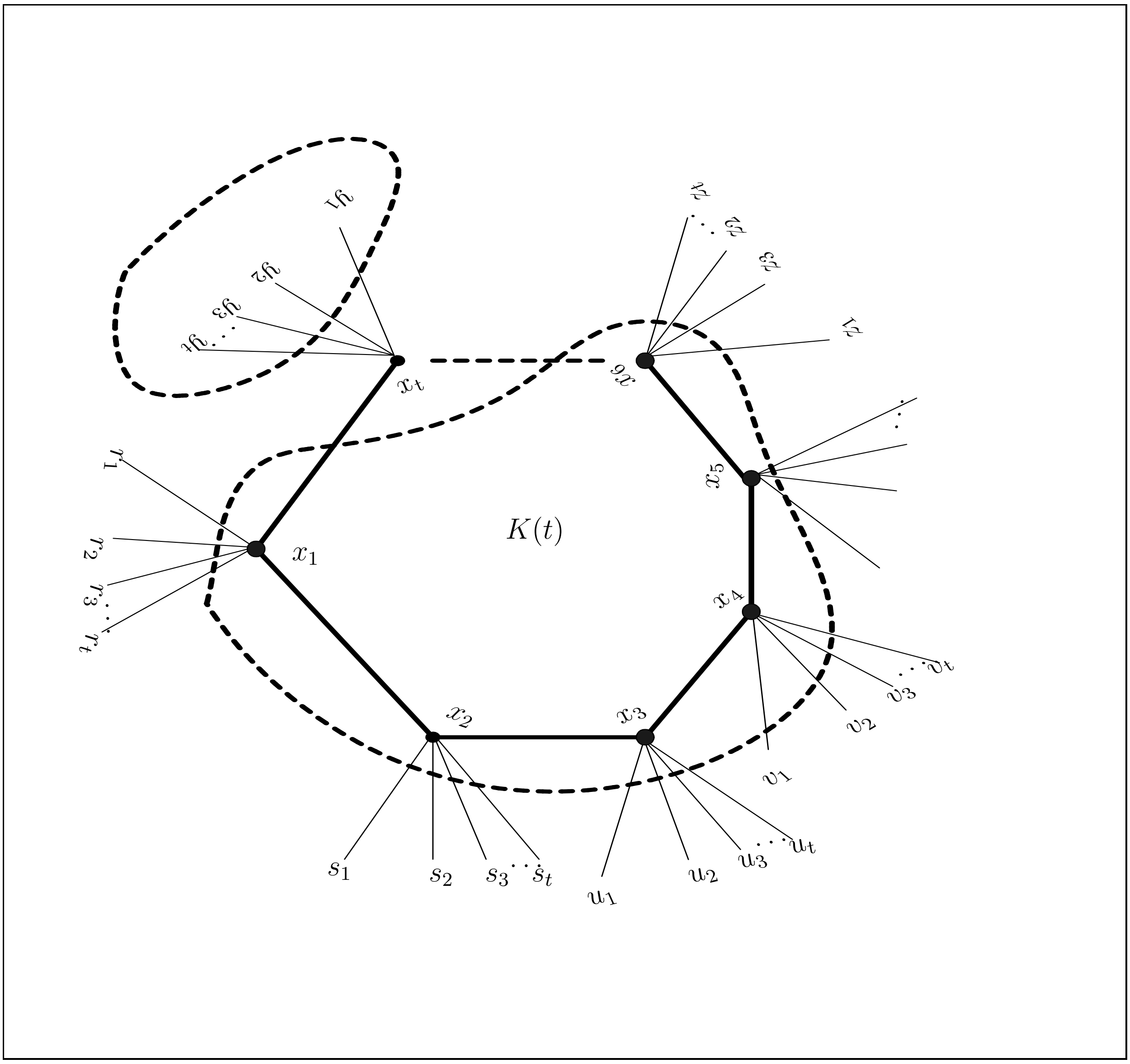

Figure 2: G(V,E) with $|V| = d(x).(t - d(x) + 1))$

References

[CL03] Ferdinando Cicalese and Eduardo Sany Laber. A new strategy for querying priced information. 2003.

[Gur77] V.A. Gurvich. On repetition-free boolean functions. *Uspkhi Matematichesckikh Nauh*, 32(1):183–184, 1977.

[Juk01] Stasys Jukna. *Extremal Combinatorics - With Applications in Computer Science*. Springer, 2001.

[KLN+93] M. Karchmer, N. Linial, I. Newman, M. Saks, and A. Wigderson. Combinatorial characterization of read-once formulae. *Discrete Math.*, 114(1-3):275–282, 1993.

[Rud74] Sergiu Rudeanu. *Boolean Functions and Equations*. North-Holland, 1974.