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ON THE SOLVABILITY OF ASYMMETRIC QUASILINEAR FINITE ELEMENT APPROXIMATE PROBLEMS IN NONLINEAR INCOMPRESSIBLE ELASTICITY

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ON THE SOLVALIBITY OF ASYMMETRIC QUASILINEAR FINITE ELEMENT APPROXIMATE PROBLEMS IN NONLINEAR INCOMPRESSIBLE ELASTICITY*

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ABSTRACT

This paper deals with a class of simplicial finite elements for solving incompressible elasticity problems in n-dimensional space, $n=2$ or 3. An asymmetric structure of the shape functions with respect to the centroid of the simplex, renders them particularly stable in the large strain case, in which the incompressibility condition is nonlinear.

We prove that under certain assembling conditions of the elements, there exists a solution to the corresponding discrete problems. Numerical examples illustrate the efficiency of the method.

KEY-WORDS: Asymmetric, compatible, compression, displacements, energy, existence, finite elements, incompressible, Mooney-Rivlin material, nonlinear elasticity, pressure, quasilinear, rubber, simplex, stability, strain, tetrahedrons, triangles.

RESUMO

Considera-se uma classe de elementos finitos de tipo simplex para a resolução de problemas relativos a materiais hiperelásticos incompressíveis como a borracha. Uma estrutura assimétrica dos elementos com respeito ao baricentro do simplex tornam a simulação numérica de grandes deformações de corpos de tais materiais particularmente estável e realista do ponto de vista físico, especialmente no caso tridimensional onde falham métodos clássicos.

Prova-se que certas construções de partições do corpo nesses elementos, conduzem a problemas discretos bem colocados matematicamente. Exemplos numéricos ilustram a eficiência do método em casos de forte compressão.

PALAVRAS-CHAVE:

Assimétricos, borracha, compressão, deformações, deslocamento, elasticidade não linear, elementos finitos, energia, estabilidade, existência, incompressível, material de Mooney-Rivlin partições compatíveis, pressão, quasilinear, simplex, tetraedros, triângulos.

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1- INTRODUCTION

In this work we discuss two finite elements of asymmetric type introduced respectively in [14] and [16], for solving finite incompressible elasticity problems.

Let us first give some notations:

Ω being a bounded set of $\mathbb{R}^n$, for every open subset D of Ω, we shall denote by $||.||_{m,r,D}$ and by $|.|_{m,r,D}$ the usual norm and semi-norm respectively, of the Sobolev space $W^{m,r}(D)$ (see e.g. [1]), $m,r \in \mathbb{R}$, $m \geq 0$ and $1 \leq r \leq \infty$, with $W^{0,r}(D) = L^r(D)$. Similarly in the case $r = 2$ we denote by $(.,.)_{m,D}$ the usual inner product of $W_0^{m,2}(D) \equiv H_0^m(D)$ and by $|.|_{m,D} = |.|_{m,2,D}$ the corresponding norm, while we will represent the norm of $W^{m,2}(\Omega) = H^m(\Omega)$ by $||.||_{m,D}$ instead of $||.||_{m,2,D}$. In all cases we shall drop the subscript D whenever D is Ω itself.

For every space of functions V defined on D, $\underset{\sim}{V}$ will represent the space of vector fields whose n components belong to V. In the case where V is $W^{m,r}(D)$ or $W_0^{m,r}(D)$, we define the norm, semi-norm and inner product (if $r=2$) for $\underset{\sim}{V}$, by introducing obvious modifications in the scalar case, and keeping the same notations.

We shall denote by $x.y$ the euclidian inner product of two vectors x and y of $\mathbb{R}^\ell$ and by $|.|$ the corresponding norm. ℓ will be either equal to n in the case of vectors of $\mathbb{R}^n$, or equal to n^2 in the case of tensors of $\mathbb{R}^{n\times n}$.

Finally for every function or vector field y defined over a certain set D, we shall denote by $y/_S$ its restriction to a subset S, $S \subset D$.

Now our problem can be described as follows:

We are given an elastic body represented by a bounded domain $\Omega \subset \mathbb{R}^n$, $n = 2,3$, with a smooth boundary Γ. Keeping fixed a part Γ_0 of Γ with $meas(\Gamma_0) \neq 0$, we consider a loading of Ω consisting of body forces $\underset{\sim}{f}$ acting on a set $\Gamma^* \subset \Gamma$, such that $meas(\overline{\Gamma}_0 \cap \overline{\Gamma}^*) = 0$ and $\overline{\Gamma}^* \cup \overline{\Gamma}_0 = \Gamma$, having a density $\underset{\sim}{g}$ per unit of measure of Γ^*. Although it is physically possible to have $\Gamma^* = \emptyset$ we will not consider this case in this paper.

The effect of $\underset{\sim}{f}$ and $\underset{\sim}{g}$ is to deform Ω into an equilibrium configuration defined by a displacement vector field that we will denote by $\underset{\sim}{u}$. In this way, the new position of every point $\underset{\sim}{x}$ of Ω is given by $\underset{\sim}{x} + \underset{\sim}{u}(\underset{\sim}{x})$.

Now the fact that every element of Ω is measure invariant in its deformed state, can be expressed mathematically by:

(1.1) $\quad J[\underset{\sim}{x} + \underset{\sim}{u}(\underset{\sim}{x})] = 1 \quad$ for almost every $\underset{\sim}{x} \in \Omega$,

where $J[\underset{\sim}{v}(\underset{\sim}{x})]$ denotes the Jacobian of a vector field $\underset{\sim}{v}$ at point $\underset{\sim}{x}$.

(1.1) is called the incompressibility condition in finite elasticity and we shall often rewrite it as:

(1.2) $\quad det(\underset{\approx}{I} + \underset{\sim}{\nabla}\underset{\sim}{u}) = 1 \quad$ a.e. in Ω ,

where $\underset{\approx}{I}$ is the identity tensor $n \times n$ and $\underset{\sim}{\nabla}$ represents the gradient operator.

<u>REMARK</u>: Condition (1.1) is obviously nonlinear but in the case of small strains, that is to say, when

$$\max_{\underset{\sim}{x} \in \Omega} |\underset{\sim}{\nabla}\, \underset{\sim}{u}(\underset{\sim}{x})| << 1$$

one can neglect products of derivatives of $\underset{\sim}{u}$ of order higher than one. (1.1) becomes then the well-known linear incompressibility condition arising in infinitesimal elasticity or in fluid mechanics, namely:

$$div\ \underset{\sim}{u}(\underset{\sim}{x}) = 0 \qquad \text{for a.e. } \underset{\sim}{x} \in \Omega. \qquad \square$$

Although there is a rather large range of incompressible materials, in this work we would like to focus our study to the case of Mooney-Rivlin materials, because they are particularly representative of the class of materials for which (1.1) holds. We note by the way that among Mooney-Rivlin materials rubber is a typical case.

For a Mooney-Rivlin material the elastic energy for a certain admissible displacement vector field $\underset{\sim}{v}$ is given by [13]:

$$(1.3)_2 \quad W(\underset{\sim}{v}) = \frac{C_1}{2} \int_\Omega |\underset{\approx}{I} + \underset{\sim}{\nabla}\,\underset{\sim}{v}\,|^2 d\underset{\sim}{x} - 2 - \int_\Omega \underset{\sim}{f} \cdot \underset{\sim}{v}\ d\underset{\sim}{x} - \int_{\Gamma^*} \underset{\sim}{g}\cdot\underset{\sim}{v}\ ds \qquad \text{for } n = 2$$

$$(1.3)_3 \quad W(\underset{\sim}{v}) = \frac{C_1}{2} \int_\Omega |\underset{\approx}{I} + \underset{\sim}{\nabla}\,\underset{\sim}{v}|^2 d\underset{\sim}{x} - 2 + \frac{C_2}{2}\int_\Omega |adj(\underset{\approx}{I}+\underset{\sim}{\nabla}\underset{\sim}{v})|\ d\underset{\sim}{x} - 3 - \int_\Omega \underset{\sim}{f}\cdot\underset{\sim}{v}\ d\underset{\sim}{x} - \int_{\Gamma^*} \underset{\sim}{g}\cdot\underset{\sim}{v}\ ds \qquad \text{for } n = 3$$

where adj A denotes the transpose of the matrix of cofactors of an $n\times n$ matrix A and C_1 and C_2 are positive physical constants.

Taking into account (1.2) and the fact that W must be finite, it is natural to choose the following set of admissible displacement vector fields:

$$X = \{\underset{\sim}{v} / \underset{\sim}{v} \in \underset{\sim}{W}^{1,r}(\Omega)\ ,\ \underset{\sim}{v}_{/\Gamma_0} = \underset{\sim}{0}\ ,\ det[\underset{\approx}{I} + \underset{\sim}{\nabla}\,\underset{\sim}{v}(\underset{\sim}{x})] = 1 \quad \text{a.e. in}\ \Omega\}$$

with $r \geq 2(n-1)$, whereas we shall assume that $\underset{\sim}{f} \in L^2(\Omega)$ and $\underset{\sim}{g} \in H^{1/2}(\Gamma^*)$.

The problem we want to solve can now be stated as follows:

$$(P) \quad \begin{cases} \text{Find } \underset{\sim}{u} \in X \text{ such that} \\ W(\underset{\sim}{u}) \leq W(\underset{\sim}{v}) \quad \forall \underset{\sim}{v} \in X \end{cases}$$

It is interesting to note that X is a non convex set and that it is a subset of the vector space $\underset{\sim}{V}$ defined by:

$$\underset{\sim}{V} = \{\underset{\sim}{v} / \underset{\sim}{v} \in \underset{\sim}{W}^{1,r}(\Omega) \; , \; \underset{\sim}{v}_{/\Gamma_0} = \underset{\sim}{0}\}$$

which can be normed by the semi-norm $|.|_{1,r}$ (Ω being connected [11]).

Instead of the minimzation problem (P) itself, we will consider the following weak formulation obtained by dualization of (1.2) with the help of a multiplier p, and by differentiation of $W(\underset{\sim}{u})$ along $\underset{\sim}{v}$ over $\underset{\sim}{V}$.

$$(P') \quad \begin{cases} \text{Find } (\underset{\sim}{u}, p) \in \underset{\sim}{V} \times Q \quad \text{such that} \\ a(\underset{\sim}{u}, \underset{\sim}{v}) + b(\underset{\sim}{u}, \underset{\sim}{v}, p) = \tilde{L}(\underset{\sim}{v}) \qquad \forall \underset{\sim}{v} \in \underset{\sim}{V} \\ \tilde{b}(\underset{\sim}{u}, q) = 0 \qquad \forall \quad q \in Q \end{cases}$$

where $Q = L^{t}(\Omega)$, with t such that $n/r + 1/t \leq 1$, and

$$(1.4) \quad a(\underset{\sim}{u}, \underset{\sim}{v}) = C_1 \int_{\Omega} \underset{\sim\sim}{\nabla u} \cdot \underset{\sim\sim}{\nabla v} \, d\underset{\sim}{x} + C_2 \int_{\Omega} adj(\underset{\approx}{I} + \underset{\sim\sim}{\nabla u}) .$$

$$.[adj(\underset{\approx}{I} + \underset{\sim\sim}{\nabla u} + \underset{\sim\sim}{\nabla v}) - adj \; \underset{\sim}{\nabla} \; \underset{\sim}{v}] \quad \text{with } C_2 = 0 \text{ if } n = 2 \; ,$$

$$(1.5) \quad b'(\underset{\sim}{u}, \underset{\sim}{v}, q) = \int_{\Omega} q[adj(\underset{\approx}{I} + \underset{\sim\sim}{\nabla u})^{T} . \; \underset{\sim}{\nabla} \; \underset{\sim}{v}] d\underset{\sim}{x}$$

$$(1.6) \quad b(\underset{\sim}{v} \; q) = \int_{\Omega} q[det(\underset{\approx}{I} + \underset{\sim}{\nabla} \; \underset{\sim}{v}) - 1] \; d\underset{\sim}{x}$$

$$(1.7) \quad L(\underset{\sim}{v}) = \int_{\Omega} \underset{\sim}{f} . \; \underset{\sim}{v} \; d\underset{\sim}{x} + \int_{\Gamma^*} \underset{\sim}{g} . \; \underset{\sim}{v} \; ds - C_1 \int_{\Omega} div \; \underset{\sim}{v} \; d\underset{\sim}{x}$$

According to results by Le Tallec [10], under reasona - ble assumptions, there exists a hydrostatic pressure p, with $p \in L^t(\Omega)$, associated with every solution $\underset{\sim}{u}$ to problem (P), and in this case $(\underset{\sim}{u},p)$ is a solution to (P').

At this stage we would like to point out that in practi ce, it seems unwise to use formulation (P') for numerical computations with mixed finite elements, such as those we are going to treat here. Indeed, there are other mixed formulation of (P) much more suitable for such a purpose and in this respect we refer to [6], for instance. However, for the sake of clearness, we prefer to consider (P') in this work, as it appears to be the most na - tural formulation of all.

Bearing in mind that our mixed finite element methods apply to other mixed formulations of (P) as well, we shall from now on, consider that we are actually going to approximate pro - blem (P'). For this purpose we will define two finite dimensional spaces $\underset{\sim}{V}_h$ and Q_h aimed at approximating $\underset{\sim}{V}$ and Q, associated with two n-simplicial finite elements for $n=2$ and $n=3$, respectively , which have an asymmetric structure with respct to the centroid of the simplex. The three-dimensional element can be viewed as a certain generalization of the two-dimensional one and it should be mentioned that the latter was first introduced in [14],whereas both were discussed in [16] for linear problems, arising in Me - chanics of incompressible media.

An outline of the paper is as follows:

In Section 2 we define in a general way a discrete analogue (P_h') of (P'), based on finite element approximations. In Section 3 we briefly recall the asymmetric elements and we describe the corresponding problem (P_h'), in connection with two kinds of partitions of Ω. In Section 4 we consider some basic properties of both elements that justify a priori their adequacy for the numerical solution of problem (P). In Section 5 we consider in detail the well-posedness of (P_h') in the case of one of the types of partition considered in Section 3. Finally in Section 6 we discuss the same question for the other type of partition in a particular case, and we give corresponding numerical results.

2. THE FINITE ELEMENT APPROXIMATE PROBLEM

Henceforth, except where otherwise specified, in this paper we consider Ω to be a domain of $\mathbb{R}^n$, $n=2,3$, having a polyhedral boundary Γ. For the case $n=3$ we also assume that $\bar{\Gamma}^*\cap\bar{\Gamma}_0$ is a set of spacial polygonal lines.

We are given a family $(\tau_h)_h$ of partitions of Ω into n-simplices, satisfying the classical assembling rules for the finite element method. Some additional compatibility conditions for $(\tau_h)_h$ related to our asymmetric elements will be specified in Section 3. We also assume that Γ^* and Γ_0 can be viewed as the union of faces of elements of τ_h and that $(\tau_h)_h$ is regular in the following sense:

Denoting by h_K the diameter of the circumscribed sphere and by ρ_K the diameter of the inscribed sphere of element K, $K \in \tau_h$ and setting

$$h = \max_{K\in \tau_h} h_K \qquad \text{and} \qquad \rho = \min_{K\in \tau_h} \rho_K ,$$

there exists a strictly positive constant c such that $\rho h^{-1} > c \ \forall h$.

With each partition τ_h we associate the finite dimensional spaces Q_h and $\underset{\sim}{V}_h$, approximations of Q and $\underset{\sim}{V}$(resp. Q^t and $\underset{\sim}{V}^r$)respectively. We assume that $Q\subset Q_h$, whereas in general a similar inclusion will not hold for V_h. Let $|.|$ be the norm of Q_h induced by $L^2(\Omega)$, and $\|.\|^2_{m,\Omega,h}$ (resp. $|\ |^2_{m,\Omega,h}$) be obtained by summation over the elements $K\in\tau_h$ of the squares of the $\|.\|_{m,K}$-norms (resp. $|.|_{m,K}$-semninorms). In particular we will use the H^1_0-discrete norm for V_h defined by:

$$(2.1) \qquad \| v_h \|^2_h = \sum_{K\in\tau_h} |v_h|^2_{1,K}$$

Now in the discrete analogue of (P'), we weaken the requirement that the approximation $\underset{\sim}{u}_h\in\underset{\sim}{V}_h$ of the solution $\underset{\sim}{u}$ to problem (P) satisfy exactly (1.1), in the following way:

The incompressibility condition is to be satisfied only at those points of Ω to which we attach the degrees of freedom of Q_h. This is equivalent to require that $\underset{\sim}{u}_h$ belong to an approximation X_h of X defined by:

$$X_h = \{ \underset{\sim}{v}_h \ / \ \underset{\sim}{v}_h \in \underset{\sim}{V}_h \ , \ b_h(\underset{\sim}{v}_h, q_h) = 0 \qquad \forall\ q_h \in Q_h \}$$

where b_h is a suitable approximation of b given by (1.6).

A natural way of defining b_h is to set

$$(2.2) \qquad b_h\ (\underset{\sim}{v}_h\ ,\ q_h) = \sum_{K\in\tau_h} b_K\ (\underset{\sim}{v}_h\ ,\ q_h)$$

where b_K corresponds to an approximation of the integral of (1.6), restricted to element K, whose quadrature points are those associated with the degrees of freedom of Q_h. We consider two possibilities of performing this numerical quadrature, according to the way of defining the elements of τ_h.

To be more specific, if the domain Ω is a polygon or a polyhedron, the elements of the partition τ_h are as prescribed above. Notice that in this case we have:

case i) Every $K \in \tau_h$ is the reciprocal image of the usual $\hat{K}$ (see Figure 2.1) by an affine transformation $A_K: K \to \hat{K}$.

In this case we define the approximation of

$$\int_K q_h \, [det(\underset{\approx}{I} + \underset{\sim}{\nabla} \, \underset{\sim}{v}_h) - 1] \, d\underset{\sim}{x} \quad \text{to be:}$$

$$(2.3) \quad b_K(\underset{\sim}{v}_h, q_h) = \sum_{j=1}^{m} w_j \, q_h(x_j^K) \, [det\,(\underset{\approx}{I} + \underset{\sim}{\nabla}\,\underset{\sim}{v}_h) - 1]_{/\underset{\sim}{x}_j^K} \; meas(K)$$

where $\{x_j^K\}_{j=1}^m$ is the set of points used to define q_h/K, and the w_j's are the weights of the numerical quadrature formula.

On the other hand, if Ω has a curved boundary and $\underset{\sim}{V}_h$ is conforming it may be interesting to partition it into curved elements defined in the classical way, namely*:

case ii) Every $K \in \tau_h$ is the reciprocal image of $\hat{K}$ by a bijective isoparametric transformation $\mathcal{R}_K : K \to \hat{K}$. This means that $\mathcal{R}_K^{-1}(\hat{\underset{\sim}{x}}) = [a_1^K(\hat{\underset{\sim}{x}}), \ldots, a_n^K(\hat{\underset{\sim}{x}})]$, where $a_i^K \in \hat{P}$, $1 \le i \le n$, $\hat{P}$ being a space of shape functions defined over $\hat{K}$, such that $\hat{v}_h = v_{h/K} \circ \mathcal{R}_K \in \hat{P}, \forall \, v_h \in V_h$ and $\forall \, K \in \tau_h$.

In this case the approximation of $\int_K q_h [det(\underset{\approx}{I} + \underset{\sim}{\nabla}\,\underset{\sim}{v}_h) - 1] \, d\underset{\sim}{x}$ is given by:

$$(2.4) \quad b_K(\underset{\sim}{v}_h, q_h) = \sum_{j=1}^{m} \omega_j \, \hat{q}_h(\hat{\underset{\sim}{x}}_j) [det \, \hat{\underset{\sim}{\nabla}}(\hat{\underset{\sim}{v}}_h + \mathcal{R}_K^{-1}) - det \, \hat{\underset{\sim}{\nabla}} \mathcal{R}_K^{-1}]_{/\hat{\underset{\sim}{x}}_j} \; meas(\hat{K})$$

where $\{\hat{\underset{\sim}{x}}_j\}_{j=1}^m$ is the set of points of $\hat{K}$ whose reciprocal images through $\mathcal{R}_K$ are the points of K to which we attach the degrees of freedom of Q_h

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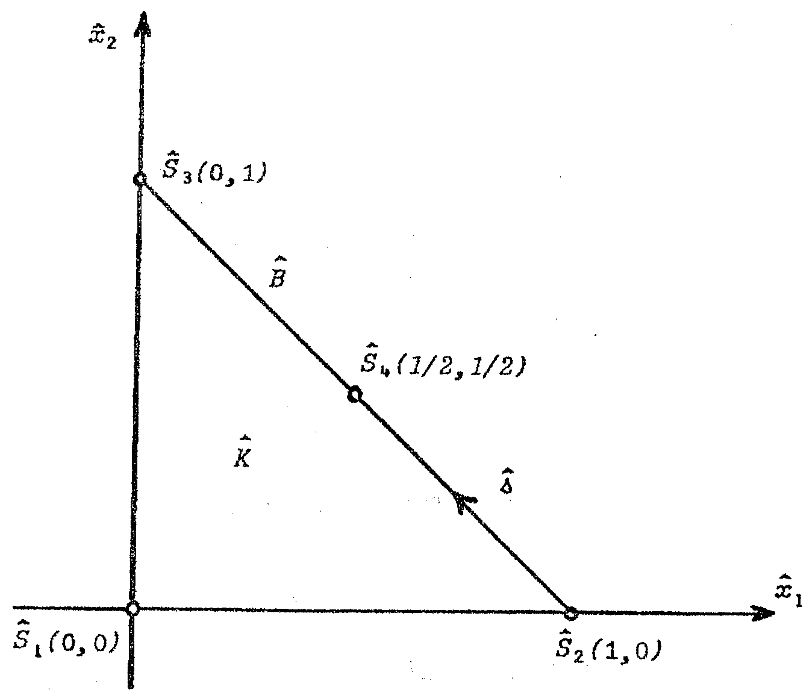
* Now both Γ_0 and Γ^* are approximated by the union of curved faces or edges of elements of τ_h.



The reference element $\hat{K}$ for $n=2$

Figure 2.1

Now, taking into consideration (2.2), we can verify that in both *cases i) and ii)* we have :

$\forall v_h \in X_h$,

(2.5) $\quad det(\underset{\approx}{I} + \underset{\sim}{\nabla}\, \underset{\sim}{v}_h)/_{\underset{\sim}{x}_j^K} = 1 \qquad \forall j,\ 1 \leq j \leq m$ and $\forall K \in \tau_h$.

Indeed, in *case i)* this is trivial provided *meas (K)* in nonzero for all $K \in \tau_h$. One the other hand, from the well-known formula of Calculus [3] we have:

(2.6) $\quad J(\underset{\sim}{v}) = \hat{J}(\hat{\underset{\sim}{v}})\, J(A)$ where $\hat{\underset{\sim}{x}} = A(\underset{\sim}{x})$ and $\hat{\underset{\sim}{v}} \circ A(\underset{\sim}{x}) = \underset{\sim}{v}(\underset{\sim}{x})$

Thus we see that (2.5) also holds for *case ii)* by setting $\underset{\sim}{v}(\underset{\sim}{x}) = \underset{\sim}{v}_h(\underset{\sim}{x}) + \underset{\sim}{x}$ and $A = \mathcal{A}_K$, and taking into account the identity $J^{-1}(A) = \hat{J}(A^{-1})$.

REMARK : If the $\hat{\underset{\sim}{x}}_j$'s are the points of a quadrature formula that integrates exactly functions of form $\hat{J}(\underset{\sim}{v}_h)$ over $\hat{K}$ $\forall \hat{\underset{\sim}{v}}_h \in \hat{P}$, then like in [15] we can draw the following conclusion :

If (2.5) holds and $\sum_{j=1}^{m} \omega_j = 1$, we have $meas(\tilde{K}) = meas(K)$ $\forall K \in \tau_h$, $\tilde{K}$ being the deformed state of K induced by $\underset{\sim}{v}_h$. □

Now we further set

$$(2.7) \quad b_h'(\underset{\sim}{u}_h, \underset{\sim}{v}_h, q_h) = \frac{\partial b_h}{\partial \underset{\sim}{\nabla} \underset{\sim}{u}_h} \cdot \underset{\sim}{\nabla} \underset{\sim}{v}_h$$

and we define the discrete mixed formulation of problem (P) to be:

$$(P_h) \quad \begin{cases} Find (\underset{\sim}{u}_h, p_h) \in \underset{\sim}{V}_h \times Q_h \text{ such that} \\ a(\underset{\sim}{u}_h, \underset{\sim}{v}_h) + b_h'(\underset{\sim}{u}_h, \underset{\sim}{v}_h, q_h) = L(\underset{\sim}{v}_h) & \forall \underset{\sim}{v}_h \in \underset{\sim}{V}_h \\ b_h(\underset{\sim}{u}_h, q_h) = 0 & \forall q_h \in Q_h \end{cases}$$

According to [9], the existence of a solution to problem (P_h) is directly dependent on the validity of a nonlinear discrete Brezzi-type compartibility condition between the spaces $\underset{\sim}{V}_h$ and Q_h. However now this condition must be expressed in terms of the vector field $\underset{\sim}{u}_h$ itself. Since $\underset{\sim}{u}_h$ is supposed to minimize the energy W in some sense, the following result proved in [10], Theorem 4.1 is of crucial importance:

The problem

(2.8) *Find* $\underset{\sim}{u}_h \in X_h$ to minimize $W(\underset{\sim}{u}_h)$ over X_h has a solution .

Now, let $\underset{\sim}{u}_h$ be a local minimum of W. Let also $\|\cdot\|$ be the norm of $\underset{\sim}{V}_h$ and $|\cdot|$ be the norm of Q_h induced respectively by $\underset{\sim}{V}$ and $L^2(\Omega)$. The nonlinear compatibility condition can be stated as follows :

There exists $\beta_h > 0$ such that

$$(2.9) \quad \sup_{\underset{\sim}{v}_h \in \underset{\sim}{V}_h} \frac{\tilde{b}'_h(\underset{\sim}{u}_h, \underset{\sim}{v}_h, q_h)}{\| \underset{\sim}{v}_h \|_h} \geq \beta_h \quad |q_h| \quad \forall q_h \in Q_h$$

According to [10], Theorema 4.3, if condition (2.9) is fulfilled, there exists a unique pressure $p_h \in Q_h$ such that $(\underset{\sim}{u}_h , p_h)$ is a solution to (P_h) .

3 - THE ASYMMETRIC FINITE ELEMENTS

We first define Q_h to be the space of functions q_h that are constant over each element of τ_h , and we clearly have $Q_h \subset Q$. For convenience we consider the degrees of freedom of Q_h to be functional values at the centroid G of the elements. V_h in turn consists of functions whose restriction to each simplex $K \in \tau_h$ belongs to a space P_a defined as follows :

Let S_i denote the vertices of a simplex $K \subset \tau_h$, $i = 1,2,\ldots,n+1$. We first assign to K a priviliedged face, say the face opposite to vertex S_{n+1}, that will be called the base B^K of K, and let F_i^K be the face opposite to vertex S_i , $i = 1,2,\ldots,n$. The F_i^K's will be called the lateral faces of K :

Let λ_i denote the area coordinate of K associated with vertex S_i, $i = 1,2,\ldots,n+1$ and S_{n+2} denote the centroid of B^K .

Now we define P_a to be the $(n+2)$-dimensional space spanned by the functions $\lambda_i = i=1,2,\ldots,n+1$ and ϕ, where :

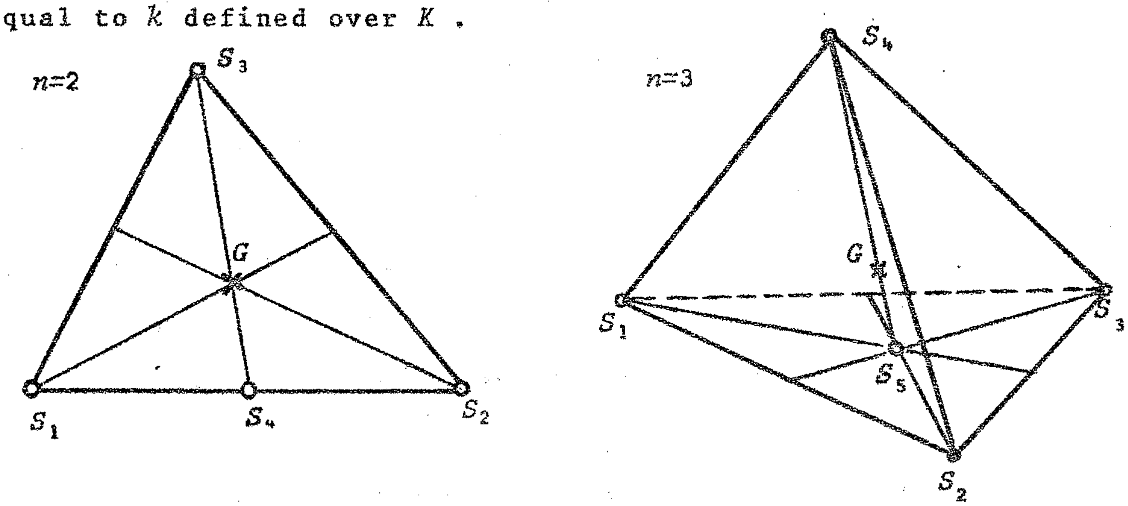
$$(3.1) \qquad \phi = \sum_{\substack{j,k=1 \\ j<k}}^{n} \lambda_j \lambda_k$$

One can easily verify that the set of degrees of freedom $\{a_i\}_{i=1}^{n+2}$, where a_i is the value of the function at point S_i, is P_a-unisolvent and that the associated basis functions are given by:

$$(3.2)\qquad \begin{cases} p_i = \lambda_i - \frac{2}{n-1}\,\phi \quad i = 1,2,\ldots,n \\ p_{n+1} = \lambda_{n+1} \\ p_{n+2} = \frac{2n}{n-1}\,\phi \end{cases}$$

In Figure 3.1. we illustrate the so-defined asymmetric finite elements where o represents degrees of freedom for $\underset{\sim}{V}_h$ and × represents those for Q_h.

Note that the following inclusions hold : $P_1 \subset P_a \subset P_2$, where P_k denotes the space of polynomials of degree less or equal to k defined over K .



The asymmetric quasilinear elements

Figure 3.1.

As remarked in [14] and [15], the elements associated with P_a must be used in connection with partitions of Ω into n-simplices constructed in a special way, which are called com - patible partitions. Let us briefly recall two kinds of such partitions given in [16] for both elements :

Partition τ_h^1 : In the two-dimensional case we first construct a partition of Ω into artibrary convex quadrilaterals (like in the case of the bilinear Q_1 element). Next, every quadrilateral is subdivided into two triangles by an arbitrarily chosen diagonal. Those diagonals will be the only bases of the elements of the so-generated triangulation.

In the three-dimensional case we first construct a partition of Ω into arbitrary convex hexahedrons having quadrilateral faces. Now we refer to figure 3.2b where we show a classical subdivision of a hexahedron into 5 tetrahedrons. We next take an arbitrary point in the interior of each central tetrahedron ABCD, say point E, and we join it to A,B,C and D, so that each hexahedron becomes the union of 8 tetrahedrons. These form partition τ_h^1 if we assign its bases to be the faces of tetrahedrons ABCD.

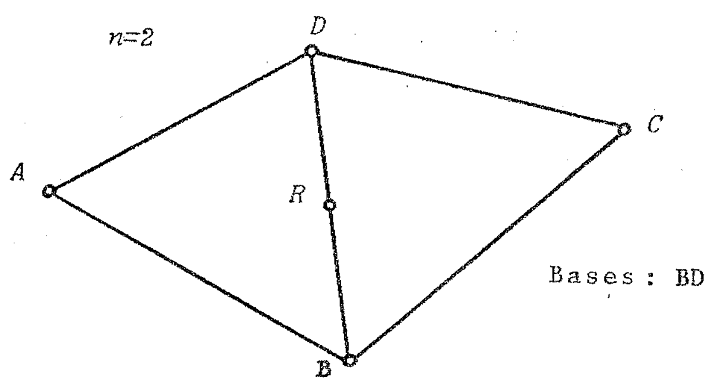


Figure 3.2a

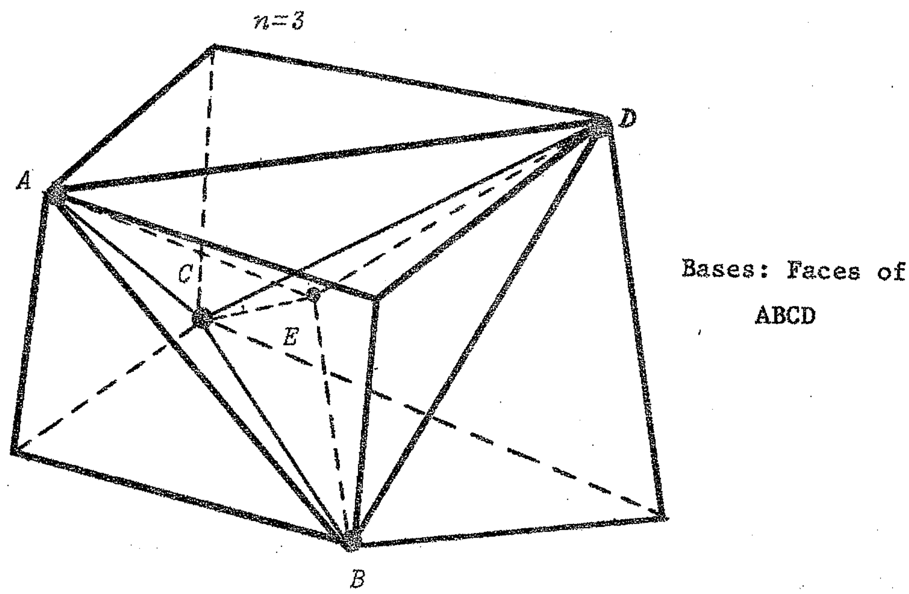


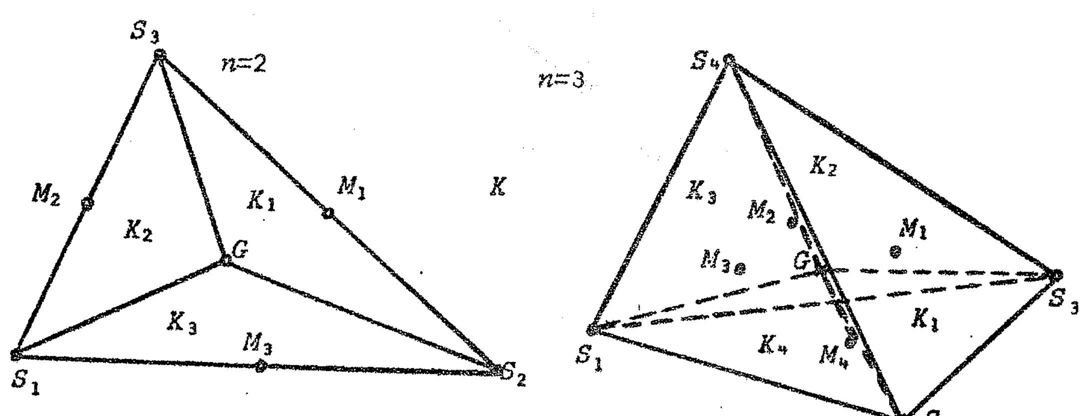
Figure 3.2b

An Illustration of compatible partition τ_h^1

Figure 3.2

Partition τ_h^2 : We first construct an arbitrary partition τ_h of Ω into n-simplices K. Then we subdivide each $K \in \tau_h$ into $n+1$ simplices having a common vertex situated in $\mathring{K}$.

This subpartition of τ_h becomes the compatible partition τ_h^2 if we define its bases to be the faces of τ_h. Note that the interior point of the simplex $K \in \tau_h$ can be arbitrary, although in this work we will choose it to be the centroid G(see figure 3.3).



An illustration of compatible partition τ_h^2 Figure 3.3

With the above considerations, we define the degrees of freedom of V_h to be the functional values at the vertices and at the centroid of the bases of a compatible partition τ_h of Ω , except the values at those nodes lying on $\overline{\Gamma}_0$, where a function of V_h vanishes necessarily.

With the above definition of V_h we can say that $\underset{\sim}{V}_h \subset C^\circ(\overline{\Omega})$ if $n=2$, but if $n=3$ this inclusion does not hold and therefore we have a nonconforming element. Nevertheless, for $n=3$, a function of V_h is necessarily continuous along the bases of the partition.

Let us now examine the particular case of problem (P_h') for the spaces $\underset{\sim}{V}_h$ and Q_h defined above:

We have $m=1$, $\omega_1 = 1$, and the quadrature point $\underset{\sim}{x}_1^k$ is centroid of K in case *i)*, and the image of the centroid of $\hat{K}$ through transformation $\mathcal{R}_k^{-1}$ in case *ii)*.

It is then possible to verify, using arguments to be developped in Section 4, that in both cases *i)* and *ii)*, the so-obtained numerical quadrature formula integrates exactly $det(\underset{\approx}{I}+\underset{\sim}{\nabla}\underset{\sim}{u}_h)$ over K, that is to say:

$$b_K(\underset{\sim}{u}_h \ , \ q_h) = \int_K q_h[det(\underset{\approx}{I} + \underset{\sim}{\nabla} \ \underset{\sim}{u}_h) - 1 \ ] \ d\underset{\sim}{x}$$

and

$$\frac{\partial b_k(\underset{\sim}{u}_h, q_h)}{\partial \underset{\sim}{\nabla} \underset{\sim}{u}_h} \cdot \underset{\sim}{\nabla} \underset{\sim}{v}_h = \int_K \ q_h \ [adj(\underset{\approx}{I} + \underset{\sim}{\nabla} \ \underset{\sim}{u}_h)]^T \cdot \underset{\sim}{\nabla} \ \underset{\sim}{v}_h \ d\underset{\sim}{x}$$

This means that, at least when $\Omega = \Omega_h \equiv \bigcup_{K \in \tau_h} K$, we have $b_h = b$ and $b_h' \equiv b'$, for V_h and Q_h defined in this section.

REMARK : Strictly speaking, if the union of the element K over τ_h is different of Ω, we should redefine problem (P'_h) by replacing a and L by approximate functionals a_h and L_h that take into account integration over Ω_h rather than over Ω.□

4. STABILITY PROPERTIES OF THE ASYMMETRIC ELEMENTS

In this section we intend to justify our proposal of the elements of asymmetric type for the numerical solution of problem (P') from the point of view of the simulation of (1.1).

First of all let us briefly recall some a priori argu - ments already considered in [14] and [15].

If a vector field of an approximation space $\underset{\sim}{V}_h$ of $\underset{\sim}{V}$ is such that each component restricted to an element K of τ_h is a polynominal of P_k , its Jacobian is a polynomial of $P_{n(k-1)}$ over K. This implies that one must satisfy constraint (1.1) in a re - latively large number of points of K in order to simulate the incompressibility phenomenon in a meaningful way. Note that this question becomes particularly critical in the three-dimensional case, where numerical instabilities are frequently observed whenever the number of these point constraints per element is taken small, specially under compression loads.

However, the total number of constraints to be satisfied in the discrete problem associated with (P') - that is precisely $dim\ Q_h$ - must not exceed the total number of displacement degrees of freedom, i.e. $dim\ \underset{\sim}{V}_h$, otherwise condition (2.9) fails to hold (see e.g. [9]). This fact is usually expressed numerically by requiring that the following *asymptotic ratio:*

$$\theta = \lim_{h\to 0} \frac{\dim Q_h}{\dim \underset{\sim}{V}_h}$$

be strictly less than one (actually in practice θ should not be too close to one).

On the other hand, from a mathematical point of view, it is not appropriate to choose a space Q_h satisfying continuity requirements at points situated on the interface of the elements. This fact prevents one from reducing the dimension of Q_h significantly like in the case of linear problems solved with the so-called Taylor-Hood elements [7] .

Let us also add that $\underset{\sim}{V}_h$ should be preferably conforming. Indeed, even if condition (1.1) is properly satisfied elementwise, the nonconformity may lead to a meaningless representation of the incompressibility phenomenon at the global level, unless one can prove that the resulting interpenetrations of neighboring deformed elements cancel each other or are negligible.

Summing up all the above considerations, we can say that, except for a very few cases, one cannot expect to approximate problem (P') by using standard spaces $\underset{\sim}{V}_h$ and Q_h, such as those that work well for fluid problems or for linear incompressible elasticity. Therefore, a solution that seems reasonable, is to construct $\underset{\sim}{V}_h$ by means of spaces of special polynomials of degree k, for which the Jacobian is of maximal degree significaticantly less than $n(k-1)$. As we show hereafter this is precisely the case of P_a.

Theorem 4.1. If $\underset{\sim}{v} = (v_1,\ldots,v_n)$ defined over K is such that $v_i \in P_a\ \forall i$, then $J[\underset{\sim}{x} + \underset{\sim}{v}(\underset{\sim}{x})]$ is a polynomial of P_1.

Proof: According to (3.2), each component v_i can be written as:

$$v_i = \sum_{j=1}^{n+1} \alpha_j^i \lambda_j + \beta^i \phi$$

where the α_j^i s and the β^i's are scalars and ϕ is the quadratic function given (3.1). We have :

$$(4.1) \quad J[\underset{\sim}{x} + \underset{\sim}{v}(\underset{\sim}{x})] = \begin{vmatrix} c_{11} + \beta^1 \frac{\partial\phi}{\partial x_1} & c_{12} + \beta^1 \frac{\partial\phi}{\partial x_2} & \cdots & c_{1n} + \beta^1 \frac{\partial\phi}{\partial x_n} \\ c_{21} + \beta^2 \frac{\partial\phi}{\partial x_1} & c_{22} + \beta^2 \frac{\partial\phi}{\partial x_2} & \cdots & c_{2n} + \beta^2 \frac{\partial\phi}{\partial x_n} \\ \vdots & \vdots & & \vdots \\ c_{n1} + \beta^n \frac{\partial\phi}{\partial x_1} & c_{n2} + \beta^n \frac{\partial\phi}{\partial x_2} & \cdots & c_{nn} + \beta^n \frac{\partial\phi}{\partial x_n} \end{vmatrix}$$

where constant c_{ij} is the x_j - derivative of the linear part of $x_i+v_i(\underset{\sim}{x})$.

Now we expand the above determinant into a sum of 2^n determinants whose j-*th* column is either $(c_{1j}, c_{2j}, \ldots, c_{nj})^\top$ or $\frac{\partial\phi}{\partial x_j}(\beta^1, \beta^2 \ldots, \beta^n)^\top$. As one can easily see, the only determinants of this expansion that do not vanish identically are those having at most one column with linear entries $\frac{\partial\phi}{\partial x_j} \underset{\sim}{\beta}^\top$, and the results follows. q.e.d.

An immediate consequence of Theorem 4.1, is the fact that it suffices to satisfy (1.1) at the centroid G of K to have incompressible elements in the following weak sense:

The measure of K in deformed state induced by $\underset{\sim}{v} \in \underset{\sim}{P}_a$ is invariant.

Indeed, if we denote by $\tilde{A}$ the deformed state induced by $\underset{\sim}{v}$ of any subset A of K, $K \in \tau_h$, according to a well-known numerical quadrature formula, we have:

$$meas(\tilde{K}) = \int_K J[x + \underset{\sim}{u}(\underset{\sim}{x})]\, d\underset{\sim}{x} = J[G + \underset{\sim}{u}(G)]\; meas(K) = meas\;(K).$$

This shows that the space Q_h defined in section 3 is a proper choice for these asymmetric elements.

REMARKS :

i) The relation above holds in the isoparametric case too, if G is replaced by the image of $\hat{G}$ under $\mathcal{R}_K$.

ii) Using the same arguments as in [15], we can conclude that for both $n=2$ and $n=3$, we have $\theta = 1/2$, in case partition τ_h^2 is used. In the case of τ_h^1, the same value of θ applies for $n=2$, while $\theta = 4/9$ for $n=3$.

iii) In the two-dimensional case, the standard $Q_1 \times P_0$ element has the same properties as the quasi-linear asymmetric element, as far as the degree of the Jacobian and θ are concerned. It can actually give satisfactory numerical results as shown by many examples in [10]. However, in the three-dimensional case, the property of Theorem 4.1 no longer holds for the Q_1 element.

iv) Another generalization to the case $n=3$ of the two-dimensional asymmetric element satisfying the property of Theorem 4.1, was presented in [15]. This element has the advantage of being conforming, but the value of the asymptotic ratio is rather high, namely $\theta = 4/5$ or $\theta = 8/11$ in the most favorable case of partitions. This explains the introduction of the present nonconforming generalization.

Now, having proved that the incompressibility can be properly treated for each element, we would like to assert that the same is true for Ω .

More precisely, letting A denote any subset of Ω, setting $A_K = A \cap K$, $K \in \tau_h$ and defining

$$\tilde{A} = \bigcup_{K \in \tau_h} \tilde{A}_K \quad \text{with} \quad \tilde{A}_K = \underset{\sim}{u}(A_K) \quad ,$$

where $\underset{\sim}{u}/K \in \underset{\sim}{P}_a$, we would like to verify that

$$meas(\tilde{K}) = meas(K) \quad \forall\, K \in \tau_h \;\Rightarrow\; meas(\tilde{\Omega}) = meas(\Omega) \quad ,$$

or yet that

$$meas(\tilde{\Omega}) = \sum_{K \in \tau_h} meas(\tilde{K})$$

Actually letting $\widehat{\Omega}$ be the deformed state of Ω induced by $\underset{\sim}{u}$ to be defined hereafter, we will prove that :

$$(4.2) \qquad meas(\widehat{\Omega}) = \sum_{K \in \tau_h} meas(\tilde{K})$$

In the two-dimensional case it will be convenient to set $\widehat{\Omega} = \tilde{\Omega}$. Indeed if $J[\underset{\sim}{x} + \underset{\sim}{u}(\underset{\sim}{x})] \geq 0 \;\; \forall \;\; \underset{\sim}{x} \in \Omega$, (3.2) is trivially satisfied since $\underset{\sim}{V}_h$ is conforming and therefore the elements in deformed state do not interpenetrate. However even under the above assumption, this is not necessarily the case of a nonconforming $\underset{\sim}{V}_h$. That is why for $n=3$ we will set $\hat{\Omega} = \bigcup_{K \in \tau_h} \bar{K}$, where $\bar{K}$ denotes the deformed state of K induced by the vector field $\pi\underset{\sim}{u}$ that interpolates $\underset{\sim}{u}$ at the vertices of the elements of τ_h. In this way $\widehat{\Omega}$ can be viewed as a certain interpolation of $\tilde{\Omega}$ at the points $\tilde{S}$, S being a vertex of an element of τ_h. In so doing we can prove that (4.2) is exactly satisfied for some kind of partitions, whereas in the general case it is satisfied up to an $O(h^2)$ term.

Before giving the proofs, let us say that, whenever the above Jacobian is negative for some $\underset{\sim}{x} \in \Omega$, we must define $\tilde{A}$ for $A \subset \Omega$, not as the union of the $\tilde{A}_K$'s, but with modifications taking into account the interpenetrations of the elements in deformed state that occur in the general case. This can be achieved by assigning a subtractive meaning to the sets $\tilde{A}$ such that $J[\underset{\sim}{x} + \underset{\sim}{u}(\underset{\sim}{x})] < 0 \ \forall \underset{\sim}{x} \in A, \ A \subset K$. In so doing, all the above assertions for the so-defined Ω would be true, and in particular (4.2) with or without a pertubation term. Bearing this in mind, for the sake of simplicity, we shall assume in this section that $\underset{\sim}{u} \in \underset{\sim}{V}_h$ is such that :

(4.3) $\quad J[\underset{\sim}{x} + \underset{\sim}{u}(\underset{\sim}{x})] \geq 0 \qquad$ for a.e. $\quad \underset{\sim}{x} \in \Omega$

(4.4) $\quad J[G_K + \underset{\sim}{u}(G_K)] = 1 \quad \forall K \in \tau_h$, where G_K is the centroid of K.

Now we note that $(x + \pi u)_{/K}$ is nothing else than the the linear part of $(\underset{\sim}{x}+\underset{\sim}{u})_{/K}$ .Therefore, since $\frac{\partial \phi}{\partial x_j}$ vanishes at vertex S_{n+1} , $j = 1,2,\dots,n$ and recalling (3.1) we have :

$$J[S_{n+1} + \underset{\sim}{u}(S_{n+1})] = J[\underset{\sim}{x} + \pi \ \underset{\sim}{u}(\underset{\sim}{x})] \quad \forall \underset{\sim}{x} \in K .$$

Since $meas(\acute{K}) = \int_K J[\underset{\sim}{x} + \pi\underset{\sim}{u}(\underset{\sim}{x})]d\underset{\sim}{x}$, assumption (4.3) implies that $meas(\acute{K}) \geq 0$, which in this case means that the $\acute{K}$'s are oriented in the same way as the K's, or yet that the $\acute{K}$'s do not interpenetrante.

Let us now consider the particular case $n=3$. We further define $\acute{A}$ to be the deformed state induced by $\pi\underset{\sim}{u}$ of every subset A of Ω. Notice that we are actually defining $\hat{\Omega} = \acute{\Omega}$.

We first need the following lemma proved in [16] :

Lemma 4.1 : Let K be a tetrahedron and $\underset{\sim}{n}_K$ denote the outer unit normal vector with respect to ∂K, the boundary of K. Let $\underset{\sim}{\psi}$ be a vector field defined over K such that $\underset{\sim}{\psi} = \underset{\sim}{\beta}\, \phi$, with $\underset{\sim}{\beta} \in \mathbb{R}^3$ and ϕ be given by (3.1). We then have :

$$\int_K div\, \underset{\sim}{\psi}\, d\underset{\sim}{x} = \frac{2}{3} \int_{B^K} \underset{\sim}{\psi} \cdot \underset{\sim}{n}_K\, ds \qquad \square$$

Now we note that since $\pi \underset{\sim}{u}$ is conforming we clearly have:

$$vol(\hat{\Omega}) = \sum_{K \in \tau_h} vol(\hat{K})$$

Actually we can prove that, under a reasonable assumption the above equality also holds if the $\hat{K}$'s are replaced by the $\tilde{K}$'s.

Theorem 4.2 : If τ_h is a compatible partition of Ω that has no base on Γ^* we have :

$$vol(\hat{\Omega}) = \sum_{K \in \tau_h} vol(\tilde{K})$$

REMARK : It is interesting to note that partition τ_h^1 satisfies the assumptions of this theorem.

Proof : A partition satisfying the assumptions of the theorem can be viewed as a subpartition of a first partition X_h of Ω, consisting of hexahedrons having triangular faces. Each hexahedron H of X_h generates two tetrahedrons of τ_h, say K_1 and K_2, having a common base lying in the interior of H, and lateral faces coinciding with the faces of the hexahedron (see Figure 4.1).

Since $\underset{\sim}{u}$ is continuous over B, the common basis of K_1 and K_2, we have:

$$vol(\tilde{H}) = vol(\tilde{K}_1) + vol\ (\tilde{K}_2)$$

Now we want to prove that we actually have

$$vol(\tilde{H}) = vol(\hat{H}) \qquad \forall\ H \in X_h$$

which will yield the result we are looking for, since

$$vol(\hat{\Omega}) = \sum_{H \in X_h} vol(\hat{H}) \ .$$

For this purpose we introduce a new variable $\hat{\underset{\sim}{x}}$ with the help of the following affine transformation over each K :

$$\underset{\sim}{x} \rightarrow \hat{\underset{\sim}{x}} = \underset{\sim}{x} + \pi \underset{\sim}{u}(\underset{\sim}{x})$$

In this way $\tilde{K}$ can be regarded as the deformed state o $\hat{K}$ obtained by the application of the displacement vector field $\hat{\underset{\sim}{\psi}}$ defined by:

$$\hat{\underset{\sim}{\psi}}(\hat{\underset{\sim}{x}}) = \underset{\sim}{\psi}(\underset{\sim}{x})$$

where $\underset{\sim}{\psi} = \underset{\sim}{\beta}\ \phi$, with $\underset{\sim}{\beta} = (\underset{\sim}{u})_5 - [\sum_{i=1}^{3} (\underset{\sim}{u})_i] \ /\ 3$, $(\underset{\sim}{u})_i$ being the value of $\underset{\sim}{u}$ at S_i, $i = 1,2...,5$.

If we denote by $\hat{\lambda}_i(\hat{\underset{\sim}{x}})$ the area coordinates of $\hat{K}$, we have necessarily $\hat{\lambda}_i(\hat{\underset{\sim}{x}}) = \lambda_i(\underset{\sim}{x})$, which means that $\hat{\underset{\sim}{\psi}} = \underset{\sim}{\beta}\ \hat{\phi}$ where

$$\hat{\phi} = \sum_{\substack{j,k=1 \\ j < k}}^{n} \hat{\lambda}_j \hat{\lambda}_k \ .$$

Now we have :

$$vol(K) = \int_{\hat{K}} \hat{J}[\hat{\underset{\sim}{x}} + \hat{\underset{\sim}{\psi}}(\hat{\underset{\sim}{x}})]\ d\hat{\underset{\sim}{x}}$$

where $\hat{J}$ represents the Jacobian with respect to the new variable $\hat{\underset{\sim}{x}}$.

Expanding the integrand above, we have :

$$vol(\tilde{K}) = vol(\hat{K}) + \int_{\hat{K}} \hat{div}\ \hat{\underset{\sim}{\psi}} d\hat{\underset{\sim}{x}} + \int_{\hat{K}} [\sum_{\ell=1}^{3} [J(\hat{\underset{\sim}{\psi}}_\ell) + \hat{J}(\hat{\underset{\sim}{\psi}})]\ d\hat{\underset{\sim}{x}}$$

where $\hat{\underset{\sim}{\psi}}_\ell$ is the vector field obtained by replacing the ℓ-th component of $\hat{\underset{\sim}{\phi}}$ by $\hat{x}_\ell$ and $\hat{div}$ represents the divergence operator associated with $\hat{\underset{\sim}{x}}$.

Since each Jacobian of the second integrand above has at least two columns of form $\underset{\sim}{\beta}\ \hat{\phi}$, they vanish identically.

On the other hand, according to Lemma 4.1 we have :

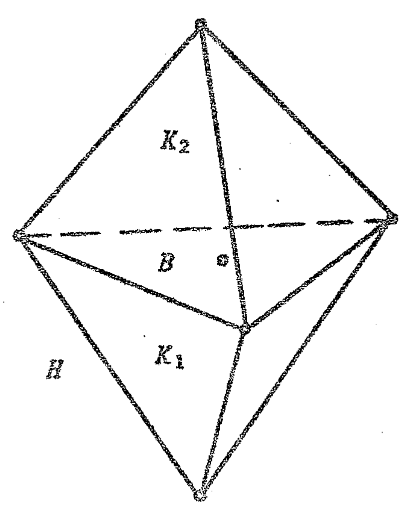
$$\int_{\hat{K}} \hat{div}\ \hat{\underset{\sim}{\psi}}\ (\hat{\underset{\sim}{x}})\ d\hat{\underset{\sim}{x}} = \frac{2}{3} \int_{\hat{B}} \hat{\underset{\sim}{\psi}} \cdot \underset{\sim}{n}_{\hat{K}}\ d\hat{s}\ .$$

However, since $\pi\underset{\sim}{u}$ is conforming, $\hat{B}$ coincides for both K_1 and K_2 together with $\hat{\psi}_{/\hat{B}}$, whereas $\underset{\sim}{n}_{/\hat{K}_1\ /\hat{B}} = -\underset{\sim}{n}_{/\hat{K}_2/\hat{B}}$

Therefore we have :

$$vol(\tilde{H}) = vol(\tilde{K}_1) + vol(\tilde{K}_2) = vol(\hat{K}_1) + vol(\hat{K}_2) = vol(\hat{H}).$$

q.e.d.



A hexahedron of partition X_h Figure 4.1

Now for the general case we have:

Theorem 4.3: For any compatible family $(\tau_h)_h$ of partitions of Ω we have:

$$\left| vol(\hat{\Omega}) - \sum_{K\in \tau_h} vol(\tilde{K}) \right| \leq C\, h^2\, |\underset{\sim}{u}|_{2,\infty}$$

where C is a constant independent of h.

Proof : According to Theorem 4.2, all we have to do is proving that

$$\left| \sum_{K\in\tau_h^*} [vol(\hat{K}) - vol(\tilde{K})] \right| \leq C\, h^2\, |\underset{\sim}{u}|_{2,\infty}$$

where $\tau_h^* = \{K/K \in \tau_h\ ,\ meas\ (B^K \cap \Gamma^*) \neq 0\}$

By a direct computation of the increments of volume of $\hat{K}$ over its faces, due to the quadratic component $\underset{\sim}{\beta}\,\phi$ of $\underset{\sim}{u}$, we get:

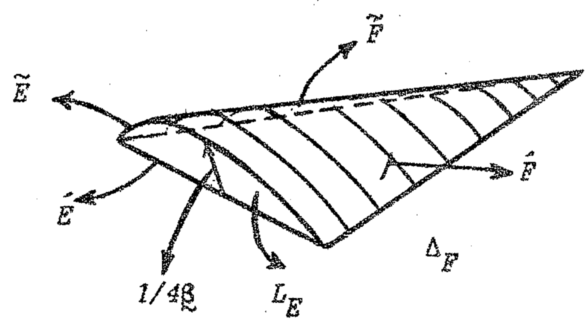
$$vol(\tilde{K}) - vol(\hat{K}) = \int_{\partial\hat{K}} \underset{\sim}{\psi}(\hat{\underset{\sim}{x}}) .\ \underset{\sim}{n}_{\hat{K}}\, d\hat{x}$$

According to Lemma 4.1 we get :

$$vol(\tilde{K}) - vol(\hat{K}) = -\,2 \sum_{i=1}^{3} \int_{\hat{F}_i} \hat{\underset{\sim}{\psi}}(\hat{\underset{\sim}{x}}) \cdot \underset{\sim}{n}_{\hat{K}}\, d\hat{\underset{\sim}{x}}$$

Now, F being a lateral face of element K, we define the set Δ_F as follows:

Let E be the edge of F belonging to the basis of te - trahedron K and let L_E be the plane surface delimited by $\hat{E}$ and $\tilde{E}$. Δ_F is defined to be the solid delimited by $\tilde{F}$, $\hat{F}$ and L_E as ilus- trated in Figure 4.2 below.



A pertubation of $\acute{F}$ due to the quadratic components of $\underset{\sim}{u}$.

Figure 4.2

Using classical arguments, if $(\tau_h)_h$ is regular we can estimate:

$$vol(\Delta_F) \leq C\ h^4\ |\underset{\sim}{u}|_{2,\infty} \qquad \forall\ F$$

Now noting that

$$\left|vol(\Delta_{F_i})\right| = \left|\int_{\hat{F}_i} \acute{\underset{\sim}{\psi}}\ (\acute{\underset{\sim}{x}})\ .\ \underset{\sim}{n}_{\acute{K}}\ d\acute{\underset{\sim}{x}}\right|$$

we have :

$$vol(\tilde{K}) - vol(\acute{K}) \leq 6\ C\ h^4\ |\underset{\sim}{u}|_{2,\infty}$$

Since card τ_h^* $\leq C\ h^{-2}$ the result follows.

q.e.d.

5. EXISTENCE RESULTS IN THE CASE OF PARTITION τ_h^2

Let us now prove that under suitable assumptions on $\underset{\sim}{u}_h$, the compatibility condition (2.9) is satisfied for any partition τ_h^2 . We treat separately cases *i)* and *ii)*, and the latter only for the two-dimensional asymmetric element. For partition τ_h^1 in some particular cases, the existence of a solution to problem (P_h') will be examined in section 6.

For the sake of simplicity we will work with the linear manifold $\underset{\sim}{V}_h^x$ of $\underset{\sim}{V}_h$, defined to be $\underset{\sim}{x} + \underset{\sim}{V}_h$. We also define the following subset of $\underset{\sim}{V}_h^x$:

$$\tilde{X}_h^x = \{\underset{\sim}{u}_h^x \;/\; \underset{\sim}{u}_h^x - \underset{\sim}{x} \in \tilde{X}_h\}$$

In both *cases i)* and *ii)* we shall prove the validity of (2.9) under the following basic assumption on $\underset{\sim}{u}_h$.

ASSUMPTION A) Let $\pi\underset{\sim}{u}_h^x$ denote the piecewise linear interpolate of $\underset{\sim}{u}_h^x$ defined in Section 3. The triangulation $\hat{\tau}_h^2$ of $\hat{\Omega}_h = \Pi\,\underset{\sim}{u}_h^x\,(\Omega_h)$. defined to be:

$$\hat{\tau}_h^2 = \{\hat{K} \;/\; \hat{K} = \Pi\,\underset{\sim}{u}_h^x\,(K)\;,\; K \in \tau_h^2\}\;,$$

is such that there exists a constant $\alpha > 0$ for which we have:

$$\frac{1}{\alpha}\quad area(K) \geq area(\hat{K}) \geq \alpha\; area(K) \qquad \forall\, K \in \tau_h^2\;. \quad \square$$

Notice that Assumption A) implies that $J(\Pi\underset{\sim}{u}_h^x) > 0$ a.e. in Ω_h. It also implies that $\hat{\tau}_h^2$ belongs to a regular family of partitions $\{\hat{\tau}_h^2\}_h$, whenever $\underset{\sim}{u}_h$ belongs to a bounded subset of $W^{1,\infty}(\Omega_h)$ $\forall\, h$.

Indeed, in this case if we set :

$$\hat{h} = \max_{K\in\tau_h^2}\;\{\,\hat{h}_K = diameter\ of\ \hat{K}\}$$

and

$$\hat{\rho} = \min_{K\in\tau_h^2}\;\{\hat{\rho}_K = \text{diameter of the inscribed circle in } \hat{K}\}$$

we have $\hat{\rho}\,\hat{h}^{-1} \geq \hat{c}\ \ \forall h$, where $\hat{c}$ is given by $\frac{2\alpha}{3}\ \frac{c^2}{U^2}$ with $U = \max_h |\underset{\sim}{u}_h|_{1,\infty}$, c being the constant such that $\rho h^{-1} \geq c > 0$ $\forall \tau_h^2 \in \{\tau_h^2\}_h$, as one can easily verify.

We now consider *case i)* :

In this case both $\Omega = \Omega_h$ and $\hat{\Omega} = \hat{\Omega}_h$ are polygons. Thus, since $\Pi \underset{\sim}{u}_h^x$ defines an affine transformation over each triangle K onto $\hat{K}$, we can define a space $\hat{\underset{\sim}{V}}_h$ over $\hat{\Omega}_h$ associated with $\hat{\tau}_h^2$ in the same way as $\underset{\sim}{V}_h$ is associated with τ_h^2, and $\hat{\underset{\sim}{V}}_h$ will have the same structure as $\underset{\sim}{V}_h$.

Also we define $\hat{Q}_h$ to be the space of pressures analo - gous to Q_h for triangulation $\hat{\tau}_h^2$.

Let us first consider the subspace $\hat{Q}_h^0$ of those pressures that are constant over $\hat{K}$, K, being a simplex of τ_h. According to [4] Lemma $C2$, if $\hat{\underset{\sim}{V}} = \{\hat{\underset{\sim}{v}} \ / \ \hat{\underset{\sim}{v}} \in H^1(\hat{\Omega}),\ \hat{\underset{\sim}{v}} = \underset{\sim}{0}$ on $\hat{\Gamma}_0 \equiv \Gamma_0\}$, $\forall\, \hat{q}_h^0 \in \hat{Q}_h^0\ \exists\, \hat{\underset{\sim}{v}} \in \hat{\underset{\sim}{V}}$ such that

$$(5.1)\qquad (\hat{div}\ \hat{\underset{\sim}{v}}\ ,\ \hat{q}_h^0)_{0,\hat{\Omega}} \geq \hat{\beta}_0\ |\hat{q}_h^0|^2_{0,\hat{\Omega}}$$

$$(5.2)\qquad |\hat{\underset{\sim}{v}}|_{1,\hat{\Omega}} \leq C_0 |\hat{q}_h^0|_{0,\hat{\Omega}}$$

where $\hat{\beta}_0 > 0$ and C_0 are independent of $\hat{q}_h^0$.

Lemma 5.1. There exist constants $\hat{\beta}_0 > 0$ and $\hat{C}_0$ such that with every $\hat{q}_h^0 \in \hat{Q}_h^0$ we can associate a $\hat{\underset{\sim}{w}}_h \in \hat{\underset{\sim}{V}}_h$ that satisfies :

(5.3) $\hat{\underset{\sim}{w}}_h(\hat{S}) = 0$ for all vertices $\hat{S}$ of a macrosimplex $\bigcup_{i=1}^{n+1} \hat{K}_i$, the K_i's being the simplices of a macrosimplex $K \subset \tau_h$, where τ_h is the first partition of Ω upon which τ_h^2 is constructed.

$$(5.4)\qquad (\hat{div}\ \hat{\underset{\sim}{w}}_h,\ \hat{q}_h^0)_{0,\hat{\Omega},h} \geq \hat{\beta}_0 |\hat{q}_h^0|_{0,\hat{\Omega}}$$

(5.5) $\quad |\hat{\underset{\sim}{w}}_h|_{1,\hat{\Omega},h} \leq \hat{C}_0 |\hat{q}_h^0|_{0,\hat{\Omega}}$

<u>*Proof*</u> : Let $\hat{\underset{\sim}{v}} \in \hat{\underset{\sim}{V}}$ satisfy (5.1) and (5.2). We associate with $\hat{\underset{\sim}{v}}$ a vector field $\hat{\underset{\sim}{w}}_h \in \hat{\underset{\sim}{V}}_h$ such that $\hat{w}_{h/\hat{K}}$ satisfies $\forall K \in \tau_h$:

$$\hat{\underset{\sim}{w}}_h(\hat{S}) = 0 \text{ if } \hat{S} \text{ is a vertex of } \hat{K}_i \quad i = 1,2,\dots,n+1$$

$$\hat{\underset{\sim}{w}}_h(\hat{M}_i) = \frac{3}{2} \quad \frac{\int_{\hat{B}_i} \hat{\underset{\sim}{v}}\, ds}{meas(\hat{B}_i)}$$

where $\hat{B}_i$ is the base of $\hat{K}_i$ and $\hat{M}_i$ is its mid-point. (5.3) is thus fulfilled.

Using Lemma 4.1, letting $\hat{\tau}_h$ be the partition of Ω into macrosimplices $\hat{K}$, $K \in \tau_h$, we obtain:

$$\int_{\hat{K}} \hat{div}\, \hat{\underset{\sim}{w}}_h\, d\underset{\sim}{x} = \int_{\hat{K}} \hat{div}\, \hat{\underset{\sim}{v}}\, d\underset{\sim}{x} \qquad \forall \hat{K} \in \hat{\tau}_h$$

This yields :

$$(\hat{div}\, \hat{\underset{\sim}{w}}_h\,,\, \hat{q}_h^0)_{0,\hat{\Omega},h} = (\hat{div}\, \hat{\underset{\sim}{v}},\, \hat{q}_h^0)_{0,\hat{\Omega}}$$

which in turn gives (5.4), taking into account (5.1).

In order to prove (5.5) we first use the Trace Theorem and we get:

$$|\hat{\underset{\sim}{w}}_h|_{1,\hat{K}_i} = \hat{\underset{\sim}{w}}_h(\hat{M}_i)\; C(\hat{K}_i) \leq C'(\hat{K}_i)\; \|\hat{\underset{\sim}{v}}\|_{1,\hat{K}_i} \quad ,$$

which according to *Assumption A)* yields :

$$|\hat{\underset{\sim}{w}}_h|_{1,\hat{\Omega},h} \leq C\,(\hat{\Omega},\, \hat{\underset{\sim}{u}}_h)\; |\hat{\underset{\sim}{v}}|_{1,\hat{\Omega}} \qquad \text{with } C < \infty$$

Thus using (5.2) we get (5.5) with $\hat{C}_0 = C_0 C$. q.e.d.

Let now $s_i = meas\ (\hat{K}_i)$, $1 \leq i \leq n+1$. Without loss of generality we can assume that $s_1 \geq s_2 \geq \dots \geq s_{n+1}$

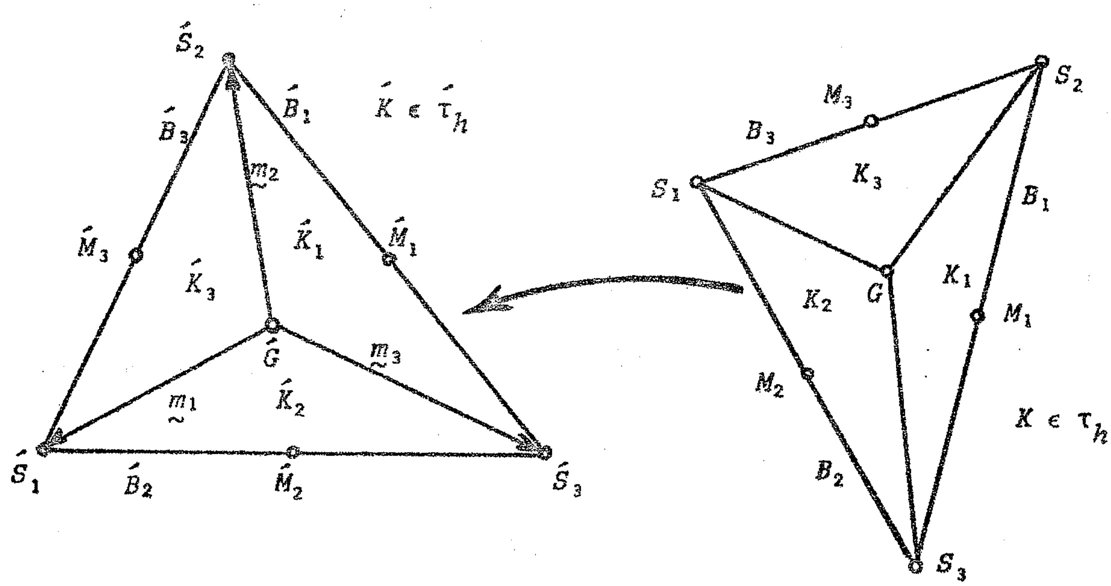
Proof : Let $\hat{q}_h^0 = q_h^0 + q_h^1$ where $q_h^0 \in \hat{Q}_h^0$ and $q_h^1 \in \hat{Q}_h^1$.

We first construct a vector field $\hat{z}_h \in \hat{Z}_h$ satisfying (5.3) in the followings way :

$z_h = 0$ at every vertex or mid-point of the bases of $\hat{K}$, $\hat{K} \in \hat{\tau}_h^2$. If $\hat{G}$ is the common vertex of $\hat{K}_i$, $i=1,2,\ldots,n+1$, we define $z_h(\hat{G})$ to be of the form:

$$\hat{z}_h(\hat{G}) = \sum_{i=2}^{n+1} \gamma_i^K \, m_i$$

where the m_i's are the oriented edges $\hat{G}\hat{S}_i$ of the $\hat{K}_i$'s, as indicated in Figure 5.1 below, and the γ_i^{K}'s are given scalars de - pending on the q_i^{K}'s only (see (5.6)).



Macrolements of partitions $\hat{\tau}_h$ and τ_h for $n=2$

Figure 5.1

First of all for $n=2$ we set

$$\gamma_2^K = -q_2^K \quad \text{and} \quad \gamma_3^K = q_3^K$$

Now using *Assumption A)* one can easily estimate :

Let $\hat{Q}_h^1$ be the subspace of $\hat{Q}_h$ generated by the set of orthogonal functions $\{\eta_2^h,\dots,\eta_{n+1}^K\}_{K\in\tau_h}$ such that $supp(\eta_i^K) \subset \hat{K}$ $i = 2,\dots,n+1$, and

$n = 2$

$$\begin{cases} \eta_2^K = -1 & \text{if } \hat{x} \in \hat{K}_1 \\ \eta_2^K = \dfrac{s_1}{s_2 + s_3} & \text{if } \hat{x} \in \hat{K}_2 \cup \hat{K}_3 \end{cases}$$

$$\begin{cases} \eta_3^K = 0 & \text{if } \hat{x} \in \hat{K}_1 \\ \eta_3^K = -1 & \text{if } \hat{x} \in \hat{K}_2 \\ \eta_3^K = \dfrac{s_2}{s_3} & \text{if } \hat{x} \in \hat{K}_3 \end{cases}$$

$n = 3$

$$\begin{cases} \eta_2^K = 1 & \text{if } \hat{x} \in \hat{K}_1 \\ \eta_2^K = \dfrac{-s_1}{s_2} & \text{if } \hat{x} \quad \hat{K}_2 \\ \eta_2^K = 0 & \text{if } \hat{x} \in \hat{K}_3 \cup \hat{K}_4 \end{cases}$$

$$\begin{cases} \eta_3^K = 0 & \text{if } \hat{x} \in \hat{K}_1 \cup \hat{K}_2 \\ \eta_3^K = \dfrac{-s_3}{s_4} & \text{if } \hat{x} \in \hat{K}_3 \\ \eta_3^K = 1 & \text{if } \hat{x} \in \hat{K}_4 \end{cases}$$

$$\begin{cases} \eta_4^K = 1 & \text{if } \hat{x} \in \hat{K}_1 \cup \hat{K}_2 \\ \eta_4^K = \dfrac{-s_1 - s_2}{s_3 + s_4} & \text{if } \hat{x} \in \hat{K}_3 \cup \hat{K}_4 \end{cases}$$

As one can easily verify we have $\eta_i^K \perp q_h^0 \;\forall q_h^0 \in \hat{Q}_h^0$, $i = 2,$ and $\hat{Q}_h = \hat{Q}_h^0 \oplus \hat{Q}_h^1$.

Let now q_h^1 be any function of $\hat{Q}_h^1$. We can write :

$$(5.6) \quad q_h^1 = \sum_{K\in\tau_h} \sum_{i=2}^{n+1} q_i^K \eta_i^K$$

where the q_i^K's are given scalars.

<u>*Lemma 5.2*</u> : If *Assumption A)* holds, for every $\hat{q}_h \in \hat{Q}_h$ there exists $\hat{v}_h \in \hat{V}_h$ satisfying (5.3) together with

$$(5.7) \quad \frac{(\hat{div}\ \hat{v}_h\ ,\ \hat{q}_h)_{0,\hat{\Omega}}}{|\hat{v}_h|_{1,\hat{\Omega}}} \geq \hat{\beta}_h\ |\hat{q}_h|_{0,\hat{\Omega}}$$

for some $\hat{\beta}_h > 0$ independent of $\hat{q}_h$

(5.8) $\quad |\hat{z}_h|_{1,\hat{\Omega}} \leq C(\underset{\sim}{u}_h) \; |q_h^1|_{0,\hat{\Omega}}$

Now dropping the superscript K we get after simple calcutaltions:

$$|q_h^1|^2_{0,\hat{K}} = q_2^2|n_2|^2_{0,\hat{K}} + q_3^2|n_3|^2_{0,\hat{K}} \leq 2\,\frac{s_1^3}{s_3^2}\,(q_2^2 + q_3^2)$$

Since *Assumption A)* implies that $s_3 \geq \alpha^2 s_1$ we have

$$|q_h^1|^2_{0,K} \leq \frac{2s_1}{\alpha^4}\,(q_2^2 + q_3^2)$$

Now we prove that

(5.9) $\quad (\hat{div}\; \hat{z}_h \,,\, q_h^1)_{0,\hat{\Omega}} \geq c'|q_h^1|^2_{0,\hat{\Omega}} \quad , \quad c' > 0 \;.$

A straighforward calculation gives :

$$\int_{\hat{K}} \hat{div}\; \hat{\underset{\sim}{z}}_h\; q_h^1\; d\hat{\underset{\sim}{x}} = (s_1 + s_3 + \frac{s_1^2 + s_1 s_3}{s_2 + s_3})\; q_2^2 \; +$$

$$+ \; (s_1 + \frac{s_1 s_2 + s_2^2}{s_3})\; q_3^2 + (\frac{s_1 s_2}{s_2 + s_3} + 2s_2 - s_1)\; q_2\, q_3 \quad .$$

Thus we have:

$$\int_{\hat{K}} \hat{div}\; \hat{\underset{\sim}{z}}_h\; q_h^1\; d\hat{\underset{\sim}{x}} \geq (2s_1 + s_3)\; q_2^2 + (2s_1 + s_2)q_3^2 + (2s_2 - s_1 + \frac{s_1 s_2}{s_2 + s_3})\; q_2\, q_3$$

Now as one can easily check, for any $\alpha > 0$ we have

$$\left| \frac{s_1 s_2}{s_2 + s_3} + 2s_2 - s_1 \right|^2 < 4(s_1 + s_2)(s_1 + s_3)$$

which yields

$$\int_{\hat{K}} \hat{div}\; \hat{\underset{\sim}{z}}_h\; q_h^1\; d\underset{\sim}{x} \geq s_1(q_2^2 + q_3^2)$$

This in turn implies (5.9) with $c' = \alpha^4/2$.

In the case $n=3$ we set
$\gamma_2^K = q_2^K \;,\; \gamma_3^K = q_4^K + q_3^K$ and $\gamma_4^K = q_4^K - q_3^K$

Like in the case $n=2$ it is straightforward to derive the estimate (5.8), and dropping again the superscripts K we also get:

$$\left| q_h^1 \right|^2_{0,\hat{K}} = \sum_{i=2}^{4} q_i^2 \, |n_i|^2_{0,K} \leq 2 \frac{s_1^2}{s_4} (q_2^2 + q_3^2 + 2 q_4^2)$$

Since *Assumption A)* implies that $s_4 \geq \alpha^2 s_1$, we have

$$(5.10) \quad |q_h^1|^2_{0,K} \leq \frac{2 \, meas(\hat{K})}{\alpha^2} (q_2^2 + q_3^2 + 2q_4^2)$$

On the other hand simple calculations yield:

$$\frac{1}{meas(\hat{K})} (\hat{div} \, \hat{z}_h, q_h^1)_{0,K} = \frac{s_1}{s_2} q_2^2 - q_2 q_4 + (1 + \frac{s_4}{s_3}) q_3^2 + (\frac{s_4}{s_3} - 1) q_3 q_4 +$$

$$+ \frac{2(s_1 + s_2)}{s_3 + s_4} q_4^2$$

Thus we have :

$$\frac{1}{meas(\hat{K})} (\hat{div} \, \hat{z}_h, q_h^1)_{0,\hat{K}} \geq (\frac{s_1}{s_2} - \frac{1}{2}) q_2^2 + (\frac{1}{2} + \frac{3s_4}{2s_3}) q_3^2 +$$

$$+ (\frac{2s_1 + 2s_2}{s_3 + s_4} - \frac{1}{2} + \frac{s_4}{2s_3}) q_4^2$$

or yet

$$(\hat{div} \, \hat{z}_h, q_h^1)_{0,\hat{K}} \geq (q_2^2 + q_3^2 + 2q_4^2) \frac{meas(\hat{K})}{2}$$

This, together with (5.10), imply again (5.9) with $c' = (\alpha/2)^2$.

Finally we proceed like in [16] Theorem 4.2, namely we set $v_h = \theta w_h + z_h$, where w_h is defined in Lemma 5.1 and $\theta > 0$. From (5.4), (5.5), (5.8) and (5.9) it is clear that for θ sufficiently small there exists $\hat{\beta}_h > 0$ such that (5.7) holds together with (5.3)

q.e.d.

Now we further prove:

Lemma 5.3 : With every $q_h \in Q_h$ we can associate $\underset{\sim}{v}_h \in \underset{\sim}{V}_h$ that satisfies

(5.11) $\underset{\sim}{v}_h(S) = \underset{\sim}{0}$ for every vertex S of a supertriangle K, $K \in \tau_H$

$$(5.12) \quad \frac{\tilde{b}'_h(\Pi \underset{\sim}{u}_h , \underset{\sim}{v}_h , q_h)}{\| \underset{\sim}{v}_h \|} \geq \beta_h \mid q_h \mid$$

where β_h is a strictly positive parameter independent of q_h.

Proof : Using an identity encountered in [, page 108] we obtain :

$$(5.13) \quad \tilde{b}_h(\Pi \underset{\sim}{u}_h , \underset{\sim}{v}_h, q_h) = \int_{\hat{\Omega}} \hat{q}_h \, \hat{div} \, \hat{\underset{\sim}{v}}_h \, d\hat{\underset{\sim}{x}}$$

where $\hat{\underset{\sim}{v}}_h(\hat{\underset{\sim}{x}}) = \underset{\sim}{v}_h(\underset{\sim}{x})$

On the other hand, from *Assumption A)* it is straight - forward to establish the existence of a constant $C(\Omega, \underset{\sim}{u}_h)$ such that :

$$\| \underset{\sim}{v}_h \| \leq C \, (\Omega, \underset{\sim}{u}_h) \, |\hat{\underset{\sim}{v}}_h|_{1,\hat{\Omega}}$$

whereas

$$|q_h| \leq \alpha^{-1/2} |\hat{q}_h|_{0,\hat{\Omega}}$$

Now, if $\hat{\underset{\sim}{v}}_h$ is the field defined in Lemma 5.2 we have (5.11) and (5.12) with $\beta_h = \alpha^{1/2} \, C^{-1} \, \beta_h > 0$. q.e.d.

As a final preparatory result we have :

Lemma 5.4 : Under *Assumption A)*, for any $\underset{\sim}{v}_h \in \underset{\sim}{V}_h$ satisfying (5.11) we have:

$$\tilde{b}'_h \, (\underset{\sim}{u}_h , \underset{\sim}{v}_h , q_h) = \tilde{b}'_h(\Pi \, \underset{\sim}{u}_h , \underset{\sim}{v}_h , q_h) \quad \forall \, q_h \in Q_h \ .$$

Proof : Taking into account the definitions of b_h' and Q_h if we prove that

$$\int_K adj^{\mathsf{T}}\, \underset{\sim}{\nabla}\, \underset{\sim}{u}_h^x \cdot \underset{\sim}{\nabla}\, \underset{\sim}{v}_h \; d\underset{\sim}{x} = \int_K adj^{\mathsf{T}}\, \underset{\sim}{\nabla} \Pi \underset{\sim}{u}_h^x \cdot \underset{\sim}{\nabla}\, \underset{\sim}{v}_h \, d\underset{\sim}{x} \quad \forall K \in \tau_h^2 \; .$$

we have the Lemma. In order to prove the above equality we rewrite :

$$\int_K adj^{\mathsf{T}}\, \underset{\sim}{\nabla} u_h^x \cdot \underset{\sim}{\nabla}\, \underset{\sim}{v}_h \, d\underset{\sim}{x} = \lim_{\theta \to 0} \frac{1}{\theta} \int_K \Big[J(\underset{\sim}{x} + \Pi \underset{\sim}{u}_h + \underset{\sim}{\beta}\, \phi + \theta\, \underset{\sim}{v}_h) - - J(\underset{\sim}{x} + \Pi \underset{\sim}{u}_h + \underset{\sim}{\beta}\, \phi) \Big] \, d\underset{\sim}{x}$$

where ϕ is given by (3.1) and $\underset{\sim}{\beta}$ is a linear combination of $\underset{\sim}{u}_i$, $i=1,2,\dots,n$ and u_{n+2}, $\underset{\sim}{u}_i$ being the value of $\underset{\sim}{u}_h$ at node S_i of $K \in \tau_h^2$ (see Figure 3.1)

Passing to element $\hat{K}$ using the affine transformation and notations already encountered in Section 4 we get:

$$\int_K adj^{\mathsf{T}}\, \underset{\sim}{\nabla}\, \underset{\sim}{u}_h^x \cdot \underset{\sim}{\nabla}\, \underset{\sim}{v}_h \, d\underset{\sim}{x} = \lim_{\theta \to 0} \frac{1}{\theta} \int_{\hat{K}} [\hat{J}(\hat{x} + \underset{\sim}{\beta}\, \hat{\phi} + \theta\, \hat{\underset{\sim}{v}}_h) - - \hat{J}(\hat{\underset{\sim}{x}} + \underset{\sim}{\beta}\, \hat{\phi})] \, d\hat{\underset{\sim}{x}}$$

Expanding the right hand side above and taking the limit one gets :

$$(5.14) \qquad \int_K adj^{\mathsf{T}}\, \underset{\sim}{\nabla} u_h^x \cdot \underset{\sim}{\nabla} \underset{\sim}{v}_h \, d\underset{\sim}{x} = \int_{\hat{K}} \hat{div}\, \hat{\underset{\sim}{v}}_h \, d\hat{\underset{\sim}{x}} + \sum_{1 \le i < j \le n} (-1)^{i+j} \int_{\hat{K}} d_{ij}(\underset{\sim}{\beta}, \hat{\phi}, \hat{\underset{\sim}{v}}_h)\, d\hat{\underset{\sim}{x}}$$

where

$$d_{ij}(\underset{\sim}{\beta}, \hat{\phi}, \hat{\underset{\sim}{v}}_h) = \frac{\partial \hat{\phi}}{\partial \hat{x}_i} \begin{vmatrix} \beta_i & \dfrac{\partial \hat{v}_i}{\partial \hat{x}_j} \\ \beta_j & \dfrac{\partial \hat{v}_j}{\partial \hat{x}_j} \end{vmatrix} + \frac{\partial \hat{\phi}}{\partial \hat{x}_j} \begin{vmatrix} \dfrac{\partial \hat{v}_i}{\partial \hat{x}_j} & \beta_i \\ \dfrac{\partial \hat{v}_j}{\partial \hat{x}_j} & \beta_j \end{vmatrix} \; , \quad \hat{\underset{\sim}{v}}_h = (\hat{v}_1, \dots, \hat{v}_n)$$

Now, according to (5.11) we can write $\hat{\underset{\sim}{v}}_h$ as the sum of two compo - nents, namely $\hat{\underset{\sim}{v}}_h = \underset{\sim}{a}\, \hat{\lambda}_{n+1} + \underset{\sim}{b}\hat{\phi}$. Then if we expand the above determinants into sums of two determinants corresponding to these components of $\underset{\sim}{v}_h$, the one associated with $\underset{\sim}{b}\hat{\phi}$ is readily seen to vanish identically . Thus we have:

$$d_{ij} = \left(\frac{\partial \hat{\lambda}_{n+1}}{\partial \hat{x}_j} \frac{\partial \hat{\phi}}{\partial \hat{x}_i} - \frac{\partial \hat{\lambda}_{n+1}}{\partial \hat{x}_i} \frac{\partial \hat{\phi}}{\partial \hat{x}_j}\right) c_{ij} \quad \text{where} \quad c_{ij} = \begin{vmatrix} \beta_i & \alpha_i \\ \beta_j & \alpha_j \end{vmatrix}$$

Now we notice that :

$$(5.15) \qquad \int_{\hat{K}} d_{ij} = \frac{\partial \hat{\lambda}_{n+1}}{\partial \hat{x}_j} \int_{\hat{K}} \frac{\partial \hat{\phi}}{\partial \hat{x}_i} - \frac{\partial \hat{\lambda}_{n+1}}{\partial \hat{x}_i} \int_{\hat{K}} \frac{\partial \hat{\phi}}{\partial \hat{x}_j}$$

But since $\dfrac{\partial \hat{\phi}}{\partial \hat{x}_k} = \sum_{\substack{i,j=1 \\ i \neq j}}^{n} \dfrac{\partial \hat{\lambda}_j}{\partial \hat{x}_k} \hat{\lambda}_i$, we have :

$$\int_{\hat{K}} \frac{\partial \hat{\phi}}{\partial \hat{x}_k} = \frac{n-1}{n+1} \; meas(\hat{K}) \sum_{j=1}^{n} \frac{\partial \hat{\lambda}_j}{\partial \hat{x}_k}$$

Taking into account the elementary identity $\sum_{j=1}^{n} \hat{\lambda}_j \equiv 1-\hat{\lambda}_{n+1}$, we finally get :

$$\int_{\hat{K}} \frac{\partial \hat{\phi}}{\partial \hat{x}_k} = -\frac{n-1}{n+1} \; meas(\hat{K}) \frac{\partial \hat{\lambda}_{n+1}}{\partial \hat{x}_k}$$

Taking the above relation to (5.15), $\int_{\hat{K}} d_{ij} \, d\hat{x}$ is readily seen to vanish.

The result then follows taking into account (5.13) and (5.14)

q.e.d.

Now, as an immediate consequence of Lemmas 5.2,5.3 and 5.4 we have :

Theorem 5.1 : If u_h satisfies Assumption A) for any $\alpha > 0$, (2.9) holds in *case i)*.

□

Let us now turn to *case ii)*. This we do for $n=2$ only.

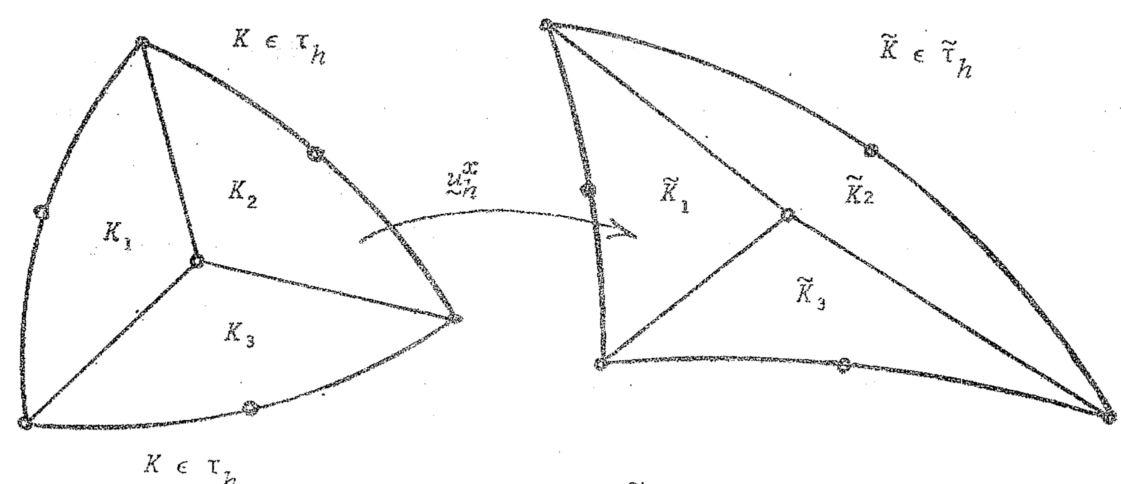
In this case Ω_h will be the union of triangles with one parabolic edge, such that its boundary Γ_h coincides with Γ at least at the nodes of those triangles that have a parabolic edge (base) on Γ_h. Let also Γ_{0_h} be the portion of Γ_h consisting of such parabolic edges that have its three nodes on $\bar{\Gamma}_0$.

Now instead of *Assumption A)* we make a stronger one, namely :

ASSUMPTION B) $J(\underset{\sim}{u}_h^x) > 0$ almost everywhere in Ω_h . □

Taking into account (4.5), the above assumption implies *Assumption A)*. Moreover, it allows us to say that $\underset{\sim}{u}_h^x$ is a bijection between Ω_h and $\tilde{\Omega}_h = \underset{\sim}{u}_h^x(\Omega_h)$. In this case $\tilde{\Omega}_h$ is a domain that has the same structure as Ω_h, in the sense that it can also be viewed as the union of isoparametric elements $\tilde{K}$, where $\tilde{K} = \underset{\sim}{u}_h^x(K)$, $K \in \tau_h^2$.

Let then $\tilde{\tau}_h^2$ be the triangulation of $\tilde{\Omega}_h$ consisting of the $\tilde{K}$'s . Similarly, let τ_h be the set of curved superelements $K = \bigcup_{i=1}^{3} K_i$ upon which τ_h^2 is constructed, and let $\tilde{\tau}_h$ be the partition of $\tilde{\Omega}_h$ into curved superelements $\tilde{K}$ where $\tilde{K} = \underset{\sim}{u}_h^x(K)$, $K \in \tau_h$ (see Figure 5.2).



$area(K_i) = area(\tilde{K}_i)$, $i = 1, 2, 3$

Supertriangles of partitions τ_h and $\tilde{\tau}_h$ Figure 5.2

For simplicity we consider the case where $\forall K \in \tau_h, area(K_1) = area(K_2) = area(K_3)$, although the more general case can be treated without major difficulties.

Now if $K_i = \Pi \underset{\sim}{u}_h^x(K)$, *Assumption B)*, hence A), implies

$$\frac{1}{\alpha}\, area(\tilde{K}) \geq 3\; area\;(K_i) \geq \alpha\;\; area\;(\tilde{K})\;, \quad 1 \leq i \leq 3\;.$$

Since $area(\tilde{K}_i) = area\;(K_i) = \frac{1}{3}\; area\;(K) \quad \forall K \in \tau_h$.

Let us now define the following spaces of functions defined over $\tilde{\Omega}_h$:

$$\tilde{Q}_h = \{\tilde{q}_h \;/\; \tilde{q}_h \circ \underset{\sim}{u}_h^x = q_h\;,\; q_h \in Q_h\}$$

$$\tilde{V}_h = \{\tilde{v}_h \;/\; \tilde{v}_h \circ \underset{\sim}{u}_h^x = \underset{\sim}{v}_h\;,\; \underset{\sim}{v}_h \in \underset{\sim}{V}_h\}$$

We equip $\underset{\sim}{\tilde{V}}_h$ and $\tilde{Q}_h$ with the norms $||\cdot||$ and $|\cdot|$ given respectively by = $||\tilde{\underset{\sim}{v}}_h|| = |\tilde{\underset{\sim}{v}}_h|_{1,\tilde{\Omega}_h}$, $\tilde{\underset{\sim}{v}}_h \in \underset{\sim}{V}_h$ and $|\tilde{q}_h| = |\tilde{q}_h|_{0,\tilde{\Omega}_h}$, $\tilde{q}_h \in \tilde{Q}_h$

(Since $\underset{\sim}{v}_h = \underset{\sim}{0}$ on $\Gamma_{0_h} \equiv \tilde{\Gamma}_{0_h}$, $||\cdot||$ is actually a norm) .

Let us also denote by $\tilde{\underset{\sim}{x}}$ the new variable $\underset{\sim}{u}_h^x\,(\underset{\sim}{x})$

More generally, for every function f defined over Ω_h we denote by $\tilde{f}$ the function defined over $\tilde{\Omega}_h$ such that $\tilde{f}\;[\underset{\sim}{u}_h^x(\underset{\sim}{x})] = f(\underset{\sim}{x}) \quad \forall \underset{\sim}{x} \in \Omega_h$.

In order to prove that (2.9) holds, we use the following theorem given by Le Tallec :

<u>*Theorem 5.2*</u> : [10 , Theor. 4.5] : Under *Assumption B)*, (2.9) is equivalent to :

$\exists\, \beta_h > 0$ such that

$$(5.16) \qquad \sup_{\tilde{v}_h \in \tilde{V}_h} \frac{\int_{\tilde{\Omega}_h} \tilde{q}_h \, \tilde{div} \, \tilde{v}_h \, d\tilde{x}}{\| v_h \|} \geq \tilde{\beta}_h \; |\tilde{q}_h| \qquad \forall \tilde{q}_h \in \tilde{Q}_h$$

where $\tilde{div}$ represents the divergence operator with respect to the $\tilde{x}$ variable .

□

The above result states that if suffices to prove the linear discrete compatibility condition between spaces $\tilde{V}_h$ and $\tilde{Q}_h$ to have existence of a solution to (P_h) in the isoparametric case.

Now, in order to prove (5.16) for the asymmetric triangle we give the following lemmas :

Lemma 5.5 : Let $\tilde{Q}_h^0$ be the subspace of $\tilde{Q}_h$ of those functions that are constant over $\tilde{K}$, $\forall \tilde{K} \in \tilde{\tau}_h$. Then for every $\tilde{q}_h \in \tilde{Q}_h^0$ there exists a vector field $\tilde{w}_h \in \tilde{V}_h$ such that :

$$(5.17) \qquad \int_{\tilde{\Omega}_h} q_h \, \tilde{div} \, \tilde{w}_h \, d\,\tilde{x} \geq \tilde{\beta}_0 \; |\tilde{q}_h|^2$$

$$(5.18) \qquad \| \tilde{w}_h \| \leq \tilde{C}_h \, |\tilde{q}_h|$$

where β_0 and C_h are strictly positive constants inde - pendent of $\tilde{q}_h$.

Proof : According to [4], Lemma *C2*, for a given $\tilde{q}_h \in \tilde{Q}_h^0$, there exists $v \in H^1(\tilde{\Omega}_h)$ with $v = 0$ on Γ_{0_h} such that

$$\int_{\tilde{\Omega}_h} \tilde{q}_h \;\; \tilde{div} \;\; v \;\; d\tilde{x} \geq \tilde{\beta}_0 | \; \tilde{q}_h|^2$$

and $\qquad |v|_{1,\tilde{\Omega}_h} \leq \tilde{C} \; |\tilde{q}_h|$.

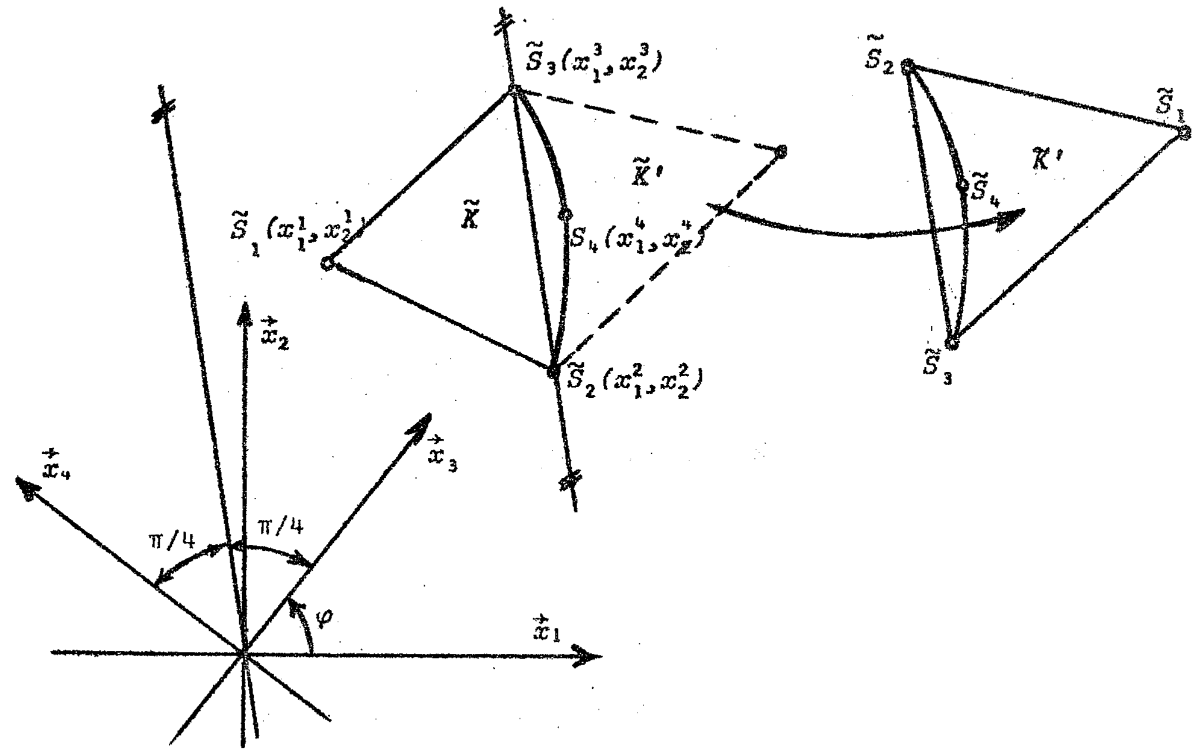
Now we construct a vector field $\tilde{\underset{\sim}{w}}_h \in \tilde{\underset{\sim}{V}}_h$ associated with v in the following way :

For each triangle $\tilde{K} \in \tilde{\tau}_h^2$, we define two perpendicular axes $\vec{x}_3^K$ and $\vec{x}_4^K$ oriented in such a way that they correspond to rotations of the reference cartesian axes $\vec{x}_1$ and $\vec{x}_2$ of an angle ϕ^K.

Dropping the supercript K for simplicity, we determine ϕ in such a way that the straight line passing through nodes $\tilde{S}_2$ and $\tilde{S}_3$ of $\tilde{K}$ forms an angle of $\pi/4$ with both $\vec{x}_3$ and $\vec{x}_4$.

Let x_j be the variable with respect to axis $\vec{x}_j$, $1 \leq j \leq 4$.

Clearly $\vec{x}_3$ and $\vec{x}_4$ will coincide for any pair of elements of $\tilde{\tau}_h^2$ that have a base $\tilde{B}$ as a common edge. Let the local numbering of the vertices of each element respect the usual permutation convention (in this way, $\tilde{S}_2$ and $\tilde{S}_3$ interchange within each element of such a pair, as shown in Figure 5.3). Now for each $\tilde{K} \in \tilde{\tau}_h^2$, let s be the curved abcissa along $\tilde{B}$ with origin in S_2 and $\underset{\sim}{n}(s)$ denote the outer unit normal vector along $\tilde{B}$ with respect to $\tilde{K}$. We also denote by $n_j(s)$ the component of $\underset{\sim}{n}$ with respect to $\vec{x}_j$.



Element $\tilde{K}$ and associated axes $\vec{x}_3$ and $\vec{x}_4$

Figure 5.3

Let $\underset{\sim}{w} = (w_1, w_2)$, $\underset{\sim}{w} = \underset{\sim}{w}_{h/K}$ and w_3 and w_4 be given by

$$w_3 = w_1 \cos \phi + w_2 \sin \phi$$

$$w_4 = w_1 \sin \phi + w_2 \cos \phi$$

Now we check that we can uniquely define w_3 and w_4 (and consequently $\underset{\sim}{w}$) in the following way :

The values of w_3 and w_4 at the vertices of $\tilde{K}$ are given by

$$w_3(\tilde{S}_i) = w_4(\tilde{S}_i) = 0 \qquad i = 1, 2, 3$$

The value of w_3 and w_4 at node $\tilde{S}_4$ are such that

$$\int_{\tilde{B}} w_j \, n_j(s) ds = \int_{\tilde{B}} v_j \, n_j(s) ds \qquad j = 3, 4$$

where v_j is the component of $\underset{\sim}{v}$ with respect to $\vec{x}_j$, $j = 3, 4$.

Since $\underset{\sim}{u}_h \in \underset{\sim}{V}_h$, we can compute the coordinates x_3 and x_4 in terms of the reference coordinates $\hat{x}_1$ and $\hat{x}_2$ (see Figure 2.1) used for defining $\hat{P}_\alpha$ over $\hat{K}$, in the following way :

$$x_3 = [4\,\xi_1^4 - 2(\xi_1^2 + \xi_1^3)]\,\hat{x}_1\,\hat{x}_2 + \xi_1^2\,\hat{x}_1 + \xi_1^3\,\hat{x}_2 + x_3^1$$

$$x_4 = [4\,\xi_2^4 - 2(\xi_2^2 + \xi_2^3)]\,\hat{x}_1\,\hat{x}_2 + \xi_2^2\,\hat{x}_1 + \xi_2^3\,\hat{x}_2 + x_4^1$$

where $\xi_1^i = x_3^i - x_3^1$ and $\xi_2^i = x_4^i - x_4^1 \qquad i = 2,3,4$ and

$$x_3^i = x_1^i \cos\phi + x_2^i \sin\phi$$

$$x_4^i = -\,x_1^i \sin\phi + x_2^i \cos\phi \qquad i = 1,2,3,4\,.$$

Using the above relations we make a change of variables in the integral $\int_{\tilde{B}} w_j\, n_j\,(s)\, ds$, $j = 3,4$, namely from s to $\hat{s}$, where s is the abcissa along the edge $\tilde{B}$ of $\tilde{K}$ with origin in $\tilde{S}_2$ (see Figure 2.1).

Since we have $n_3(s) = \dfrac{d\,x_4}{ds}$ and $n_4(s) = -\dfrac{d\,x_3}{ds}$, for a vector field $\underset{\sim}{f}$ defined over $\tilde{B}$, whose components with respect to $\vec{x}_j$ are f_j , $j = 3,4$, we have for the x_3 -component :

$$\int_{\tilde{B}} f_3\, n_3(s)\, ds = \int_{\tilde{S}_2}^{\tilde{S}_3} f_3\,(s)\,\frac{dx_4}{ds}\,ds = \int_{\hat{S}_2}^{\hat{S}_3} \hat{f}_3\,(\hat{s}) \left[\frac{\partial x_4}{\partial \hat{x}_1}\,\frac{d\hat{x}_1}{ds} + \frac{\partial x_4}{\partial \hat{x}_2}\,\frac{d\hat{x}_2}{d\hat{s}}\right]_{\hat{x}_1+\hat{x}_2=1} d\hat{s}$$

where $\hat{f}_j(\hat{s}) = f_j(s)$. Since $\dfrac{d\hat{x}_i}{d\hat{s}} = (-1)^i\,\dfrac{\sqrt{2}}{2}$ we have :

$$\int_{\tilde{B}} f_3\, n_3(s)\,ds = \int_0^{\sqrt{2}} \frac{1}{\sqrt{2}}\,\hat{f}_3\,(\hat{s})\{(\xi_2^3 - \xi_2^3) + [4\xi_2^4 - 2(\xi_2^3 + \xi_2^2)](1 - \hat{s}\sqrt{2})\}\,d\hat{s}$$

whereas an entirely analogous relation holds for the x_4 - component.

Now since $\hat{w}_{j/\hat{B}} = 2\ \hat{w}_j(\hat{S})\ \hat{s}\ (\sqrt{2} - \hat{s})$, we have :

(5.19) $$\int_{\hat{B}} w_3\ n_3\ (s)\ ds = \frac{2}{3}\ (\xi_2^3 - \xi_2^2)\ w_3(\tilde{S}_4)$$

and analogously

(5.20) $$\int_{\hat{B}} w_4 n_4(s) ds = \frac{2}{3}\ (\xi_1^2 - \xi_1^3\ )\ w_4(\tilde{S}_4)\ .$$

Since by construction $|\xi_1^3 - \xi_1^2| = |\xi_2^3 - \xi_2^2| = \frac{\sqrt{2}}{2}\ length(\hat{B}) \neq 0$, $\underset{\sim}{w}$ can be defined uniquely.

Furthermore, proceeding in the same way for every element we can define a vector field $\tilde{\underset{\sim}{w}}_h \in \tilde{\underset{\sim}{V}}_h$ such that :

$$\int_{\tilde{B}} \tilde{w}_h \cdot \underset{\sim}{n}(s)\ ds = \int_{\tilde{B}} \underset{\sim}{v}\cdot\underset{\sim}{n}(s) ds \quad \text{for every base } \tilde{B} \text{ of } \tilde{K} \in \tilde{\tau}_h$$

This yields :

$$\int_{\tilde{\Omega}_h} \tilde{q}_h\ \tilde{div}\ \tilde{\underset{\sim}{w}}_h\ d\tilde{\underset{\sim}{x}} = \int_{\tilde{\Omega}_h} \tilde{q}_h\ \tilde{div}\ \tilde{\underset{\sim}{v}}\ d\tilde{\underset{\sim}{x}} \quad \forall\ \tilde{q}_h \in \tilde{Q}_h^0 \ ,$$

and consequently (5.17) holds.

On the other hand we have

$$\|\tilde{\underset{\sim}{w}}_h\|^2 = \sum_{\tilde{K}\in\tilde{\tau}_h^2} \int_{\tilde{K}} (|\underset{\sim}{\nabla}\ w_1|^2 + |\underset{\sim}{\nabla}\ w_2|^2)\ d\tilde{\underset{\sim}{x}} = \sum_{\tilde{K}\in\tilde{\tau}_h^2} \int_{\tilde{K}} (|\underset{\sim}{\nabla} w_3|^2 + |\underset{\sim}{\nabla} w_4|^2) d\tilde{\underset{\sim}{x}}$$

But

$$\int_{\tilde{K}} |\underset{\sim}{\nabla} wj|^2\ d\tilde{\underset{\sim}{x}} = w_j^2\ (\tilde{S}_4) \int_{\tilde{K}} |\underset{\sim}{\nabla}\tilde{p}_4\ |^2\ d\tilde{\underset{\sim}{x}}\ , \quad j = 3,4 \ ,$$

where $\tilde{p}_4(\tilde{\underset{\sim}{x}}) = \hat{p}_4(\hat{\underset{\sim}{x}})$, $\hat{p}_4(\hat{\underset{\sim}{x}}) = 4\ \hat{x}_1\hat{x}_2$, $\hat{\underset{\sim}{x}} = \underset{\sim}{u}^x_{h/K}\ [\mathcal{A}_K^{-1}(\underset{\sim}{x})]$

Now, according to *Assumption B)* and standard estimates we have :

$$\int_{\tilde K} |\nabla \tilde p_4|^2 \, d\tilde x \le C \frac{h^2 |u_h|^2_{1,\infty}}{\tilde\rho_K^2}$$

where $\tilde\rho_K$ denotes the diameter of the inscribed circle in $\tilde K$.

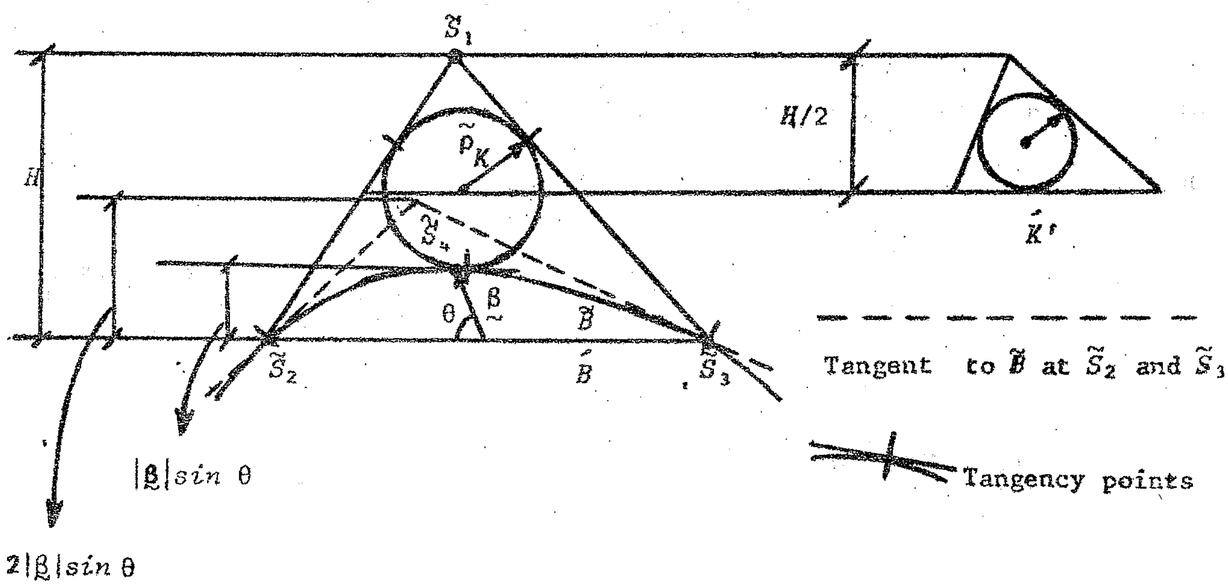
Now if $area(\tilde K) \ge area(\acute K)$ we clearly have :

$$\tilde\rho_K \ge \acute\rho_K \ge \frac{2area(\acute K)}{3\acute h_K} \ge \frac{2\alpha \; area(K)}{3h|u_h|_{1,\infty}}$$

If $area(\tilde K) \le area(\acute K)$ we use *Assumption B)* together with geometrical arguments sketched in self-explanatory Figure 5.4 (we omit details for the sake of conciseness). It is then possible to prove that $\tilde\rho_K$ is greater than the diameter of the inscribed circle in a triangle $\acute K'$, defined to be the homotetical reduction of $\acute K$ with ratio 1/2 .

Hence we have in this case :

$$\tilde\rho_K \ge \frac{\frac{1}{4} \, area(\acute K)}{\frac{3}{4} \, \acute h_K} \ge \frac{1}{3} \frac{area(K)}{h|u_h|_{1,\infty}}$$



Triangles $\tilde K$ and $\acute K$ when $area(\tilde K) \le area(\acute K)$

Figure 5.4

This gives :

$$\int_{\tilde{K}} |\nabla \tilde{p}_4|^2 \, d\tilde{x} \le \frac{C}{\frac{2}{3} \min(\alpha, 1/2)} \; \frac{h^4 |u_h|^4_{1,\infty}}{[area(K)]^2} \le \frac{C'}{c^4} |u_h|_{1,\infty}^4$$

where c is the constant of regularity of $\{\tau_h^2\}_h$ (see Section 2) .

On the other hand, by construction , (5.19), (5.20) and the Trace Theorem we have :

$$|w_j(\tilde{S}_4)| \le \frac{\int_{\tilde{B}} |v_j| \, ds}{\frac{\sqrt{2}}{3} \acute{\rho}_K} \le C \; \frac{\| v \|_{1,\tilde{K}} \; |u_h|_{1,\infty}}{h}$$

Therefore

$$\| \tilde{w}_h \| \le C \, h^{-1} |u_h|^3_{1,\infty} \; \| v \|_{1,\tilde{\Omega}_h} \le C(\tilde{\Omega}_h) \; h^{-1} |u_h|^3_{1,\infty} |\tilde{v}|_{1,\tilde{\Omega}_h}$$

which proves (5.23) with $\tilde{C}_h = C \; (\tilde{\Omega}_h) \; |u_h|^3_{1,\infty} / h.$

q.e.d.

Let us now construct a vector field $\tilde{z}_h \in \tilde{V}_h$ associated with the subspace $\tilde{Q}_h^1$ of $\tilde{Q}_h$, such that $\tilde{Q}_h = \tilde{Q}_h^0 \oplus \tilde{Q}_h^1$. Like space $\tilde{Q}_h^1$ of *case i)*, $\tilde{Q}_h^1$ is spanned by a set of orthogonal basis functions $\{\gamma_2^K , \gamma_3^K\}_{\tilde{K} \in \tilde{\tau}_h}$ defined in an entirely analogous way (for $s_1 = s_2 = s_3$). Now we prove :

<u>*Lemma 5.6*</u> : Let $\tilde{q}_h^1$ be a function of $\tilde{Q}_h^1$ whose components with respect to γ_2^K and γ_3^K are respectively ξ_2^K and ξ_3^K, $K \in \tau_h$. Under *Assumption B)*, the vector field $\tilde{z}_h \in \tilde{V}_h$ that vanishes at all the vertices of τ_h and whose value at the common vertex G of K_i, $1,2,3$, $K_i \subset K$ is given by (refer to Figure 5.5).

$$\tilde{z}_h(\tilde{G}) = - \xi_2^K \, m_2 + \xi_3^K \, m_3$$

satisfies :

(5.21) $$\| \tilde{\underset{\sim}{z}}_h \| \leq C(\underset{\sim}{u}_h) |\tilde{q}_h^1|, \qquad C(\underset{\sim}{u}_h) < \infty$$

(5.22) $$\int_{\tilde{\Omega}_h} \tilde{q}_h^1 \, \tilde{div} \, \tilde{\underset{\sim}{z}}_h \, d\tilde{\underset{\sim}{x}} \geq \tilde{\beta}_1 |\tilde{q}_h^1|^2 \qquad \text{with } \tilde{\beta}_1 > 0$$

Proof : (5.21) is a trivial consequence of the definition of $\tilde{\underset{\sim}{z}}_h$.

On the other hand a straightforward computation gives:

$$\int_{\tilde{K}} \tilde{q}_h^1 \, \tilde{div} \, \tilde{\underset{\sim}{z}}_h \, d\tilde{\underset{\sim}{x}} = (s_3+s_1) \frac{3\xi_2^2}{2} + (2s_1+s_2)\xi_3^2 + (3s_2+s_3-s_1)\xi_2 \xi_3$$

where $s_i = area(\hat{K}_i) \qquad i = 1,2,3$.

Assuming again that the local numbering of the nodes of K is such that $s_1 \geq s_2 \geq s_3$ we have :

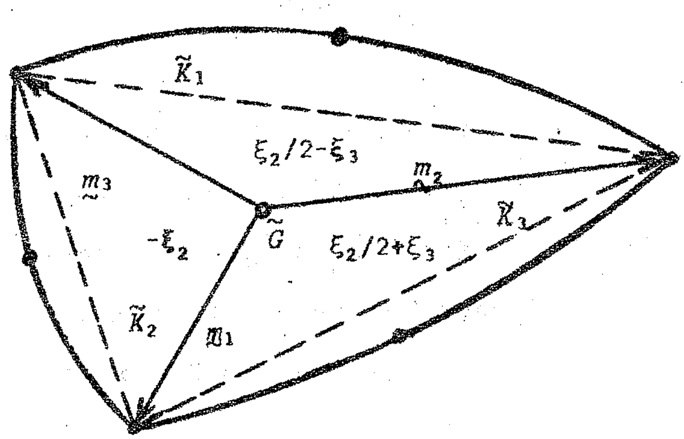
$$\frac{1}{4} \left(\frac{3}{2} s_2 + s_3 + s_1\right)^2 - \frac{1}{2} (s_1 + 3s_3)(s_1 + s_2) < 0$$

if we just have $s_1 \geq \alpha \; area(K)/3 > 0$

Thus we can write :

$$\int_{\tilde{K}} \tilde{q}_h^1 \, \tilde{div} \, \tilde{\underset{\sim}{z}}_h \, d\tilde{\underset{\sim}{x}} \geq \frac{\alpha}{3} (\xi_2^2 + \xi_3^2) \; area(K)$$

which yields (5.27) with $\tilde{\beta}_1 = \frac{2}{9}\alpha$ q.e.d.



Superelement $\tilde{K}$ Figure 5.5

Now defining $\tilde{v}_h = \theta\, \tilde{w}_h + \tilde{z}_h$, from (5.22), (5.23) (5.26) and (5.27) we have (5.18) just like in Lemma 5.2, for a sufficiently small θ. Hence, as an immediate consequence of Theorem 5.2 and Lemmas 5.5 and 5.6 we have :

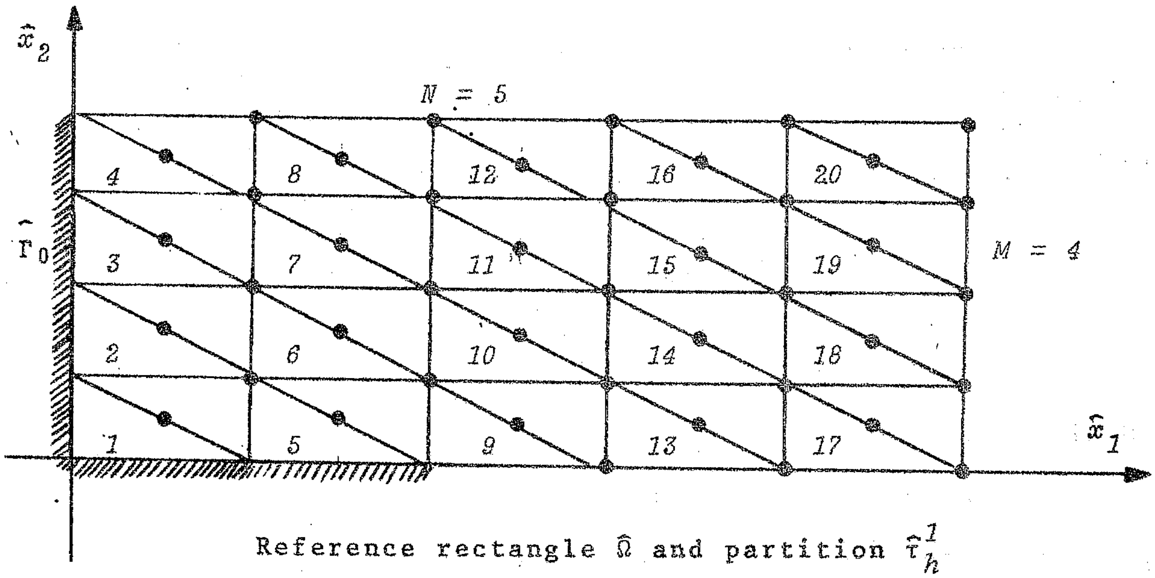
Theorem 5.3 : Under *Assumption B)* the compatibility condition (5.16) holds for *case ii)*. □

REMARK : *Assumption B)* and *A)* with $\alpha > 0$ express in particular the fact that the area delimited by the base of the triangle in deformed states $\tilde{B}$ and $\hat{B}$ do not account for the whole of *area(K) =area*$(\tilde{K})$. This fact was crucial for the assertion of the existence results in both *cases i)* and *ii)* . □

6. THE CASE OF PARTITION τ_h^1 AND NUMERICAL EXAMPLES

Let us finally consider the existence of a solution to problem (P_h') when one uses a partition of type τ_h^1 for the special case described below :

Let Ω be a domain that can be viewed as the images of a rectangle $\hat{\Omega}$ with boundary $\hat{\Gamma}$, through a mapping $A : \hat{x} \to \hat{w}\,(\hat{x})$. Here $\hat{w}$ is an element of a reference vector space $\hat{V}_h$ such that $det\,(I + \hat{\nabla}\,\hat{w}(x)) > 0$. a.e. in $\hat{\Omega}$. $\hat{V}_h$ is defined in the same way as V_h in Section 3, for a compatible partition τ_h^1 of $\hat{\Omega}$ into equal triangles illustrated in Fig. 6.1. $\hat{\tau}_h^1$ is constructed upon a first partition $\hat{\chi}_h$ of $\hat{\Omega}$ into rectangles by means of a uniform $M \times N$ grid, in such a way that the edges of $\hat{\tau}_h^1$ over which $\hat{z}_h \in \hat{V}_h$ is necessarily linear, are the edges parallel to the reference axes $\vec{\hat{x}}_1$ and $\vec{\hat{x}}_2$



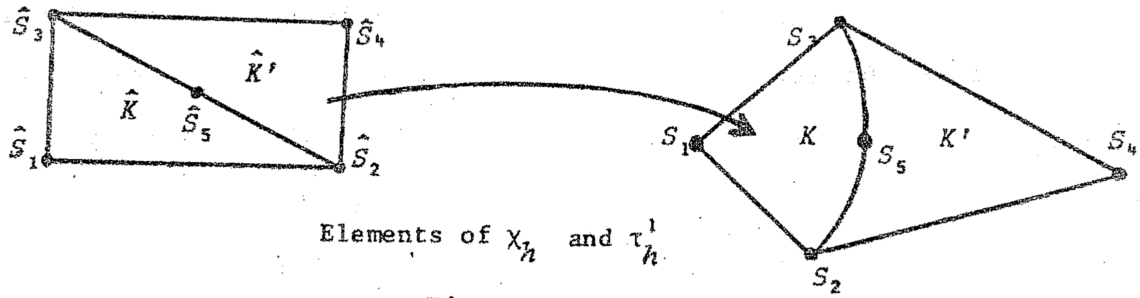
Reference rectangle $\hat{\Omega}$ and partition $\hat{\tau}_h^1$

Figure 6.1

We assume that the fixed portion $\hat{\Gamma}_0$ of $\hat{\Gamma}$ over which $\hat{v}_h \in \hat{V}_h$ vanishes, is the union of edges of rectangles of $\hat{\chi}_h$. If we define Γ_0 to be the image through A of $\hat{\Gamma}_0$ it is clear that Γ_0 consists of polygonal lines (eventually disjoint), just like $\Gamma = A(\hat{\Gamma})$

Now we define τ_h^1 to be the partition of Ω into isopa - rametric elements K that are the image of $\hat{K}$ through A, $\forall \hat{K} \in \hat{\tau}_h^1$. Similarly we define χ_h to be the partition of Ω into elements that are the images through A of rectangles of $\hat{\chi}_h$.

Notice that the union of a pair of elements K and K' of τ_h^1 that are the images of two triangles of $\hat{\tau}_h^1$ contained in a given rectangle of $\hat{\chi}_h$, is a quadrilateral (with four straight edges). Therefore every element of χ_h is a quadrilateral (see Figure 6.2).



Elements of χ_h and τ_h^1

Figure 6.2

Now, according to [10], Theorem 4.5, it suffices to prove (5.16) to assert the existence of a solution to (P_h'), assuming of course that $J(u_h^x) > 0$ a.e. in Ω.

Let us denote the quadrilaterals of χ_h by R_i, $i = 1, 2, \ldots, M \times N$, where $R_i = A(\hat{R}_i)$. $\hat{R}_i$ are the rectangles of $\hat{\chi}_h$ that we number in a systematic way along the columns, row by row, as indicated in Fig. 6.1

Let $\eta = \{\eta_i, \eta_{i+M\times N}\}_{i=1}^{M\times N}$ be the basis of the space of pressures Q_h associated with τ_h^1, in such a way that $supp(\eta_i) \subset R_i$ $supp(\eta_{i+M\times N}) \subset R_i$, $1 \le i \le M \times N$, with :

$$\eta_i(\underset{\sim}{x}) = 1 \qquad \forall\, \underset{\sim}{x} \in K_i$$

$$\eta_i(\underset{\sim}{x}) = -1 \qquad \forall\, \underset{\sim}{x} \in K_i'$$

$$\eta_{i+M\times N}(\underset{\sim}{x}) = 1 \qquad \forall\, \underset{\sim}{x} \in R_i\,,$$

where K_i and K_i' are the curved triangles into which R_i is sub - divided.

Let also $\nu = \{\underset{\sim}{\nu}_i\}_{i=1}^{2L}$ be the usual basis of $\underset{\sim}{V}_h$, where L is the number of free nodes of τ_h^1. Each $\underset{\sim}{\nu}_i$ is associated with a degree of freedom of $\underset{\sim}{V}_h$ which are assigned to two diffe - rent blocks. The first one corresponds to the $2M\times N$ components of a field of $\underset{\sim}{V}_h$ that are associated with the nodes lying in the interior of $R_i \in \chi_h$, while the remaining degrees of freedom are assigned to the second block. Now we number the degrees of freedom of $\underset{\sim}{V}_h$ in such a way that those in the first block carry the number from one to $2M\times N$ and those in the second block the numbers from $2M\times N+1$ to $2L$.

Finally, let B_h be the $(2L\ ) \times (2M\times N)$ matrix whose entry at the i -th row and j-th column is given by

$$\int_\Omega n_i \, div \, \underset{\sim}{v}_j \, d\underset{\sim}{x} \, .$$

According to [9], Lemma 5.1, the existence of $\tilde{\beta}_h > 0$ such that (5.16) holds is equivalent to the rank of B_h being equal to $dim\ Q_h = 2M \times N$.

In order to examine this rank condition, it is convenient to split B_h into four rectangular matrices, according to the pattern below:

i = \ j =	$1 \ldots\ldots 2M\times N$	$2M\times N+1 \ldots\ldots 2L$
$1 \ldots M\times N$	B_h^1	B_h^4
$M\times N+1 \ldots 2M\times N$	B_h^3	B_h^2

First we notice that all the terms of B_h^3 vanish, since the basis functions of $\underline{V}_h$ associated with nodes lying in the interior of the quadrilaterals have zero flux along its boundary.

Secondly, recalling (5.19) and (5.20), we can say that the entries of B_h^1 in the positions $j = 2i-1$ or $j = 2i$, $1 \leq i \leq M \times N$, are given by expressions of the form $\pm\ 2(x_k^3 - x_k^2)/3$, $k = 1,2$, where (x_1^ℓ, x_2^ℓ) , $\ell = 2,3$ are the coordinates of the vertices of the curved diagonal of R_i . Since those vertices are necessarily distinct, at least one of the above terms of B_h^1 is nonzero.

Finally we notice that matrix B_h^2 has exactly the same entries as the matrix studied by Le Tallec for the $Q_1 \times P_0$ element associated with a partition of Ω into quadrilaterals, like χ_h.

With the above considerations it is easy to conclude that the rank of B_h is $2M \times N$, provided the rank of B_h^2 is $M \times N$. Therefore the condition of existence and uniqueness of p_h such that $(\underline{u}_h\ , p_h)$ is a solution to (P_h') becomes the same as in the case of the $Q_1 \times P_0$, at least for domains defined as above. That is why we refer to the work of Le Tallec [9] for the proper answer to this question in various situations depending on the shape of Γ_0.

Nevertheless, with the purpose of giving a brief illustration of his results we mention here the following case :

If Γ_0 is contained in a set that is the image through A of two non disjoint edges of $\hat{\Omega}$, then the above existence and uniqueness result is guaranteed. If on the other hand Γ_0 does not fall in this category this can only be asserted under some restrictive condition.

Finally, we would like to give a short account of the three-dimensional case, when one uses a partition of type τ_h^1 .

First we note that, as far as linear problems related to incompressible media are concerned, we can prove that, under identical assumptions on Γ and on the first hexahedral mesh upon which τ_h^1 is constructed, the existence and convergence results that hold for the tetrahedral element considered in this paper , are the same that apply to the mixed element defined as follows:

V_h is the space associated with the classical isoparametric trilinear functions (Q_1) defined on the hexahedral mesh, and Q_h is the space of constant functions (P_0) over each hexahedron.

Corresponding proofs will be given in a forthcoming paper on the Stokes problem. Note that in the case of the $Q_1 \times P_0$ element, the existence and convergence analysis of two-dimensional linear problems for a rectangular Ω, due to Pitkäranta [8], have later been shown to apply in an analogues way to the case where Ω is a parallelepiped [12]. For this reason one can expect that the existence results given above for our nonlinear problem, can be easily extended to the case of a parallelepipedal domain, if one uses a partition τ_h^1 constructed upon a partition of Ω also consisting of parallelepipedal elements.

In [15] one can find numerical results related to the two-dimensional element treated in this paper. Let us now illustrate the superiority of the three-dimensional one, compared to classical methods(*). Indeed, for $n=3$, the appropriate numerical

(*) Comparison has actually been made with standard elements such as the $Q_1 \times P_0$ element.

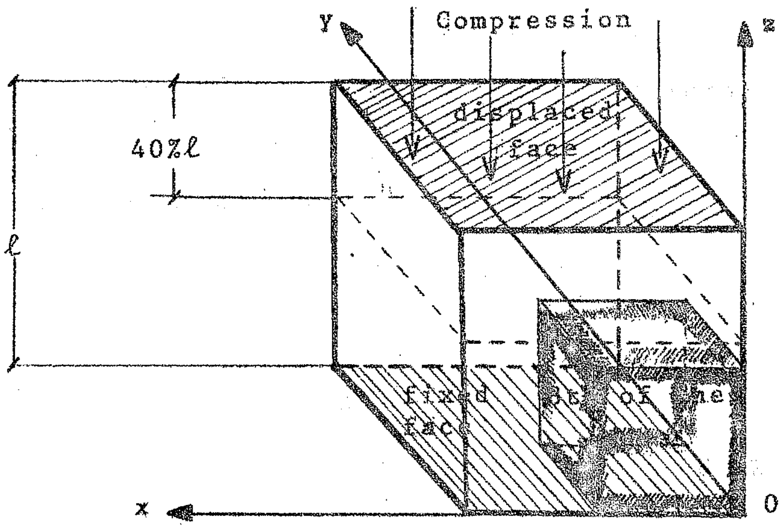
solution of *(P)* becomes a critical issue, as the Jacobian is a high order polynomial for standard elements. We have taken a compression test-problem, which is precisely one of the most difficult cases to simulate correctly from a numerical point of view. As one will see, particularly stable and realistic results are obtained.

In our test-problem we take Ω to be a cube having a fixed face Γ_0. We bring the face opposite Γ_0 closer to it parallelely to itself of a certain percentage of the edge length ℓ of Ω. Due to symmetry only the eight of the cube shown in Figure 6.3 is taken into account in the computations.

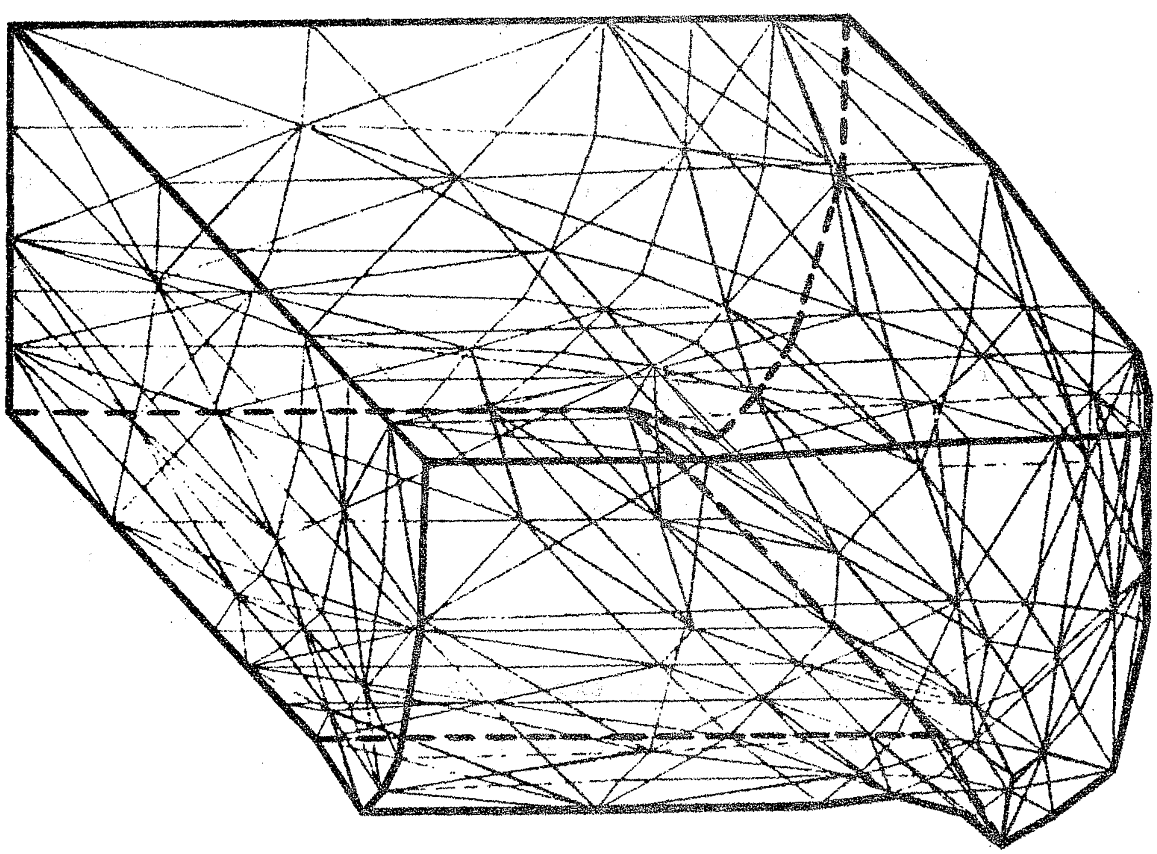
The τ_h^1 partition is obtained in the following way:

We first subdivide Ω into 27 equal cubes. Next the slices of cubes adjacent the faces $x_i = 0$, $i = 1,2,3$, are subdivided into three equal slimmer slices, parallel to these faces. This yields a mesh consisting of 125 parallelepipeds. Finally each parallelepiped is subdivided into eight tetrahedrons, in the way shown in Figure 3.2b.

We show in Figure 6.4 the boundary of the eighth of the cube in deformed state induced by a compression of 40%. It is interesting to notice that this deformed configuration corresponds to what one can expect to obtain, by performing a similar experience with a rubber cube.



Initial configuration of the cube   Figure 6.3



Deformed configuration of 1/8 of the cube induced by a compression of 40%   Figure 6.4

ACKNOWLEDGEMENT :

The numerical results given in this paper were obtained by combining the author's finite element methods, with an algorithm of augmented lagrangian type due to Glowinski and Le Tallec (see e.g. [6]) for solving the nonlinear problem *(P)*. The author wishes to thank Dr. Le Tallec for having supplied him with FORTRAN programs corresponding to this algorithm.

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