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SOME RESULTS ON THE CURVED PLATE BENDING PROBLEM SOLVED WITH NON-CONFORMING FINITE ELEMENTS

by

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CONTENTS

ABSTRACT:

Convergence properties are studied for two non conforming finite elements of degree two and three for the plate bending problem, which were introduced respectively by MORLEY and FRAEIJS DE VEUBEKE. Unlike some other non conforming elements, such as the so-called Zienkiewicz triangle, they turn out to be very suitable for the case of curved boundaries since success in the patch-test is guaranteed without any restrictions on the shape of the plate. Furthermore, one is allowed to suppose that no kind of curved elements are needed to attain the same rates of convergence derived by LASCAUX & LESAINT for the polygonal case, because of the low degree of the corresponding space of polynomials. Two kind of support conditions are examined: For the clamped plate, the above assuptions are verified for both elements. This provides a simple alternative to avoid the complexity involved in handling curved boundaries with standard conforming elements. In the case of the simply supported plate, Morley's element preserves the same convergence properties as in the polygonal case without curved elements. For Fraeijs de Veubeke's triangle however this is only possible with the use of second degree parabolas to approximate the boundary. In addition to the above results, an indication is given for the optimal choice of Poisson's coefficient to be used in the variational formulation for clamped plates and numerical examples are shown.

KEY WORDS:

Finite elements, curved boundary, convergence in Sobolev norms, non-conforming, patch-test, plate bending, Fraeijs de Veubeke, Strang.

RESUMO:

Dois elementos triangulares não conformes conhecidos como elementos de Morley e Fraeijs de Veubeke, introduzidos para a resolução do problema de flexão de placas são tratados. Sua relativa simplicidade devida ao baixo grau dos polinômios correspondentes, aliada à flexibilidade traduzida pela não existência das restrições à aplicação com sucesso de outros elementos correntemente utilizados para placas retangulares, tornam seu uso altamente indicado ao caso de bordo curvo. Para ambos os elementos demonstra-se que nesse, caso não é necessário se recorrer aos elementos curvos para se atingir resultados comparáveis aos da placa de bordo retilíneo, quando este é engastado. Se o bordo curvo é simplesmente apoiado, o elemento de Morley mantém a mesma propriedade. Entretanto para o triângulo de Fraeijs de Veubeke a aproximação do bordo por arcos de parábola do 2º grau se faz aconselhável. Enfim, uma indicação quanto à escolha do valor ótimo do coeficiente de Poisson a ser utilizado na formulação variacional para placas engastadas é fornecida. Todos os resultados são acompanhados de exemplos numéricos.

PALAVRAS-CHAVE:

Elementos finitos, bordo curvo, convergência em normas de Sobolev, não conforme, "patch-test", flexão de placas, Fraeijs de Veubeke, Strang.

1 - INTRODUCTION

Our starting point is the proposal by FRAEIJS DE VEUBEKE [1] of a family of non-conforming triangles for solving the plate bending problem, whose degrees of fredom are functional values, normal derivatives or mean values of normal derivatives along the edges.

These elements have the interesting property of providing convergence with no restrictions on the shape of the domain, unlike many other non-conforming elements frequently used for rectangles or parallelograms, In fact, the case of such plates has been extensively studied and many efficient methods, including conforming finite element methods are available for general polygonal plates.

For the case of curved plates however, limitations or complexity involved render the use of most of these methods unrealistic. So, if it is possible to verify that the convergence properties derived for the polygonal case by LASCAUX & LESAINT [2] for some elements of low degree belonging to this family remain valid if the curved boundary is approximated by a polygon, then they provide an extremely pratical and simple possible solution.

The analysis in the following sections will show that this assumption is true for the so-called Morley triangle (quadratic) and, in some cases, for the Fraeijs de Veubeke triangle (cubic).

2 - SOME BASIC RESULTS

In this paper we consider $W^{m,p}(\Omega)$ as the Sobolev space of functions defined over an open set Ω of R^2, where $p \in [1,\infty)$, and its norm and semi-norm.

$$||\cdot||_{m,p,\Omega} = \{\sum_{|\alpha|\leq m} \int_\Omega (\partial^\alpha \cdot)^p \, dx_1 dx_2\}^{1/p}$$

$$|\cdot|_{m,p,\Omega} = \{\sum_{|\alpha|=m} \int_\Omega (\partial^\alpha \cdot)^p \, dx_1 dx_2\}^{1/p}$$

whit the usual modification if $p = \infty$.

If $p = 2$, i.e., if $W^{m,p}(\Omega) = H^m(\Omega)$ we drop this index.

Let the plate be represented by the bounded domain Ω of R^2 with boundary Γ and μ be the Poisson coefficient of the material of the plate which is supposed homogeneous.

If the load is proportional to a function , which we suppose to be bounded $(f \in L^\infty(\Omega) \Rightarrow f \in L^2(\Omega))$, then the solution of this problem is the displacement function u that minimizes the functional $J(v)$:

$$(1) \quad J(v) = \int_\Omega (\Delta v)^2 dx_1 dx_2 + 2(1-\mu) \int_\Omega [(\frac{\partial^2 v}{\partial x_1 \partial x_2})^2 - \frac{\partial^2 v}{\partial x_1^2} \frac{\partial^2 v}{\partial x_2^2}] \, dx_1 dx_2 - 2 \int_\Omega f v \, dx_1 dx_2$$

over the space of admissible functions v - a subspace of $H^2(\Omega)$ - which is the same as finding u satisfying suitable boundary conditions and such that:

$$\Delta^2 u = f$$

We next define the two elements treated here:

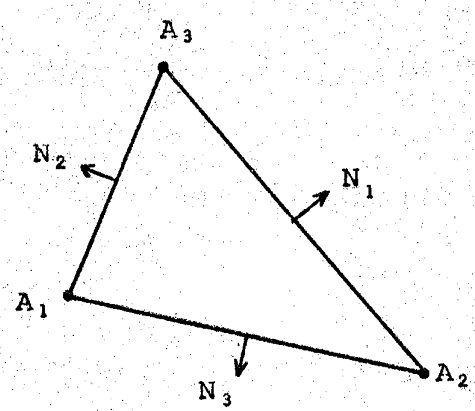
Morley's element

- Degrees of freedom:

3 functional values at the vertices;

3 first normal derivatives at the mid points of the edges.

- Degree of the polynomial: 2 (complete)



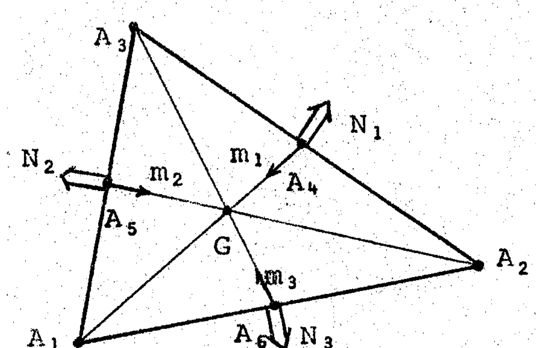
Fraeijs de Veubeke's element

- Degrees of freedom:

3 functional values at the vertices;

3 functional values at the mid points of the edges;

3 mean values of first normal derivatives along the edges.



- Degree of the polynomial: 3 (incomplete) satisfying the following condition:

  Its value at the centroid G is given by a fixed linear combination of its values at the vertices and mid poits of the edges, and mean values of first derivatives taken in the direction of the medians [2] .

Let τ_h be a given triangulation of the domain Ω , and V_h be the space of trial functions whose restrictions over each triangle K belonging to τ_h is a polynomial of degree 2 or 3, respectively according to the element, defined by the degrees of freedom given above. The degrees of freedom lying on the boundary Γ of functions belonging to these spaces must vanish according to support conditions. For instance, if the plate is clamped, the degrees of freedom related to normal derivatives taken at the boundary edges vanish as well as those corresponding to functional values at boundary points. If the plate is simply supported only the latter vanish.

We define additionally

$$h = \max_{K\epsilon\tau_h} \{ \text{ diameter of } K \}$$

$v_h \epsilon V_h$ does not generally imply $v_h \epsilon H^2(\Omega)$ [1] which compels us to the definition of another norm for V_h.

It can be proved [2] that the expression

$$(2) \quad ||v_h||_h = \{ \sum_{K\epsilon\tau_h} \int_K [(\frac{\partial^2 v_h}{\partial x_1^2})^2 + 2(\frac{\partial^2 v_h}{\partial x_1 \partial x_2})^2 + (\frac{\partial^2 v_h}{\partial x_2^2})^2] dx_1 dx_2 \}^{1/2}$$

---

[1] Generally the inclusion $V_h \epsilon C^0(\Omega)$ is not satisfied either.

is a norm over V_h for both clamped and simply supported plates. This norm is often called the discrete $H^2(\Omega)$ norm and it coincides with the usual norm whenever $v_h \in H^2(\Omega)$.

Let now u_h be the unique solution of the discrete problem:

$$(P_h)\ \begin{cases} \text{Find } u_h \in V_h \\ \text{such that} \\ a_h(u_h,v_h) = \sum\limits_{K\in\tau_h} a_K(u_h,v_h) = (f,v_h) \quad \forall\ v_h \in V_h. \end{cases}$$

where

$$(f,v_h) = \int_\Omega f v_h \, dx_1 dx_2$$

and

$$a_K(u,v) = \int_K \Delta u \Delta v dx_1 dx_2 + (1-\mu)\int_K \left[2\frac{\partial^2 u}{\partial x_1 \partial x_2}\frac{\partial^2 v}{\partial x_1 \partial x_2} - \frac{\partial^2 u}{\partial x_1^2}\frac{\partial^2 v}{\partial x_2^2} - \frac{\partial^2 u}{\partial x_2^2}\frac{\partial^2 v}{\partial x_1^2}\right]dx_1 dx_2$$

The uniqueness of the solution is due to the V_h - ellipticity of the bilinear form $a_h(u_h, v_h)$ for physically admissible values of μ,

$$0 \le \mu < \frac{1}{2}$$

So, for both cases of clamped and simply supported plates we have [2] :

$$||u-u_h||_h \le C[h|u|_{3,\Omega} + h^2|u|_{4,\Omega}]$$

where C is a constant independent of h.[2]

---

[2] Everywhere in this text the letter C will represent constants independents of h.

The above result is basically derived from the so-called Strang's inequality [6] :

(3) $$||u-u_h||_h \le C \left[\inf_{v_h \in V_h} ||u-v_h||_h + \sup_{w_h \in V_h} \frac{|E_h(u,w_h)|}{||w_h||_h} \right]$$

where:

$$E_h(u,w_h) = a_h(u,w_h) - (f,w_h)$$

(4) $$E_h(u,w_h) = \int_{\partial K} \left\{ \left[\Delta u + (1-\mu)\frac{\partial^2 u}{\partial s_K^2} \right] \frac{\partial w_h}{\partial n_K} + (1-\mu)\frac{\partial^2 u}{\partial s_K \partial n_K} \frac{\partial w_h}{\partial s_K} + \frac{\partial \Delta u}{\partial n_K} w_h \right\} ds_K$$

∂K represents in (4) the boundary of K; n_K and s_K its outer normal and tangent respectively.

The term $E_h(u,w_h)$ is also called the term of non-conformity since it vanishes in the case of conforming elements.

The first part of the right side of (3) is usually estimated by taking $v_h = r_h u$, where $r_h u$ is the function of V_h that interpolates u at the degrees of freedom of the mesh.

3 - CASE OF CURVED BOUNDARIES

If we are to approximate a curved boundary Γ by the boundary Γ_h a polygon Ω_h :

$$\Omega_h = \bigcup_{K \in \tau_h} K$$

then the vanishing values of a function $v_h \in V_h$ at known degrees of freedom are no longer taken on Γ but on Γ_h.

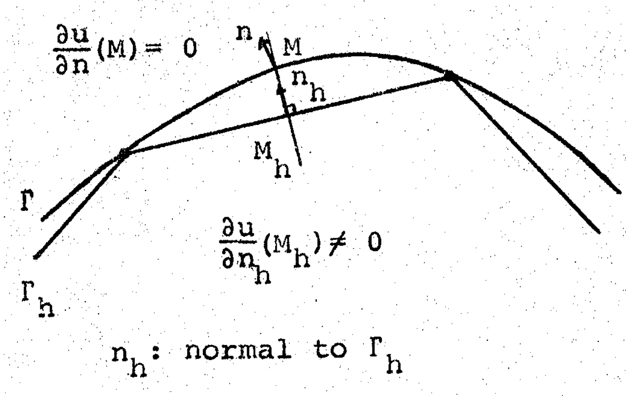
At this point it is convenient to treat separately two important cases of support conditions:

1^{st} Case: Clamped plate

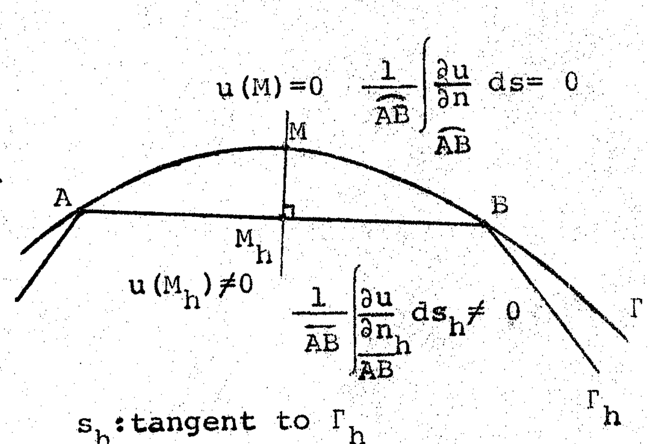
In this case we assert first of all that the shift of vanishing values from Γ to Γ_h does not cause any alteration in the estimate for the term $|E_h(u,w_h)|$ of Strang's inequality already given for the polygonal case.

So, all we have to do for the clamped plate is to examine the error involved in interpolating the function u with a function $r_h u$ belonging to V_h. This implies now a different interpolation since the values of $r_h u$ related to degrees of freedom on Γ_h are forced to vanish whereas the corresponding values of u are not necessarily zero. This more restrictive interpolation however provides the same estimates in the discrete $H^2(\Omega_h)$ - norm as those obtained for the polygonal case, as far as rates of convergence are concerned. The proof of this fact can be found in the author's work [5] [3]. The key to the problem is the estimate of the difference of the values of u and its first normal derivatives taken at Γ_h and Γ respectively:

Morley's element



Fraeijs de Veubeke's element



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[3] Actually, these results are still valid for the non homogeneous case $(\partial u/\partial n \neq 0)$ [5] for Morley's element. This has also been numerically verified during some tests recently performed.

The final convergence results are, for both elements :

(5) $||u-u_h||_h \leq C[h|u|_{3,\Omega} + h^{3/2}||u||_{4,\Omega}]$

Now we give some numerical results.

In this case the exact solution u is independent of μ since it is also the solution of

$$(P_C)\begin{cases} \Delta^2 u = f \\ u/\Gamma = \frac{\partial u}{\partial n}/\Gamma = 0 \end{cases}$$

So we can solve the approximate problem (P_h) using different values of Poisson's coefficient.

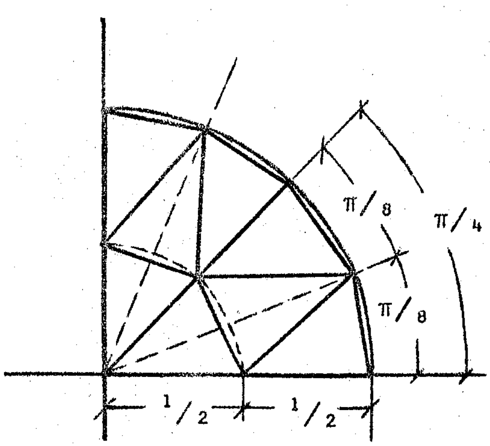
In our example we take a disk of unit radius uniformly loaded.

The exact solution for $f = 64$ is given by

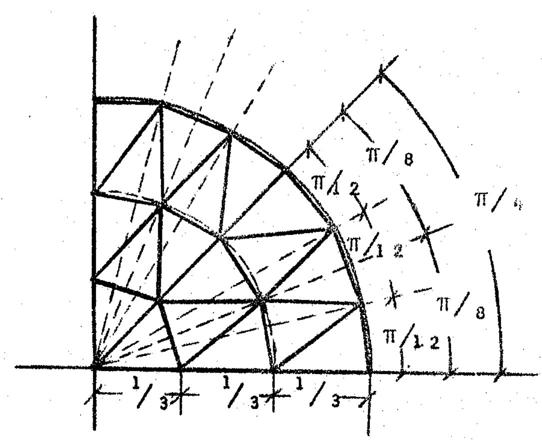
$$u = (1-x_1-x_2)^2$$

The mesh we used defined by an integer positive parameter p is illustrated below for a quarter of the disk.

p = 2:



p = 3:



In the following table we give the approximate values of the displacement at the centre of the disk calculated for different values of μ.

p \ μ	MORLEY 0.00	MORLEY 0.25	MORLEY 0.50	FRAEIJS DE VEUBEKE 0.00	FRAEIJS DE VEUBEKE 0.25	FRAEIJS DE VEUBEKE 0.50
2	1.3483	1.5705	2.0075	0.9497	0.9541	0.9644
4	1.0950	1.1740	1.2700	0.9851	0.9851	0.9857
6	1.0434	1.0699	1.1221	0.9931	0.9930	0.9930
8	1.0251	1.0393	1.0692	0.9961	0.9960	0.9959

Some important conclusions can be drawn from the above results:

1st) Taking into account the number of degrees of freedom (N.D.F.) and the total number of elements (T.N.E.) of the symmetric band matrix of the resulting linear system (which we solved by the Cholesky method) given below:

	MORLEY		FRAEIJS DE VEUBEKE	
p	N.D.F.=2p(2p-1)	T.N.E.=$8p^2(2p-1)$	N.D.F.=p(7p-2)	T.N.E.=$7p^2(7p-2)$
2	12	96	24	336
4	56	896	104	2,912
6	132	3,168	240	10,080
8	240	7,680	432	24,192

One can see that Fraeijs de Veubeke's element is better than Morley's element from the computational point of view.

2nd) The observed rate of convergence in terms of powers of h in the $L^{\infty}(\Omega)$ - norm is 2 for both elements. This is also the case of polygonal plates.

3nd) The approximate values calculated with $\mu = 0$ are more accurate than those obtained with $\mu > 0$. This fact is even more remarkable for Morley's element.

We conjecture that the first two conclusions are quite natural and can be easily explained by the theoretical treatement of the problem. The third one however seems to be an interesting result.

Actually, if we examine more carefully inequality (3) we find [5] :

$$(6) \quad ||u-u_h||_h \leq \frac{2}{1-\mu} \inf_{v_h \in V_h} ||v_h-u||_h + \frac{1}{1-\mu} \sup_{w_h \in V_h} \frac{|E_h(u,w_h)|}{||w_h||_h}$$

The first term of the right side of inequality (6) attains a minimum for $\mu = 0$ in the interval $[0,1/2]$ and its value increases with increasing μ .

The analysis of the second term is more complex since $E_h(u,w_h)$ given by (4) depends on μ.

Nevertheless we can write:

$$\frac{1}{1-\mu} E_h(u,w_h) = \frac{1}{1-\mu} \sum_{K\in \tau_h} \int_{\partial K} \left[\frac{\partial \Delta u}{\partial n_K} w_h - \Delta u \frac{\partial w_h}{\partial n_K}\right] ds_K + E_h^1(u,w_h)$$

where $E_h^1(u,w_h)$ is an expression independent of μ .

If we bound the right side again we get :

$$\frac{1}{1-\mu}\, |E_h(u,w_h)| \le \frac{1}{1-\mu} \Big| \sum_{K\in\tau_h} \int_{\partial K} \Big[\frac{\partial \Delta u}{\partial n_K} w_h - \Delta u \frac{\partial w_h}{\partial n_K}\Big]\, ds_K \Big| + |E_h^1(u,w_n)|$$

The right side of the above inequality also attains a minimum for $\mu = 0$.

In spite of this conclusion a word of caution is in order:

Since the above procedure was essentially based upon bounds the result should not be regarded as a definitive one. In other words, one should not expect that for an arbitrary mesh the results with $\mu = 0$ will be more accurate than those calculated with $\mu > 0$.

Nevertheless the above analysis can be very useful since one can expect that, at least assimptotically, $\mu = 0$ should be the optimal value of μ from an accuracy point of view. Furthermore for this value the bilinear form $a_h(u_h, v_h)$ becomes obviously simpler.

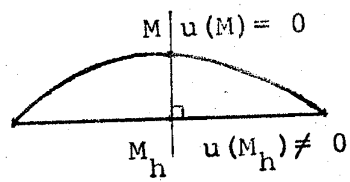
2nd Case: Simply supported plate

If the plate is simply supported, the displacement - function u is the solution of:

$$(P_{SS}) \quad \begin{cases} \Delta^2 u = f \\ u/_\Gamma = 0 \\ [\mu \Delta u + (1-\mu)\dfrac{\partial^2 u}{\partial n^2}]/_\Gamma = 0 \end{cases}$$

For Morley's element the interpolation error estimate of the 1^{st} case remains valid. This is due to the fact that now there is no value of u to be "replaced" at the boundary of the polygon Ω_h since in this case the normal derivatives on Γ_h are not previously assigned.

For Fraeijs de Veubeke's triangle however only this part of the error estimate indicates a lowering from 1 to 1/2 in the rate of convergence in discrete $H^2(\Omega_h)$ - norm. This is essentially because of the changes in values at the mid points of the edges of Γ_h.



The final result for this element is [5] :

$$||u-r_h u||_h \le C[h^{1/2}||u||_{3,\Omega} + h^{3/2}||u||_{4,\Omega}]$$

In order to achieve final error estimates for both elements however it is still necessarily to study the term:

$$\sup_{w_h \in V_h} \frac{|E_h(u,w_h)|}{||w_h||_h} ,$$

since the following term of $E_h(u,w_h)$ cannot be bounded using the same arguments as in [2] :

$$\int_{\Gamma_h} [\Delta u-(1-\mu)\frac{\partial^2 u}{\partial s_h}]\frac{\partial w_h}{\partial n_h}\ ds_h$$

where n_h and s_h represents the normal and the tangent to Γ_h respectively.

This term vanishes in the polygonal case for

$$[\Delta u-(1-\mu)\ \frac{\partial^2 u}{\partial s_h^2}]_{/\Gamma_h} = [\mu\Delta u+(1-\mu)\frac{\partial^2 u}{\partial n_h^2}]_{/\Gamma_h} = 0 \qquad \text{if} \quad \Gamma_h = \Gamma$$

We could try to bound this term using the fact that the value given above is not far from zero, combined with similar arguments to those used for clamped plates. It seems to us however that we would have to assume $u \in H^5(\Omega)$ which is too much to expect of the solution u. It is well known [4] that we only have $u \in H^4(\Omega)$ for $f \in L^2(\Omega)$ and sufficiently smooth boundaries.

In order to be able to obtain a realistic estimate we have adapted for the present case an analysis due to STRANG [6] for the Neumann problem solved with conforming finite elements . This consists basically of considering another discrete problem analogous to (P_h) but this time using a formulation which corresponds to integrating over the true domain Ω instead of the approximate one Ω_h.

Let us first prove the validity of discrete Poincaré inequalities for the spaces V_h .

<u>Lemma 1:</u> $\exists$ two constants C_1 and C_2 independents of h such that:

(7) $||v_h||_{0,\Omega_h} \leq C_1 ||v_h||_h \quad \forall\, v_h \in V_h$

(8) $||v_h||_{1,h} \leq C_2 ||v_h||_h \quad \forall\, v_h \in V_h$

where

$$||v_h||_{1,h} = \Big(\sum_{K \in \tau_h} \int_K |\text{grad } v_h|^2 dx_1 dx_2 \Big)^{1/2}$$

Proof: We first derive inequality (7)

We have:

$$||v_h||_{0,\Omega_h} = \sup_{g \in L^2(\Omega_h)} \frac{|(v_h, g)|}{||g||_{0,\Omega_h}}$$

where $(.,.)$ is the scalar product in $L^2(\Omega_h)$.

For every $g \in L^2(\Omega_h)$, $\exists$ $w \in H^2(\Omega_h) \cap H_0^1(\Omega_h)$ such that [3] :

$$\begin{cases} -\Delta w = g & \text{in} \quad \Omega_h \\ w/_{\Gamma_h} = 0 & \end{cases}$$

Furthermore $\exists$ C, C' and C'' such that

$$||\Delta w||_{0,\Omega_h} \geq C|w|_{2,\Omega_h}$$

$$||w||_{0,\Omega_h} \leq C'|w|_{2,\Omega_h}$$

$$|w|_{1,\Omega_h} \leq C''|w|_{2,\Omega_h}$$

Thus we have:

$$||v_h||_{0\Omega_h} \leq \sup_{w \in H^2(\Omega_h)} \frac{|(v_h, \Delta w)|}{|w|_{2,\Omega_h}}$$

On the other hand

$$(v_h, \Delta w) = \sum_{K\in\tau_h} [\int_{\partial K} \frac{\partial w}{\partial n_K} v_h \, ds_K - \int_{\partial K} \frac{\partial v_h}{\partial n_K} w ds_K + \int_K \Delta v_h w dx_1 dx_2]$$

Using similar arguments as in [2, p. 20,21] we can write:

$$|\sum_{K\in\tau_h} \int_{\partial K} \frac{\partial w}{\partial n_K} v_h \, ds_K| \leq C[h|w|_{1,\Omega_h} + h^2 |w|_{2,\Omega_h}] ||v_h||_h$$

and

$$|\sum_{K\in\tau_h} \int_{\partial K} \frac{\partial v_h}{\partial n_K} w ds_K| \leq C\, h|w|_{1,\Omega_h} ||v_h||_h$$

Thus we have:

$$|(v_h, \Delta w)| \leq C[h|w|_{2,\Omega_h} + |w|_{2,\Omega_h}] ||v_h||_h$$

which proves inequality (7).

Inequality (8) can be derived in the following way:

We first have:

$$||v_h||^2_{1,h} = \sum_{K\in\tau_h} \int_K |\text{grad } v_h|^2 dx_1 dx_2 = \sum_{K\in\tau_h} [-\int_K \Delta v_h v_h dx_1 dx_2 + \int_{\partial K} \frac{\partial v_h}{\partial n_K} v_h ds_K]$$

Using once more the same arguments as in [2, p. 21] we have:

$$\sum_{K\in\tau_h} \int_{\partial K} \frac{\partial v_h}{\partial n_K} v_h ds_K \leq C \sum_{K\in\tau_h} [h|v_h|_{1,K} + h^2 |v_h|_{2,K}] |v_h|_{2,K}$$

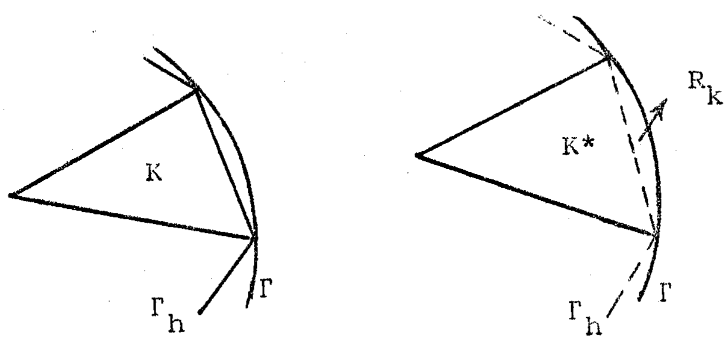
which gives:

$$||v_h||_{1,h} \le C\{||v_h||_h||v_h||_{0,\Omega_h} + [h||v_h||_{1,h} + h^2||v_h||_h]||v_h||_h\}$$

The above inequality and (7) yields (8) for sufficiently small h □

Let us now define the following sets:

- $G_h = \{K/K\in\tau_h$ and ∂K has a common edge with $\Gamma_h\}$
- If $K\in G_h$, than K^* is the region limited by the boundary Γ and the edges of K not belonging to Γ_h



- $R_K = K^* - K$
- $G^*_h = \bigcup_{K\in G_h} (K\cup R_K)$
- $\bar{G}_h = \tau_h - G_h$
- $S_h = \Omega - \Omega_h$

Now we define the space V_h^* as identical to V_h except that we extend to K^* each polynomial defined over the corresponding K.

Let

$$||v_h||_h^* = [\sum_{K\in\bar{G}_h} |v_h|_{2,K} + \sum_{K^*\in G_h^*} |v_h|^2_{2,K^*}]^{1/2} \quad \forall\, v_h \in V_h^*$$

be the norm over V_h^*,

$$a_h^*(u_h,v_h) = \sum_{K\in\bar{G}_h} a_K(u_h,v_h) + \sum_{K^*\in G^*_h} a_{K^*}(u_h,v_h)$$

be a V_h^* - elliptic bilinear form over $V_h^* \times V_h^*$, and

$$(f,v_h)^* = \int_\Omega f v_h dx_1 dx_2$$

be a continuous linear form over V_h^*.

Suppose now that we have to solve problem

$$(P_h^*) \begin{cases} \text{Find } u_h^* \in V_h^* \\ \text{such that} \\ a_h^*(u_h^*,v_h) = (f,v_h)^* \ \forall\, v_h \in V_h^* \end{cases}$$

which has a unique solution.

For problem (P_h^*) we could analogously derive:

$$||u_h^* - u_h||_h^* \le C \, [\inf_{v_h\in V_h^*} ||u - v_h||_h^* + \sup_{w_h\in V_h^*} \frac{|E_h^*(u,w_h)|}{||w_h||_h^*}]$$

where:

$$E_h^*(u,w_h) = a_h^*(u,w_h) - (f,w_h)^*$$

With this new approach, however, all the terms of $E^*_h(u,w_h)$ can be bounded in the same way as those of $E_h(u,w_h)$ for the polygonal case [4] except for the integral:

$$\int_{\Gamma_h} [\mu\Delta u+(1-\mu)\frac{\partial^2 u}{\partial n_h}]\frac{\partial w_h}{\partial n_h}\, ds_h$$

whose analogue along Γ vanishes, since u is the solution of (P_{SS}).

So we can write:

$$|E^*_h(u,w_h)| \le C[h|u|_{3,\Omega} + h^2|u|_{4,\Omega}]||w_h||^*_h \,.$$

Thus we have:

(9) $$||u^*_h - u||^*_h \le C[h|u|_{3,\Omega} + h^2|u|_{4,\Omega}]$$

since the interpolation error can still be estimated in the same way as before.

The only problem left is to estimate:

$$||e_h||_h = ||u_h - u^*_h||_h \qquad \text{for}$$

$$||u_h - u||_h \le ||u^*_h - u||^*_h + ||e_h||_h.$$

4 For terms representing integrals along Γ this is not so obvious. In order to shorten our text however we prefer to omit this part of it.

In order to do this let us first consider the expression:

$$a_h(e_h,e_h) = a_h(u_h^* - u,e_h) + a_h(u,e_h) - a_h^*(u,e_h) - E_h^*(u,e_h) + (f,e_h)^* - (f,e_h)$$

So we have:

(10) $$||e_h||_h^2 \le C[||u_h^* - u||_h^*||e_h||_h + |A_h(u,e_h)| + |E_h^*(u,e_h)|]$$

where:

$$A_h(u,e_h) = a_h(u,e_h) - a_h^*(u,e_h) + (f,e_h)^* - (f,e_h)$$

The term above is actually a sum of integrals over R_k, $K \in G_h^*$.

Thus:

$$|A_h(u,e_h)| \le C \max[||f||_{0,\infty,\Omega}, ||u||_{2,\infty,\Omega}][\text{area}(S_h)]^{1/2}\{\int_{S_h}[|\frac{\partial^2 e_h}{\partial x_1^2}|^2 + 2|\frac{\partial^2 e_h}{\partial x_1 \partial x_2}|^2 + |\frac{\partial^2 e_h}{\partial x_2^2}|^2 + |e_h|^2]dx_1dx_2\}^{1/2}$$

Here we use a lemma given by [6, p. 199]:

If p is a polynomial, then:

(11) $$\int_{R_K}[|\frac{\partial p}{\partial x_1}|^2 + |\frac{\partial p}{\partial x_2}|^2 + p^2]dx_1dx_2 \le C\,\frac{\text{area}(R_K)}{\text{area}(K)}\int_K[|\frac{\partial p}{\partial x_1}|^2 + |\frac{\partial p}{\partial x_2}|^2 + p^2]dx_1dx_2$$

where the constant C only depends on the degree of p.
Setting in (11) p successively equal to

$$\frac{\partial e_h}{\partial x_1}, \frac{\partial e_h}{\partial x_2} \text{ and } e_h ,$$

adding up the resulting inequalities and using Sobolev's inclusion theorem:

$$||u||_{2,\infty,\Omega} \leq C||u||_{4,\Omega} \quad \text{we get:}$$

$$|A_h(u,e_h)| \leq C \max[||f||_{0,\infty,\Omega}, ||u||_{4,\Omega}][\text{area}(S_h)]^{1/2}$$

$$\max_{K \in G_h} [\frac{\text{area}(R_K)}{\text{area}(K)}]^{1/2} \{ \sum_{K\in\tau_h} \int_K [|\frac{\partial^2 e_h}{\partial x_1^2}|^2 + 2|\frac{\partial^2 e_h}{\partial x_1 \partial x_2}|^2 + |\frac{\partial^2 e_h}{\partial x_2^2}|^2 + |\frac{\partial e_h}{\partial x_1}|^2 +$$

$$+ |\frac{\partial e_h}{\partial x_2}|^2 + |e_h|^2]dx_1 dx_2\}^{1/2}$$

The sum in the right side can be bounded by $C||e_h||_h^2$ by Lemma 1. Furthermore

$$\max_{K\in\tau_h} [\frac{\text{area}(R_K)}{\text{area}(K)}]^{1/2} \leq C h^{1/2}$$

and

$$[\text{area}(S_h)]^{1/2} \leq C h$$

Going back to (10) and using (9) we get:

$$||e_h||_h^2 \leq C\{[h|u|_{3,\Omega} + h^2|u|_{4,\Omega} + h^{3/2} \max(||f||_{0,\infty,\Omega}, ||u||_{4,\Omega})]||e_h|| +$$

$$+ |E_h^*(u,e_h)|\}$$

which finally gives:

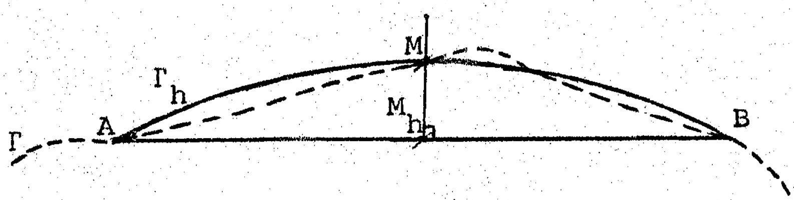
$$(12) \quad ||e_h||_h \leq C[h|u|_{3,\Omega} + h^{3/2} \max(||f||_{0,\infty,\Omega}, ||u||_{4,\Omega})]$$

Inequality (12) allows us to assert that the non-conformity term of Strang's inequality does not bring about any loss in rate of convergence for curved simply supported plates compared to the polygonal case.

An important consequence of this fact is the following:

If the curved boundary is simply approximated by a polygon, then Morley's element provides results of the same degree of accuracy as if the plate were actually a polygon.

For Fraeijs de Veubeke's triangle however, we can only expect comparable results if we approximate the boundary of curved simply supported plates using parabolas arcs interpolating three points of Γ .



The following numerical examples illustrate the above statements. We take again a circular plate of unit radius, uniformly loaded, with $\mu = 0.5$. The mesh we have used is the same as for the clamped plate.

The exact solution for $f = 64$ is:

$$u = (1-x_1^2- x_2^2)(11/3-x_1^2-x_2^2)$$

Its value at the centre of the plate is 3.66667 whose approximations are given at the table below:

MORLEY				
p	Approximate value	Absolute error	N.D.F.= $4p^2$	T.N.E.= $4p^2(4p+1)$
2	4.74175	1.075	16	144
4	3.95342	0.287	64	1,088
6	3.79477	0.128	144	3,600
8	3.73766	0.071	256	8,448

FRAEIJS DE VEUBEKE				
p	Approximate value	Absolute error	N.D.F.= $7p^2$	T.N.E.= $7p^2(7p+1)$
2	3.10562	0.561	28	420
4	3.44341	0.223	112	3,248
6	3.54654	0.120	252	10,836
8	3.58824	0.078	448	25,536

It can be easily seen that for equivalent absolute errors the amount of calculation performed with Morley's element is rather smaller. Furthermore the observed rate of convergence for this element is 2 whereas it is reduced to 3/2 for Fraeijs de Veubeke's triangle.

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