PUC

Series: Monographs in Computer Science
and Computer Applications

Nº 9/72

THE PRIMAL FACTORIZATION OF COMPLETE GRAPHS

by

Larry Kerschberg

Computer Science Department - Rio Datacenter

Pontifícia Universidade Católica do Rio de Janeiro
Rua Marquês de São Vicente, 209 — ZC-20
Rio de Janeiro — Brasil

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Larry Kerschberg
Associate Professor

Computer Science Department
PUC/RJ

Series Editor: A. L. Furtado November/1972

ABSTRACT

The Kronecker sum and Kronecker product of graphs are defined. For the class of complete graphs. K_m, on m vertices, a primal factorization is obtained in terms of the Kronecker sum and product of prime complete graphs, K_p, p a prime.

I. INTRODUCTION

Complete graphs play a special role in Graph Theory in that their structure helps to characterize important properties of graphs which are not necessarily complete. For example, Zykov [1] calculates the chromatic polynomial of a graph as the sum of the chromatic polynomials associated with complete graphs. Kuratowski's Theorem [2] states that a graph is planar if and only if it has no subgraph homeomorphic to K_5 or $K(3, 3)$.

The motivation for this study is based on the author's research into the multilinear structures which abound in Algebraic Systems Theory. An important property of these structures is that they admit a canonical "factorization" in terms of the tensor product map [3] followed by a linear map. Examples of this decomposition occur in Systems Theory [4], the finite Fourier Transform [5], and Image Processing [6].

The tensor product can be used to decompose multilinear structures as well as to create new ones. In our study we will factor complete graphs by means of the tensor product. The tensor product of graphs has appeared in the literature, and Weichsel [7], Mc Andrew [8], and Harary and Trauth [9] have studied their connectedness properties. Our objective is to factor complete graphs into simpler, prime factors.

We begin by presenting basic definitions and well known results which will be useful in the sequel . Italicized definitions can be found in Harary [10], while more important notions will be numbered.

An undirected *graph* $G = (V, E)$ consists of a finite nonempty set of *vertices*, V, together with a possibly empty set of *edges*, E, consisting of unordered pairs of vertices.

Two vertices u and v are *adjacent* if $\{u, v\}$ is in E. The *adjacency matrix*, A, of a graph G = (V, E) is an nxn binary valued symmetric matrix whose elements are of the form:

$$\alpha_{ij} = \begin{cases} 0 \text{ if } \{v_i, v_j\} \notin E \\ 1 \text{ if } \{v_i, v_j\} \in E \end{cases}$$

Notice that n is the cardinality of V.

Let $\underline{n} = \{1, 2, \ldots, n\}$. The *complete graph* $K_n = (\underline{n}, E)$ has every pair of distinct vertices adjacent. The *complete bipartite graph* $K(m, n) = (\underline{m} \mathring{\cup} \underline{n}, E)$ has each vertex in $\underline{m}$ adjacent to every vertex in $\underline{n}$. A *star* is a complete bipartite graph K(1, n) with vertex "1" as its *center*.

1. Definition (Sum of Graphs) Set $G_i = (V_i, E_i)$, i = 1, 2, be graphs such that $V \equiv V_1 \equiv V_2$. The *sum* of G_1 and G_2 written $G_1 + G_2$, is defined as the sum modulo 2 of the adjacency matrices.

$$G_1 + G_2 = (V, E) \text{ such that } A = (A_1 + A_2) \bmod 2.$$

If G_1 and G_2 are *edge - disjoint*, $E_1 \cap E_2$ = null set, then the sum of G_1 and G_2 is called the *direct sum* and written $G_1 \dotplus G_2$.

Notice that the sum defined above avoids multigraphs. Moreover, it is usually assumed that an undirected graph has no self-loops, but in this study we admit a very special graph with self - loops.

2. <u>Definition</u> (Identity Graph) Let $I_k = (\underline{k}, E)$ denote the graph with k vertices and k edges whose adjacency matrix is the k x k identity matrix. Thus, I_k is a graph with k isolated vertices, each having a self - loop.

3. <u>Definition</u> (Tensor Product of Graphs) Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ such that

$$V_1 = \{p_1, p_2, \dots, p_n\} \text{ and } V_2 = \{q_1, q_2, \dots, q_m\}$$

The tensor product $G_1 \otimes G_2$ of G_1 and G_2 is the graph $G_1 \otimes G_2 = (V = V_1 \times V_2, E)$ such that

$$V = \{p_i\, q_j \mid p_i \in V_1 \text{ and } q_j \in V_2,\ 1 \leq i \leq n,\ 1 \leq j \leq m\}$$

and

$$\{p_i\, q_j, p_\ell\, q_k\} \in E \Longleftrightarrow \{p_i, p_\ell\} \in E_1 \text{ and } \{q_j, q_k\} \in E_2$$

Clearly, the cardinality of V is $n \cdot m$ and the adjacency matrix is the *Kronecker product*, $A = A_1 \otimes A_2$. Throughout the sequel we choose as matrix indices the lexicographical ordering of the vertices. Thus,

$$V = \{p_1\, q_1, p_1\, q_2, \dots, p_1\, q_m, p_2\, q_1, \dots, p_2\, q_m, \dots, p_n\, q_1, \dots, p_n\, q_m\}$$

With this ordering of V, the Kronecker product becomes

$$A = A_1 \otimes A_2 = \begin{bmatrix} \alpha_{11} A_2 & \cdots & \alpha_{1n} A_2 \\ \vdots & & \vdots \\ \alpha_{n1} A_2 & \cdots & \alpha_{nn} A_2 \end{bmatrix} \qquad (1)$$

such that every element of A_1 is multiplied by the whole of A_2.

For example, $K_2 \otimes K_3$ has adjacency matrix

$$A_{K_2 \otimes K_3} = A_{K_2} \otimes A_{K_3} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \otimes \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} =$$

	11'	12'	13'	21'	22'	23'
11'	0	0	0	0	1	1
12'	0	0	0	1	0	1
13'	0	0	0	1	1	0
21'	0	1	1	0	0	0
22'	1	0	1	0	0	0
23'	1	1	0	0	0	0

4. <u>Definition</u> (Kronecker Sum of Graphs). Let $G_1 = (\underline{n}, E_1)$ and $G_2 = (\underline{m}, E_2)$ with respective adjacency matrices A and B. The *Kronecker sum* of G_1 and G_2, written $G_1 \hat{+} G_2$ is,

$$G_1 \hat{+} G_2 = G_1 \otimes I_m + I_n \otimes G_2 \qquad (2)$$

and the adjacency matrix is,

$$A_{G_1 \hat{+} G_2} = A \otimes I_m + I_n \otimes B \qquad (3)$$

Notice that in Definitions 1 through 4 we can treat operations on graphs or on their adjacency matrices interchangably*. With this in mind, one can easily verify the following identities:

i) $I_p \otimes I_q = I_{pq} = I_q \otimes I_p$

ii) $G_1 + G_2 = G_2 + G_1$

iii) $(G_1 + G_2) \otimes G_3 = G_1 \otimes G_3 + G_2 \otimes G_3$

iv) $G_3 \otimes (G_1 + G_2) = G_3 \otimes G_1 + G_3 \otimes G_2$

Unfortunately, $\hat{+}$ and $\otimes$ are *not* commutative operations, but they are associative,

v) $G_1 \hat{+} G_2 \hat{+} G_3 = G_1 \hat{+} (G_2 \hat{+} G_3) = (G_1 \hat{+} G_2) \hat{+} G_3$

vi) $G_1 \otimes G_2 \otimes G_3 = G_1 \otimes (G_2 \otimes G_3) = (G_1 \otimes G_2) \otimes G_3$

* Thus, we use the terms tensor porduct and Kronecker product interchangably.

Lastly, we present several notions about factorizations. A *factor* of a graph G is a spanning subgraph of G which is not totally disconnected. A *factorization* of G is the *direct sum* of factors G_i. König [11] has studied n-factors and n-factorizations, where an n-factor is regular of degree n. We now present what appears to be a new type of factorization.

5. Definition (Primal Factorization) Let G = (V, E) be a graph. A factorization of G is *primal* if G can be expressed in terms of the Kronecker sum and Kronecker product of prime complete graphs, K_{p_i}, p_i prime, i = {1, 2, ... , k}

Our objective is to study the class of graphs which admit a primal factorization. Before doing so, we present a known result from Number Theory.

6. Theorem ([3], p.p. 155 - 156) Any natural number m can be decomposed as the product of powers of primes

$$m = p_1^{e_1} \cdot p_2^{e_2} \cdots\cdots p_k^{e_k} \qquad (4)$$

where $2 \leq p_1 < p_2 \cdots\cdots < p_k$

II. MAIN RESULTS

We now state and prove the main theorem of this study.

1. **Theorem** Every complete graph K_m, $m \geq 1$, admits a primal factorization.

Proof: In order to prove the theorem we use the following

2. **Lemma** Let p and q be relatively prime such that $m = pq$. Then

$$K_m = K_p \dotplus K_q \dotplus K_p \otimes K_q \tag{5}$$

$$= K_p \otimes I_q \dotplus I_p \otimes K_q \dotplus K_p \otimes K_q \tag{6}$$

Proof: Since $m = pq$ we may write the identity

$$K_m \dotplus I_m = (K_p \dotplus I_p) \otimes (K_q \dotplus I_q) \tag{7}$$

The bilinearity of the Kronecker product permits an expansion of the right hand side of (7)

$$K_m \dotplus I_m = K_p \otimes K_q \dotplus K_p \otimes I_q \dotplus I_p \otimes K_q \dotplus I_p \otimes I_q$$

Also, $I_m = I_p \otimes I_q$ which permits cancellation of that term on both sides yielding,

$$K_m = K_p \otimes K_q \dotplus K_p \otimes I_q \dotplus I_p \otimes K_q \tag{8}$$

The terms on the r.h.s. of (8) are edge-disjoint because of the structure of the adjacency matrices. Thus

$$K_m = K_p \otimes K_q \dotplus K_p \otimes I_q \dotplus I_p \otimes K_q = K_p \,\hat{+}\, K_q \dotplus K_p \otimes K_q$$

□

One can also verify, by means of the adjacency matrix, that

$$K_m = K_q \,\hat{+}\, K_p \dotplus K_q \otimes K_p$$

Although the Kronecker sum and product of graphs are not commutative operations, the primal factorization for $m = pq = qp$ "commutes".

Now we return to the proof of the theorem. Given a natural number m, we obtain its decomposition as the product of powers of primes

$$m = p_1^{e_1} \cdot p_2^{e_2} \cdots\cdots p_k^{e_k} \tag{9}$$

where the primes are ordered lexicographically into a string of length $\sum_{i=1}^{k} e_i$.

K_m is calculated recursively as follows:

Set $m_0 = 1$.

For $j = 0, 1, \ldots, \sum_{i=1}^{k} ei - 1$,

let $m_{j+1} = m_j \cdot p_{j+1}$,

where p_{j+1} is the $(j + 1)^{st}$ prime in the string (9).

Using the lemma, we compute

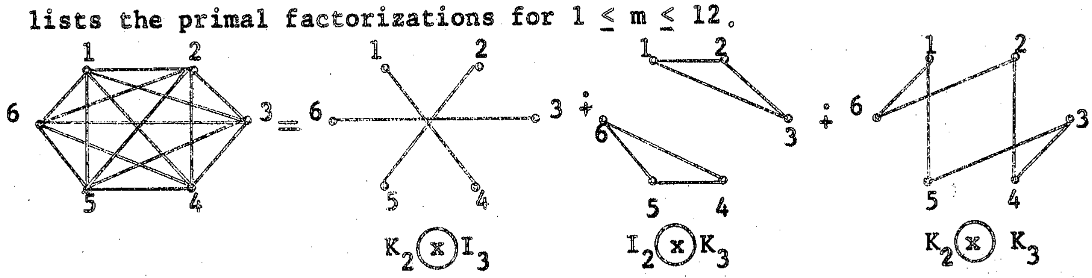
$$K_{m_{j+1}} = K_{p_{j+1}} \hat{+} K_{m_j} \dot{+} K_{p_{j+1}} \otimes K_{m_j} \qquad (10)$$

The algorithm terminates on the $(\sum_{i=1}^{k} e_k)^{th}$ iteration, yeilding the desired primal factorization. □

Notice that if m is a prime, then the algorithm will calculate K_m on the first iteration

$$\begin{aligned} K_{m_1} &= K_m \hat{+} K_{m_0} \dot{+} K_m \otimes K_{m_0} \\ &= K_m \hat{+} K_1 \dot{+} K_m \otimes K_1 = K_m \otimes I_1 \dot{+} I_m \otimes K_1 \dot{+} K_m \otimes K_1 \\ &= K_m \dot{+} I_m \otimes \bigcirc \dot{+} K_m \otimes \bigcirc \\ &= K_m. \end{aligned}$$

Figure 1 shows the primal factorization for K_6 and Table 1 lists the primal factorizations for $1 \leq m \leq 12$.



Primal Factorization of K_6
Figure 1

TABLE 1 – PRIMAL FACTORIZATIONS FOR K_m

m	Prime Decomposition	Primal Factorization for K_m
1	1	K_1
2	2	K_2
3	3	K_3
4	2^2	$K_4 = K_2 \hat{+} K_2 \dot{+} K_2 \otimes K_2 = K_2 \otimes I_2 \mathring{+} I_2 \otimes K_2 \mathring{+} K_2 \otimes K_2$
5	5	K_5
6	$2 \cdot 3$	$K_6 = K_2 \hat{+} K_3 \dot{+} K_2 \otimes K_3 = K_2 \otimes I_3 \dot{+} I_2 \otimes K_3 \dot{+} K_2 \otimes K_2$
7	7	K_7
8	2^3	$K_8 = K_2 \hat{+} K_4 \dot{+} K_2 \otimes K_4$ $= K_2 \otimes I_2 \otimes I_2 \dot{+} I_2 \otimes K_2 \otimes I_2 \mathring{+} I_2 \otimes I_2 \otimes K_2$ $\dot{+} K_2 \otimes K_2 \otimes I_2 \dot{+} K_2 \otimes I_2 \otimes K_2 \dot{+} I_2 \otimes K_2 \otimes K_2$ $\dot{+} K_2 \otimes K_2 \otimes K_2$
9	3^2	$K_9 = K_3 \hat{+} K_3 \dot{+} K_3 \otimes K_3 = K_3 \otimes I_3 \mathring{+} I_3 \otimes K_3 \mathring{+} K_3 \otimes K_3$
10	$2 \cdot 5$	$K_{10} = K_2 \hat{+} K_5 \dot{+} K_2 \otimes K_5 = K_2 \otimes I_5 \dot{+} I_2 \otimes K_5 \dot{+} K_2 \otimes K_5$
11	11	K_{11}
12	$2^2 3$	$K_{12} = K_2 \otimes I_2 \otimes I_3 \dot{+} I_2 \otimes K_2 \otimes I_3 \dot{+} I_2 \otimes I_2 \otimes K_3$ $\dot{+} K_2 \otimes K_2 \otimes I_3 \dot{+} I_2 \otimes K_2 \otimes K_3 \dot{+} K_2 \otimes I_2 \otimes K_3$ $\dot{+} K_2 \otimes K_2 \otimes K_3$

By observing the primal factorization for K_8 and K_{12} in Table 1, one sees the motivation for the

3. **Fact** To each factor of K_m, $m = p_1^{e_1} \cdots\cdots p_k^{e_k}$, one can associate a non-zero binary representation of length $\sum_{i=1}^{k} e_i$ in the following manner: define the mappings $0 \longmapsto I_{p_i}$ and $1 \mapsto K_{p_i}$ such that the subscripts, p_i, match the lexicographical ordering of product of powers of primes for m. Insert a Kronecker product between each two digits of the binary representation.

For example K_{12} has $12 = 2 \cdot 2 \cdot 3$; its binary representation and corresponding factors are shown below:

$001 \mapsto I_2 \otimes I_2 \otimes K_3$; $010 \mapsto I_2 \otimes K_2 \otimes I_3$; $100 \mapsto K_2 \otimes I_2 \otimes I_3$

$011 \mapsto I_2 \otimes K_2 \otimes K_3$; $110 \mapsto K_2 \otimes K_2 \otimes I_3$; $101 \mapsto K_2 \otimes I_2 \otimes K_3$

$111 \mapsto K_2 \otimes K_2 \otimes K_3$.

4. **Fact** The binary representation (and the primal factorization) can be expressed as $Q_n - \{0000 \ldots 0\}$, where Q_n is an n-cube [10] and $n = \sum_{i=1}^{k} e_i$.

The next result shows that the Kronecker sum and the Kronecker product in a primal factorization are complements of one another. The *complement* of a graph $G = (V, E)$ is the graph $\bar{G} = (V, \bar{E})$ in which two vertices are adjacent in $\bar{G}$ if and only they are not adjacent in G. It is well know that for any graph $G = (V, E)$ such that the cardinality $(V) = n$, $K_n = G \dot{+} \bar{G}$.

5. <u>Proposition:</u> Let $m = pq$ where p and q are relatively prime

Then in the primal factorization of K_m:

$$\text{i)} \quad K_p \otimes K_q = \overline{K_p \hat{+} K_q} \tag{11}$$

and

$$\text{ii)} \quad K_p \hat{+} K_q = \overline{K_p \otimes K_q} \tag{12}$$

<u>Demonstration:</u> We use the fact that $K_m = G \dot{+} \bar{G}$ and note that $K_m = K_p \hat{+} K_q \dot{+} K_p \otimes K_q$. The conclusion follows. By using the recursion relation (10) the result is extended to any decomposition for m as the product of powers of primes.

III - <u>CONCLUDING REMARKS</u>

The main result of this study is that every complete graph admits a primal factorization. Moreover, the Kronecker sum and product of graphs in the factorization are complements of one another.

These facts, while interesting in their own right, fit into a much richer mathematical framework: The Representation Theory of Finite Groups [12, 13] and Lie Algebras [14]. Harary [10] states that the group of automorphisms associated with K_m is the symmetric group S_m of all permutions on <u>m</u>. Thus, the primal factorization of K_m is a representation of S_m over the field of binary numbers. Since the tensor product plays an important role in representation theory, it might be possible to find factorizations for other groups associated with graphs in terms of "primitive generators" and the Kronecker operations.

ACKNOWLEDGEMENTS

The author is grateful to Professor A. L. Furtado for several conversations which helped to enhance the presentation. Thanks also go to Dr. S. K. Mitter of MIT who first introduced the author to the tensor product.

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