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QUASILINEAR FINITE ELEMENTS FOR THE STOKES PROBLEM IN $\mathbb{R}^3$

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ABSTRACT

We discuss a special kind of finite element approximation of the three-dimensional Stokes problem, giving rise to quasi-solenoidal velocity fields. Convergence results for the case of a parallelepipedal domain are derived, and numerical examples are shown.

KEY-WORDS: Asymmetric, convergence, divergence, finite elements, flow, incompressible, LBB-condition, pressure, quasi-linear, solenoidal, Stokes problem, tetrahedrons, three-dimensional, velocity, viscosity.

RESUMO

Apresenta-se um método de elementos finitos para a resolução de problemas de escoamento de fluidos viscosos tridimensionais, com campos de velocidade discretos quase solenoidais. Um estudo de convergência é desenvolvido e exemplos numéricos atestam a eficiência do método.

PALAVRA-CHAVE: Assimétricos, condição LBB, convergência, divergência, elementos finitos, fluido, incompressível, pressão, problema de Stokes, quasilinear, solenoidal, tetrahedros, tridimensional, velocidade, viscosidade.

C O N T E N T S

1 - INTRODUCTION

This paper deals with a special kind of finite element approximation of the Stokes problem in three-dimension space. This finite element was first introduced in [7] for the numerical treatment of nonlinear incompressible media. In that very delicate case it has been applied successfully (see e.g. [8]) and, as matter of fact, it appeared that its quasilinear and asymmetric structure was the key to its good numerical simulation properties, for a problem where most of the classical methods fail.

In this work we show that the same element is also particularly suitable for solving problems of Fluid Mechanics, specially in the framework of viscous flow. The main reason for this assertion is the fact that optimal convergence rates can be attained, whereas the total number of velocity degrees of freedom for a given degree of accuracy is relatively low, and the divergence free condition is satisfied almost exactly in the discrete problem. The latter aspect is far from irrelevant, taking into account the present state of the art for 3-D flow problems.

It is well-known that, as far as the finite element method is concerned, the approximation properties for a Stokes flow remain unchanged as one switches to a Navier - Stokes flow (see e.g. [5]). Thus, we shall confine the discussion of our finite element method to the case of the former problem, namely:

Given a bounded domain Ω of $\mathbb{R}^3$ with boundary Γ, find a velocity field $\underset{\sim}{u}$ and a pressure p up to an additive constant, such that:

$$
\begin{aligned}
-\nu\Delta\underset{\sim}{u} + \underset{\sim}{\nabla}p &= \underset{\sim}{f} && \text{in } \Omega \\
div\ \underset{\sim}{u} &= 0 && \text{in } \Omega \\
\underset{\sim}{u} &= 0 && \text{on } \Gamma
\end{aligned}
$$

where $\nu>0$ is the viscosity of the fluid, and $\underset{\sim}{f}$ represents given external forces, that we assume to be in $\underset{\sim}{L}^2(\Omega)$.

As usual, we shall work with the weak formulation of this problem, namely:

$$
(P)\quad \begin{cases} \text{Find } (\underset{\sim}{u},p)\in \underset{\sim}{H}^1_0(\Omega)\times L^2_0(\Omega) & \text{such that} \\ \nu(\underset{\sim}{\nabla}\underset{\sim}{u},\underset{\sim}{\nabla}\underset{\sim}{v})+(p,div\ v) = (\underset{\sim}{f},\underset{\sim}{v}) & \forall \underset{\sim}{v}\in \underset{\sim}{H}^1_0(\Omega) \\ (q,div\ \underset{\sim}{u}) = 0 & \forall q\in L^2_0(\Omega) \end{cases}
$$

Here $L^2_0(\Omega)$ denotes the subspace of $L^2(\Omega)$ of functions q such that $\int_\Omega q\,d\underset{\sim}{x}=0$, and $(\cdot\,,\,\cdot)$ denotes the standard inner product of $L^2(\Omega)$. $(\underset{\sim}{\nabla}u,\underset{\sim}{\nabla}v) = \sum_{i=1}^{3}\int_\Omega \frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_i}\,dx$ is the inner product of $H^1_0(\Omega)$, the subspace of $H^1(\Omega)$ of functions that vanish a.e. on Γ, with associated norm $|v|_{1,\Omega}=(\underset{\sim}{\nabla}v,\underset{\sim}{\nabla}v)^{1/2}$

<u>Remark 1.1</u>: We refer to [1] for a precise definition of the Sobolev space $H^m(\Omega)$ for $m\geq 0$, and its norm and semi-norm, denoted by $\|\cdot\|_{m,\Omega}$ and $|\ |_{m,\Omega}$, respectively.

Remark 1.2: As usual, $\underset{\sim}{V}$ denotes the space of vector fields $\underset{\sim}{v}=(v_1 v_2, v_3)$, such that $v_i \in V$, $i=1,2,3$, equipped with the euclidian norm associated with the norm of V. □

An outline of the paper is as follows:

In Section 2 we define the finite approximation of problem (P), and we make some general remarks leading to the particular case of a parallelepipedal domain. In Section 3 we give existence results related to our finite element solution applying to the particular case above. In Section 4 we show that with a slight change our method yields a sequence of velocity and pressure fields converging with order one to the exact solution of (P), in the sense of $\underset{\sim}{H}_0^1(\Omega)\times L_0^2(\Omega)$. Finally in Section 5 computational aspects are discussed and a numerical example is given.

2 - THE FINITE ELEMENT APPROXIMATE PROBLEM

We assume that Ω is a polyhedron and we let $\mathcal{C}_h$ be a suitable partition of Ω into tetrahedrons with maximal edge length equal to h. We also assume that $\mathcal{C}_h$ belongs to a regular famity of partitions $(\mathcal{C}_h)_h$ in the usual sense [2].

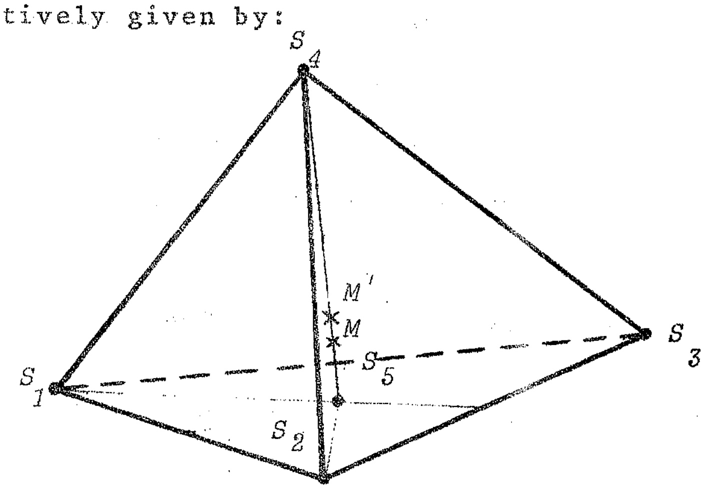
Let then K be a tetrahedron of $\mathcal{C}_h$ with vertices S_i, and associated area coordinates λ_i, $i=1,2,3,4$. Let us call the face opposite S_4 the *base* of K, and let S_5 be the centroid of this face (see Figure 2.1). The other faces of K are called *lateral faces*.

We now define P_a to be the space of functions spanned by $\lambda_1,\lambda_2,\lambda_3,\lambda_4$ and ϕ, where ϕ is a function that vanishes at all the vertices of K, and whose value is nonzero at S_5. We shall consider in this paper two simple choices of ϕ, namely:

(2.1) $\qquad \phi = \lambda_1\lambda_2\lambda_3$

and (2.1)' $\qquad \phi = \lambda_1\lambda_2+\lambda_1\lambda_3+\lambda_2\lambda_3$

and in this way the basis functions of P_a associated with the unisolvent set of functional values at S_i, i=1,2,3,4,5, are respectively given by:



The quasilinear tetrahedral finite element

Figure 2.1

(2.2) $$\begin{cases} p_i = \lambda_i - 9\phi & i=1,2,3 \\ p_4 = \lambda_4 \\ p_5 = 27\phi \end{cases}$$

if ϕ is given by (2.1), and

$$(2.2)' \quad \begin{cases} p_i = \lambda_i - \phi \quad , \quad i=1,2,3 \\ p_4 = \lambda_4 \\ p_5 = 3\phi \end{cases}$$

if ϕ is given by $(2.1)'$.

Let now V_h be the space of functions, whose restriction to each tetrahedron K of $\mathcal{C}_h$ belongs to P_a and that are continuous at each one of the five nodes situated on the boundary of K, or that vanish if this node happens to lie on Γ. The actual possibility of constructing V_h in this way will be discussed below.

We further define Q_h to be the subspace of $L_0^2(\Omega)$ of such functions that are constant over each $K \in \mathcal{C}_h$, and that are defined by its value at point M or M', according to the definition of ϕ (2.1) and $(2.1)'$, respectively. In area coordinates M and M' are given by:

$$M = (1/3-\sqrt{5}/30,\ 1/3-\sqrt{5}/30,\ 1/3-\sqrt{5}/30,\ \sqrt{5}/10)$$

$$M' = (1/4, 1/4, 1/4, 1/4).$$

We assume that partition $\mathcal{C}_h$ is constructed in such a way that a face being common to two neighboring elements is either a base or a lateral face of both elements. In so doing, one can easily check that if (2.1) is used we have $V_h \subset H_0^1(\Omega)$, in which case the discrete analogue of (P) is defined as follows:

$$(P_h)\quad \begin{cases} \text{Find } (\underset{\sim}{u}_h, p_h) \in \underset{\sim}{V}_h \times Q_h \text{ such that} \\ \nu(\underset{\sim}{\nabla}\underset{\sim}{u}_h, \underset{\sim}{\nabla}\underset{\sim}{v}_h) + (p_h, div\ \underset{\sim}{v}_h) = (\underset{\sim}{f}, \underset{\sim}{v}_h) & \forall \underset{\sim}{v}_h \in \underset{\sim}{V}_h \\ (q_h, div\ u_h) = 0 & \forall q_h \in Q_h \end{cases}$$

In the case where $(2.1)'$ is used we have $V_h \not\subset H_0^1(\Omega)$, and the approximate problem becomes:

$$(P_h)'\quad \begin{cases} \text{Find } (\underset{\sim}{u}_h, p_h) \in \underset{\sim}{V}_h \times Q_h \text{ such that} \\ \nu(\underset{\sim}{\nabla}\underset{\sim}{u}_h, \underset{\sim}{\nabla}\underset{\sim}{v}_h)_h + (p_h, div\ \underset{\sim}{v}_h)_h = (\underset{\sim}{f}, \underset{\sim}{v}_h) & \forall \underset{\sim}{v}_h \in \underset{\sim}{V}_h \\ (q_h, div\ \underset{\sim}{u}_h)_h = 0 & \forall q_h \in Q_h \end{cases}$$

where $(u,v)_h = \sum_{K \in \mathcal{T}_h} \int_K uv d\underset{\sim}{x}$, with obvious modifications in the vectorial case. We also define the associated norm of V_h to be:

$$|v|_{1,h} = (\nabla v, \nabla v)_h^{1/2}.$$

We discuss in detail the case of problem (P_h), *i.e.*, the case of a cubic ϕ. As it should be pointed out however, all these results will apply to the case of a quadratic ϕ. The essential modifications to be introduced are those taking into account non-conformity, and can be carried out in the same way as in [9].

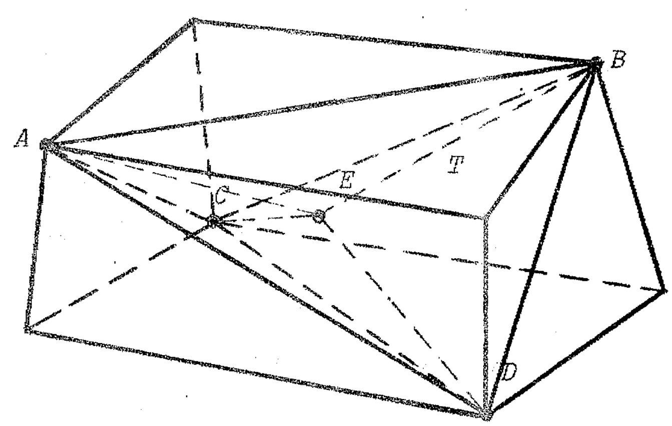
Incidentally, in that work we had given a detailed analysis of convergence of $(\underset{\sim}{u}_h, p_h)$ to $(\underset{\sim}{u}, p)$, by considering a construction of $\mathcal{T}_h$ based on macrotetrahedrons, and the optimal error bound:

$$|\underset{\sim}{u}-\underset{\sim}{u}_h|_{1,h} + \| p-p_h \|_{0,\Omega} \leq Ch \, [\, |\underset{\sim}{u}|_{2,\Omega} + |p|_{1,\Omega}]$$

was demonstrated. In this paper we derive existence and convergence results for (P_h) associated with another kind of partition $\mathcal{C}_h$, introduced in [8], that we recall:

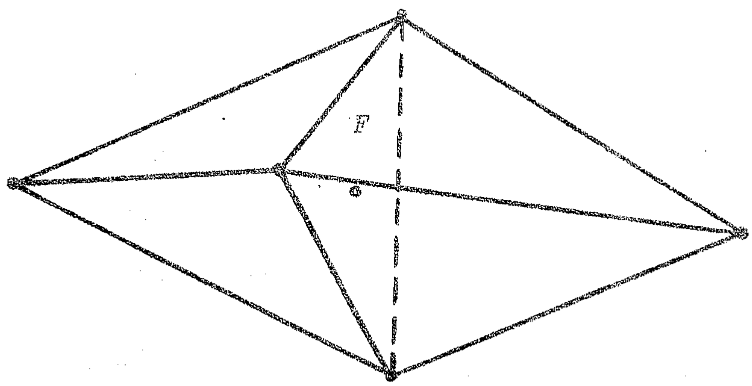
First we are given a partition $\mathcal{S}_h$ of Ω into hexahedrons with quadrilateral faces. Next each hexahedron H of $\mathcal{S}_h$ is subdivided into five tetrahedrons as shown in Figure 2.2. Finally we take the centroid E of the inner tetrahedron T and we join it to its four vertices A, B, C and D also illustrated in Figure 2.2, thereby generating a subdivision of H into eight tetrahedrons. The bases of the partition denoted by F, are precisely the faces of T.

It is interesting to notice that such a $\mathcal{C}_h$ can also be viewed as a partition χ_h of Ω into hexahedrons having triangular faces. Each one of these is the union of two tetrahedrons having a common base F, as shown in Figure 2.3.



Construction of partition $\mathcal{C}_h$ based on $\mathcal{S}_h$

Figure 2.2

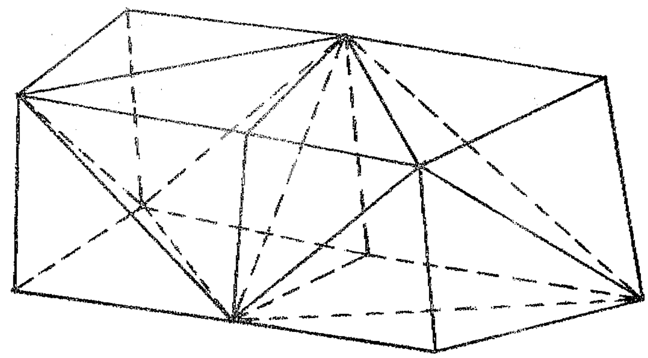


An element of partition χ_h with a pair of elements of $\mathcal{T}_h$

Figure 2.3

At this stage a word of caution is in order:

In general the subdivision of a domain into tetrahedrons based on a first partition $\mathcal{S}_h$ into hexahedrons, which are next subdivided into five tetrahedrons, is a procedure that requires some care in assembling the hexahedrons of $\mathcal{S}_h$ (see e.g. [12], p.99). This is aimed at satisfying in turn the basic assembling conditions for tetrahedral finite elements,namely: Two different tetrahedrons intersect either through a common face, edge or vertex, or an empty set. This requirement is fulfilled, for instance, if around each inner vertex of $\mathcal{S}_h$ there are exactly eight hexahedrons (four or two if the vertex lies on Γ), and we construct the inner tetrahedrons T in such a way that whenever two hexahedrons have a common face, they are disposed symmetrically with respect to this face, as shown in Figure 2.4.



Tetrahedrons of two neighboring hexahedrons of $\mathcal{S}_h$

Figure 2.4

Due to the above restriction, in order to simplify things, we shall consider henceforth that Ω is a rectangular parallelepiped and that partition $\mathcal{S}_h$ consists of equal parallelepipeds obtained by subdividing Ω in each direction x_1, x_2 and x_3 into M_1, M_2, and M_3 equal slices respectively. Notice that in so doing we have placed ourselves in the above described case of partitions into tetrahedrons. Moreover in this case we can derive existence and convergence results for the solution to problem (P_h) based on similar arguments as those used by Pitkäranta [6] for the tri-linear-constant $Q_1 \times Q_0$ velocity-pressure approximation for a rectangular domain, associated with partition $\mathcal{S}_h$, as shown in the next two sections.

Before going into this analysis, it is important to point out that, even in the particular case above, our finite element method is not a mere complicated version of the $Q_1 \times Q_0$ element, as one might think at a first glance. As a matter of fact it has two a priori advantages over that method, namely:

1) The divergence free condition is better approximated: Indeed, in both cases the pressure is taken constant over each element, which corresponds to the exact fulfillment of the divergence free condition whenever the velocity happens to be linear over the element. However, in the case of the $Q_1 \times Q_0$ element, the velocity is a linear function perturbed by a cubic function depending on four additional parameters, whereas in the case of our element the linear function is perturbed by only one additional component.

2) The pressure is more finely approximated in the following sense:
For the same number of velocity degrees of freedom (on which depends essentially the computational effort for solving (P_h)), the number of pressure degrees of freedom of our method increases by a factor of 1/3 with respect to the $Q_1 \times Q_0$ method.

<u>Remark 2.1</u>: A significant reduction of the computational effort of our method can be achieved by a simple and straighforward procedure to be described in Section 5.

We further comment that, as far as three-dimensional finite elements are concerned, methods leading to conforming solenoidal velocity fields are not known for the time being. In this sense, our method is an attempt to come as close as possible to this ideal situation.

Finally we would like to make a comment on our choice of points M and M' for the definition of the pressure over each element. Of course, since the pressure is discontinuous at element interfaces, the choice of this point is irrelevant. Nevertheless we can write:

$$(q_h, div\ \underset{\sim}{v}_h) = \sum_{K\in\mathcal{T}_h} q_h(M_K)\ div\ \underset{\sim}{v}_h(M_K) meas(K)$$

or

$$(q_h, div\ \underset{\sim}{v}_h)_h = \sum_{K\in\mathcal{T}_h} q_h(M'_K) div\ \underset{\sim}{v}_h(M'_K) meas(K)$$

$\forall q_h \in Q_h$ and $\forall\ \underset{\sim}{v}_h \in V_h$, where M_K and M'_K are points M and M' of tetrahedron K, respectively.

Indeed, in the case where ϕ is given by (2.1) $div\ \underset{\sim}{v}_h$ over K is a function of the form [10] :

$$c + \sum_{i=1}^{3} c_i \phi/\lambda_i \quad , \qquad c \text{ and } c_i \text{ being scalars,}$$

which can be integrated exactly over K by multiplying its value at point M by $meas(K)$.

Similarly, if ϕ is given by (2.1)', $div\ \underset{\sim}{v}_h$ over K is a linear function, which can be integrated exactly over K by multiplying its value at the centroid M' of K by $meas(K)$.

Thus in both cases it suffices to have $div\ \underset{\sim}{u}_h(M_K)= 0$ (resp. $div\ \underset{\sim}{u}_h(M'_K)=0$ $\forall K\in \mathcal{T}_h$ to have $(q_h, div\ \underset{\sim}{u}_h)= 0$ (resp. $(q_h, div\ \underset{\sim}{u}_h)_h = 0$) $\forall q_h \in Q_h$, and definitively, the points M and M' correspond to quadrature points related to reduced integration of the term $(div\ \underset{\sim}{v}_h, div\ \underset{\sim}{v}_h)$ (resp. $(div\ \underset{\sim}{v}_h, div\ \underset{\sim}{v}_h)_h$) in a penalized minimization problem associated with (P_h) (see e.g. [6]) .

3 - THE LBB-CONDITION

It is well-known that there exists a solution to problem (P_h) only if the LBB (Ladyzhenskaya - Babuška -Brezzi) - condition holds, namely:

$\exists\ \beta_h > 0$ such that $\forall q_h \in Q_h$ we have :

$$(3.1) \qquad \sup_{\underset{\sim}{v}_h \in \underset{\sim}{V}_h} \frac{(div\ \underset{\sim}{v}_h, q_h)}{\|\underset{\sim}{v}_h\|_h} \geq \beta_h\ |q_h|_h \quad ,$$

where $\|\cdot\|_h$ and $|\cdot|_h$ are any norms of $\underset{\sim}{V}_h$ and Q_h, respectively. It will be natural to take $|\cdot|_{1,\Omega}$ (or $|\cdot|_{1,h}$ if $V_h \not\subset H^1_0(\Omega)$) as a norm for $\underset{\sim}{V}_h$ and $\|\cdot\|_{0,\Omega}$ as a norm for Q_h.

<u>Remark 3.1</u>: Although for the purpose of establishing the existence (and eventually the uniqueness) of a solution of (P_h), it is not necessary to derive very sharp estimates of

β_h in terms of h, this is by no means the case of the analysis of convergence of $(\underset{\sim}{u}_h, p_h)$ to $(\underset{\sim}{u}, p)$. In particular , optimal convergence results are to be expected if β_h is independent of h. In the next section we will show that such a situation only occurs if we change a bit space Q_h, in which case the pressure p_h becomes also uniquely defined. □

Let us now introduce a decomposition of Q_h with respect to the $L^2(\Omega)$ - inner product, namely:

$$Q_h = Q_h^o \oplus Q_h^1 ,$$

where Q_h^o is the space of pressures that are constant over each element of $\mathcal{T}_h$, and belong to $L_0^2(\Omega)$. In this way $\forall q_h \in Q_h$ let $q_h^o \in Q_h^o$ and $q_h^1 \in Q_h^1$ be such that

$$q_h = q_h^o + q_h^1 .$$

We introduce below a suitable basis $\mathcal{B}$ of $Q_h \oplus \mathcal{C}$, $\mathcal{C}$ being the space of constant functions over Ω, and for this purpose we denote by K_i , $i=1,2,\ldots,8$, the eight tetrahedrons lying in $H \in \mathcal{S}_h$, in such a way that K_i and K_{i+4} have a common base, K_{i+4} being the tetrahedron lying in T, $1 \leq i \leq 4$

Now we set $= \bigcup_{H \in \mathcal{S}_h} \left\{ \gamma_i^H \right\}_{i=1}^{8}$, where $\gamma_i^H(\underset{\sim}{x}) = 0$

if $x \notin H$, $i=1,2,\ldots 8$ and are otherwise defined by:

- $\gamma_1^H(\underset{\sim}{x})=1$ if $\underset{\sim}{x}\in H$

- For $2\le i\le 4$

$$\gamma_i^H(\underset{\sim}{x})=1 \qquad \text{if } \underset{\sim}{x}\in K_1\cup K_i\cup K_5\cup K_{i+4}$$

$$\gamma_i^H(\underset{\sim}{x})= -1 \qquad \text{if } \underset{\sim}{x}\in K_{[i+1]}\cup K_{[i+2]}\cup K_{[i+1]+4}\cup K_{[i+2]+4}$$

where $[i+j] = \begin{cases} i+j & \text{if } i+j<5 \\ i+j-3 & \text{if } i+j\ge 5 \end{cases}$

- For $5\le i\le 8$

$$\gamma_i^H(\underset{\sim}{x}) = \gamma_{i-4}(\underset{\sim}{x}) \qquad \text{if } \underset{\sim}{x}\notin T$$

$$\gamma_i^H(\underset{\sim}{x}) = -2\ \gamma_{i-4}(\underset{\sim}{x}) \qquad \text{if } \underset{\sim}{x}\in T$$

Notice that $\gamma_i^H\in L_0^2(\Omega)$ $\forall H\in\mathcal{S}_h$ and $2\le i\le 8$

In this way Q_h^0 is the intersection of the space spanned by $\bigcup_{H\in\mathcal{S}_h} \gamma_1^h$ with $L_0^2(\Omega)$, and since $vol(K_{i+4})=0.5\ vol(K_i)$, $\mathcal{B}$ is orthogonal with respect to the L^2- inner product. As a consequence, Q_h^1 is spanned by $\bigcup_{H\in\mathcal{S}_h} \{\gamma_i^H\}_{i=2}^{8}$, and we have:

<u>Lemma 3.1</u>: For every $q_h^1 \in Q_h^1$ one can find $\underset{\sim}{v}_h^1 \in \underset{\sim}{V}_h$ such that

$$(3.2) \quad (div\ \underset{\sim}{v}_h^1,\ q_h^1) \geq C_1 \|\, q_h^1 \,\|^2_{0,\Omega}$$

$$(3.3) \quad |\underset{\sim}{v}_h|_{1,\Omega} \leq C_2 \|\, q_h^1 \|_{0,\Omega}$$

with C_1 and C_2 strictly positive and independent of h.

<u>Proof</u>: $\forall H \in \mathcal{S}_h$ we can write:

$$q^1_{h/H} = \sum_{j=2}^{8} q_j, \quad \text{where} \quad q_j = \sum_{H\in\mathcal{S}_h} \eta_j \gamma_j^H \ ,$$

η_j being scalars, $1 \leq j \leq 8$.

Let now $H \in \mathcal{S}_h$ and M_i be the centroid of the common base F_i of K_i and $K_{i+4} \subset H$, $1 \leq i \leq 4$. We define $\underset{\sim}{w}_j \in \underset{\sim}{V}_j$, $j=5,6,7,8$, to be such that

(i) $\underset{\sim}{w}_j(M_i)$ is orthogonal to F_i, $1 \leq i \leq 4$, pointing outwards F_i if $\gamma_j > 0$ over K_i and inwards F_i if $\gamma_j < 0$ over K_j.

(ii) For $i=1,2,3,4$ we set:

$$|\underset{\sim}{w}_5(M_i)| = vol(H)(\eta_5+\eta_6+\eta_7+\eta_8)\ /2\alpha_i$$

$$|\underset{\sim}{w}_6(M_i)| = vol(H)(-\eta_5+\eta_6-\eta_7+\eta_8)\ /2\alpha_i$$

$$|\underset{\sim}{w}_7(M_i)| = vol(H)(-\eta_5+\eta_6+\eta_7-\eta_8)\ /2\alpha_i$$

$$|\underset{\sim}{w}_8(M_i)| = vol(H)(-\eta_5-\eta_6+\eta_7+\eta_8)\ /2\alpha_i$$

where $\alpha_i = \int_{F_i} 27\phi_i\, d\underset{\sim}{x}$, ϕ_i being the restriction of ϕ given by (2.1) over the base F_i .

(iii) $|\underset{\sim}{w}_j(S)| = 0$ if S is any other node of $\mathcal{T}_h$, $5 \leq j \leq 8$.

It is then straightforward to check that:

(3.4) $$\int_H q_k \, div \, \underset{\sim}{w}_j \, d\underset{\sim}{x} = 0 \qquad \text{if } j \neq k, \; 1 \leq k \leq 8 \text{ and } 5 \leq j \leq 8,$$

(3.5) $$\int_H q_j \, div \, \underset{\sim}{w}_j \, d\underset{\sim}{x} = \int_H q_j^2 \, d\underset{\sim}{x} \qquad , \; 5 \leq j \leq 8$$

On the other hand, we define a vector field $\underset{\sim}{z} \in \underset{\sim}{V}_h$ to be such that:

(iv) $$\underset{\sim}{z}(G) = \frac{4}{3} [-(\eta_2 - 2\eta_6)\underset{\sim}{m}_2 + (\eta_3 - 2\eta_7)\underset{\sim}{m}_3 + (\eta_4 - 2\eta_8)\underset{\sim}{m}_4]$$

where G is the centroid of H (or yet the centroid of T) and $\underset{\sim}{m}_i$ is the oriented segment joining the vertex of T opposite K_{i+4} to G, $1 \leq i \leq 4$.

(v) $\underset{\sim}{z}(S)=0$ if S is any other node of $\mathcal{T}_h$.

Using the same arguments as in [9], Theorem 4.1, we get:

(3.6) $$\sum_{j=2}^{4} \int_H (q_j + q_{j+4}) div \, \underset{\sim}{z} \, d\underset{\sim}{x} \geq \frac{2}{3} \int_T \Big[\sum_{j=2}^{4} (q_j + q_{j+4})\Big]^2 d\underset{\sim}{x}$$

(3.7) $$\int_H q_i \, div \, \underset{\sim}{z} \, d\underset{\sim}{x} = 0 \qquad \text{for } i = 1 \text{ or } 5.$$

Now setting $\underset{\sim}{v}_h^1 = \underset{\sim}{w} + \underset{\sim}{z}$ where $w = \sum_{j=5}^{8} \underset{\sim}{w}_j$ we have

$$\int_H q_h^1 \; div \; \underset{\sim}{v}_h^1 \; d\underset{\sim}{x} = \int_H (\sum_{j=2}^{8} q_j) \; div(\underset{\sim}{w} + \underset{\sim}{z}) \; d\underset{\sim}{x}$$

Taking into account (3.4), (3.5), (3.6) and (3.7) we get

$$(3.8) \quad \int_H q_h^1 \; div \; \underset{\sim}{v}_h^1 \; d\underset{\sim}{x} \geq \sum_{j=5}^{8} \int_T q_j^2 + \frac{2}{3} \sum_{j=2}^{4} \int_T (q_j + q_{j+4})^2 \, d\underset{\sim}{x}$$

On the other hand, for $2 \leq j \leq 4$, by a straighforward computation we obtain:

$$\int_T q_{j+4}^2 \; d\underset{\sim}{x} + \frac{2}{3} \int_T (q_j + q_{j+4})^2 d\underset{\sim}{x} \geq \frac{1}{3} \int_T (q_j^2 + q_{j+4}^2) \; d\underset{\sim}{x}$$

$$\geq \frac{1}{9} \quad \int_H q_j^2 \; d\underset{\sim}{x} + \int_H q_{j+4}^2 \; d\underset{\sim}{x}$$

This, together with (3.8), yields (3.2) with $C_1 = 1/9$.

Finally, taking into account that $(\mathcal{C}_h)_h$ is regu - lar,(3.3) can be readily established. □ q.e.d.

Let now $\underset{\sim}{W}_h$ be the space of velocity fields whose restriction to each $H \in \mathcal{S}_h$ belongs to $\underset{\sim}{Q}_1$.

<u>Lemma 3.2</u>: Assume that $\forall q_h^o \in Q_h^o$ there exists a vector field $\underset{\sim}{w}_h \in \underset{\sim}{W}_h$ such that:

$$(3.9) \quad (div \; \underset{\sim}{w}_h, \; q_h^o) \geq C_3(h) \; \| \, q_h^o \, \|_{0,\Omega}^2$$

$$(3.10) \quad |\underset{\sim}{w}_h|_{1,\Omega} \leq C_4 \; \| \, q_h^o \, \|_{0,\Omega}$$

with C_4 independent of h and $C_3(h) > 0 \;\; \forall h$.

Then there exists a vector field $\underset{\sim}{v}_h^o \in \underset{\sim}{V}_h$ such that

(3.11) $\quad (div\ \underset{\sim}{v}_h^o,\ q_h^o) = (div\ \underset{\sim}{w}_h,\ q_h^o)$

(3.12) $\quad |\underset{\sim}{v}_h^o|_{1,\Omega} \leq C_5\ \| q_h^o \|_{0,\Omega}$

with C_5 independent of h and positive.

Proof: Let $\underset{\sim}{v}_h^o$ be the vector field $\underset{\sim}{V}_h$ whose value at a vertex S of $\mathcal{C}_h$ is $3/2\ \underset{\sim}{w}_h(S)$, if S is not a vertex of T and $3/4\ \underset{\sim}{w}_h(S)$ if S is a vertex of T. We further set $\underset{\sim}{v}_h^o(M)=3/2\ \underset{\sim}{w}_h(M)$, where M is any node of $\mathcal{C}_h$ lying in the interior of H, $H\in\mathcal{S}_h$

One can easily verify that

$$\int_H div\ \underset{\sim}{v}_h^o\ d\underset{\sim}{x} = \int_H div\ \underset{\sim}{w}_h\ d\underset{\sim}{x} \qquad \forall H\in\mathcal{S}_h$$, which yields (3.11).

On the other hand,

$$\underset{\sim}{v}_h^o = \frac{3}{2}\ \pi\ \underset{\sim}{w}_h - \underset{\sim}{r}_h$$

where $\pi\underset{\sim}{w}_h$ is the $\underset{\sim}{V}_h$- interpolant of $\underset{\sim}{w}_h$ at the nodes of $\mathcal{C}_h$ and $\underset{\sim}{r}_h \in \underset{\sim}{V}_h$ is the field such that:

$\underset{\sim}{r}_h(S) = \frac{3}{4}\ \underset{\sim}{w}_h(S)$ if S is a vertex of both H and T, $\forall H\in\mathcal{S}_h$, and $\underset{\sim}{r}_h(S)=0$ at any other vertex of $\mathcal{C}_h$.

We then have by standard arguments [2] :

(3.13) $$|\underset{\sim}{v}_h^o|_{1,\Omega} \leq \frac{3}{2}\ |\pi\ \underset{\sim}{w}_h|_{1,\Omega} + |\underset{\sim}{r}_h|_{1,\Omega} \leq C_6\ [\,|\underset{\sim}{w}_h|_{1,\Omega} + h\|\underset{\sim}{r}_h\|_{0,\Omega}\,]$$

On the other hand one can check that

$$\|\underset{\sim}{r}_h\|_{0,\Omega} \leq C_7\ \|\pi\ \underset{\sim}{w}_h\|_{0,\Omega}$$

where C_7 depends on space P_a but not on h.

Hence, using Poincaré's inequality we obtain:

$$\| \underset{\sim}{r}_h \|_{0,\Omega} \leq C_8 \; |\underset{\sim}{w}_h|_{1,\Omega}$$

which combined with (3.13) gives

$$|\underset{\sim}{v}_h^0|_{1,\Omega} \leq C_6(1 + C_8 h)|\underset{\sim}{w}_h|_{1,\Omega}$$

This estimate combined with (3.10) yields (3.12) with $C_5 = C_4 C_6 (1+C_8 \ell), \ell$ being the maximal edge length of Ω.□

q.e.d.

As seen below, an immediate consequence of Lemma 3.2 is the fact that the LBB-condition holds for our element whenever it holds for the $Q_1 \times Q_0$ element associated with $\mathcal{S}_h$. As a matter of fact, the existence of $\underset{\sim}{w}_h$ satisfying both (3.9) and (3.10) is guaranteed in most of the cases (see Remark 3.2 below).

In any case we have:

<u>Theorem 3.1</u>: Under the assumptions of Lemma 3.2, $\forall q_h \in Q_h$ there exists a vector field $\underset{\sim}{v}_h \in \underset{\sim}{V}_h$ such that condition (3.1) holds.

<u>Proof:</u> Let $\underset{\sim}{v}_h^0$ and $\underset{\sim}{v}_h^1$ be the vector fields of $\underset{\sim}{V}_h$ defined in Lemmas 3.2 and 3.1, respectively. For a certain $\theta>0$ to be defined below we set:

$$\underset{\sim}{v}_h = \underset{\sim}{v}_h^0 + \theta v_h^1 .$$

We have:

$$(div\ \underset{\sim}{v}_h, q_h) = \theta\ (div\ \underset{\sim}{v}_h^1, q_h^1) + (div\ \underset{\sim}{v}_h^o, q_h^1) + (div\ \underset{\sim}{v}_h^o, q_h^o)$$

Using (3.2), (3.11) and (3.9) we obtain:

$$(div\ \underset{\sim}{v}_h, q_h) \geq \theta\ C_1 \| q_h^1 \|^2_{0,\Omega} + C_3(h) \| q_h^o \|^2_{0,\Omega} - 2\ |\underset{\sim}{v}_h^o|_{1,\Omega} \| q_h^1 \|_{0,\Omega}$$

Taking into account (3.12) we see that if we choose

$$\theta = \frac{4C_5^2}{C_1 C_3(h)} \quad \text{then} \quad C(h) = min\ (\frac{2C_5^2}{C_1 C_3(h)}, \frac{C_3(h)}{2}) \text{ is such that}$$

$$(div\ \underset{\sim}{v}_h, q_h) \geq C(h) \| q_h \|^2_{0,\Omega}$$

On the other hand , under these circumstances we have:

$$|\underset{\sim}{v}_h|_{1,\Omega} \leq C'(h) \| q_h \|_{0,\Omega}$$

where $C'(h) = max\ [\frac{4C_5^3}{C_1 C_3(h)}, C_2]$, which leads to (3.1) with $\beta_h = C(h)/C'(h)$. □

q.e.d.

Remark 3.2: As one can easily infer from Theorem 3.1, the way β_h depends on h is strictly related to the way C_3 depends on h. Notice that the latter constant is associated with the $Q_1 \times Q_0$ element and,as proved in [6], $C_3(h)=0(h)$ if M_1, M_2 and M_3 are even. Thus in this particular case β_h is readily seen to be an $O(h^2)$ at most. Further comments on this dependence of h will be made in the next section. □

4. CONVERGENCE RESULTS

In this section we discuss sufficient conditions for our finite element method to provide a sequence of convergent velocity and pressure fields towards the solution of *(P)*.

As we saw in the previous section, the constant β_h appearing in the LBB-condition depends effectively on h, and this is essentially caused by the component $q_h^o \in Q_h^o$ of a pressure $q_h \in Q_h$.

According to [6], if M_1, M_2 and M_3 are even, say , equal to $2m_1, 2m_2$ and $2m_3$ respectively, we can find a suitable decomposition of Q_h^o into the direct sum of eight subspaces R_h^ℓ defined in a way that we recall:

We first assign to each element of $\mathcal{S}_h$ a triple subscript $i_1 i_2 i_3$ where $1 \le i_1 \le 2m_1$, $1 \le i_2 \le 2m_2$ and $1 \le i_3 \le 2m_3$, in such a way that $H_{i_1 i_2 i_3}$ is the element lying in the i_k-th slice of the x_k -direction, $k=1,2,3$.

We next consider another partition $\mathcal{S}_h^o$ of Ω into equal parallelepipeds by changing M_1, M_2 and M_3 with m_1, m_2 and m_3 , respectively.

We denote each element of $\mathcal{S}_h^o$ by $H^0_{i_1 i_2 i_3}$, $1 \le i_k \le m_k$, $k=1,2,3$, which clearly satisfies:

$$\overline{H}^o_{i_1 i_2 i_3} = \bigcup_{\ell_1=0}^{1} \bigcup_{\ell_2=0}^{1} \bigcup_{\ell_3=0}^{1} \overline{H}_{2i_1-\ell_1,\ 2i_2-\ell_2,\ 2i_3-\ell_3}$$

The R_h^{ℓ}'s are then the spaces spanned by $\xi^{\ell}_{i_1 i_2 i_3}$, each one being a function whose support is $B^0_{i_1 i_2 i_3} \in \mathcal{S}^0_h$, and whose values are otherwise defined to be:

$$\xi^1_{i_1 i_2 i_3}(\underset{\sim}{x}) = 1 \qquad\qquad \xi^5_{i_1 i_2 i_3}(\underset{\sim}{x}) = (-1)^{\ell_1+\ell_2}$$

$$\xi^2_{i_1 i_2 i_3}(\underset{\sim}{x}) = (-1)^{\ell_1} \qquad\qquad \xi^6_{i_1 i_2 i_3}(\underset{\sim}{x}) = (-1)^{\ell_1+\ell_3}$$

$$\xi^3_{i_1 i_2 i_3}(\underset{\sim}{x}) = (-1)^{\ell_2} \qquad\qquad \xi^7_{i_1 i_2 i_3}(\underset{\sim}{x}) = (-1)^{\ell_2+\ell_3}$$

$$\xi^4_{i_1 i_2 i_3}(\underset{\sim}{x}) = (-1)^{\ell_3} \qquad\qquad \xi^8_{i_1 i_2 i_3}(\underset{\sim}{x}) = (-1)^{\ell_1+\ell_2+\ell_3}$$

Now according to [6], if we consider the subspace $Q_h^{\delta} \oplus \mathcal{C}$ of $Q_h^{o} \oplus \mathcal{C}$ spanned by $\{\xi^{\ell}_{i_1 i_2 i_3}, 1 \le i_k \le m_k \,,\, k=1,2,3 \,,\, \ell=1,2,3,4\}$, we can prove:

Lemma 4.1 [6]: $\forall q_h^{\delta} \in Q_h^{\delta}$ there exists a vector field $\underset{\sim}{w}_h^{\delta} \in \underset{\sim}{W}_h$ such that:

$$(div\ \underset{\sim}{w}_h^{\delta}\ ,\ q_h^{\delta}) \ge c \| q_h^{\delta} \|^2_{0,\Omega}$$

$$|w_h^{\delta}|_{1,\Omega} \le C\ \| q_h^{\delta} \|_{0,\Omega}$$

where c and C are strictly positive and independent of h. □

Taking into account the above Lemma and proceding in the same way as in Lemma 3.2 one can prove:

Lemma 4.2: $\forall q_h^{\delta} \in Q_h^{\delta}$ there exists a vector field $\underset{\sim}{v}_h^{\delta} \in \underset{\sim}{V}_h$ such that

$$(div\ \underset{\sim}{v}_h^{\Delta}\ ,\ q_h^{\Delta}) \geq c\ \| q_h^{\Delta} \|^2_{0,\Omega}$$

and

$$|\underset{\sim}{v}_h^{\Delta}|_{1,\Omega} \leq \overline{C}\ \| q_h^{\Delta} \|_{0,\Omega}$$

where $\overline{C}$ is independent of h, and c is the constant of Lemma 4.1. □

Let us now define a subspace Q_h^S of Q_h consisting of suitably *smothed pressures* given by:

$$Q_h^S = Q_h^{\Delta} \oplus Q_h^1 \quad .$$

Notice that $dimQ_h^S + 1 = 15/16\ (dimQ_h + 1)$ and that combining Lemma 3.1 and Lemma 4.2 we conclude in the same way as in Theorem 3.1 the following:

Theorem 4.1 : There exists a strictly positive constant β independent of h such that:

$$\forall q_h \in Q_h^S \qquad \exists\ \underset{\sim}{v}_h \in \underset{\sim}{V}_h \qquad \text{such that}$$

$$\sup_{\underset{\sim}{v}_h \in \underset{\sim}{V}_h} \frac{(q_h,\ div\ v_h)}{|\underset{\sim}{v}_h|_{1,\Omega}} \geq \beta\ \| q_h \|_{0,\Omega} \qquad \square$$

Let us now define a modified approximate problem , namely:

$$(P_h^S) \quad \begin{cases} \text{Find} \quad (\underset{\sim}{u}_h^S, p_h^S) \in \underset{\sim}{V}_h \times Q_h^S \quad \text{such that} \\ \nu(\underset{\sim}{\nabla}\underset{\sim}{u}_h^S, \underset{\sim}{\nabla}\underset{\sim}{v}_h) + (p_h^S, div\, \underset{\sim}{v}_h) = (\underset{\sim}{f}, \underset{\sim}{v}_h) \quad \forall \underset{\sim}{v}_h \in \underset{\sim}{V}_h \\ (div\, \underset{\sim}{u}_h^S, q_h) = 0 \quad \forall q_h \in Q_h^S \end{cases}$$

Again recalling the arguments of [6], it is possible to prove that the solution to problem (P_h^S) is unique, which was not the case of (P_h), due to the components $\gamma^{\ell}_{i_1 i_2 i_3}$, $5 \leq \ell \leq 8$, of the basis of Q_h.

Since the LBB-condition now holds with β independent of h, it is possible to use standard estimates given in [11] and we get:

<u>Theorem 4.3</u> : If Ω is a rectangular parallelepiped and the mesh $\mathcal{C}_h$ is constructed in the way prescribed in Section 2 with $\mathcal{S}_h$ consisting of $M_1 \times M_2 \times M_3$ equal elements, where M_1, M_2 and M_3 are even, then if $\underset{\sim}{u} \in \underset{\sim}{H}^2(\Omega)$ and $p \in H^1(\Omega)$ we have:

$$|\underset{\sim}{u} - \underset{\sim}{u}_h^S|_{1,\Omega} + \| p - p_h^S \|_{0,\Omega} \leq Ch\, [|\underset{\sim}{u}|_{2,\Omega} + |p|_{1,\Omega}] \qquad \square$$

<u>Remark 4.1</u> : If the whole space Q_h is used instead of Q_h^S, like in the case of the $Q_1 \times Q_0$ element studied in [6] one can expect the velocity field $\underset{\sim}{u}_h$ to satisfy:

$$|\underset{\sim}{u} - \underset{\sim}{u}_h|_{1,\Omega} \leq Ch\, [\| \underset{\sim}{u} \|_{9/2,6/5,\Omega} + \| p \|_{1,\Omega}]$$

if $\underset{\sim}{u}$ belongs to the Sobolev space $\underset{\sim}{W}^{9/2,6/5}(\Omega)$.

On the other hand, one can only expect the pressure p_h to converge to p in $L^2(\Omega)$ in case Q_h is replaced by Q_h^S. $\square$

Notice however that, in the case of the $Q_1 \times Q_0$ element, the corresponding change is more severe, as it reduces the dimension of Q_h by one half, thereby implying a much worse approximation of the divergence free condition.

Remark 4.2 : The use of the space Q_h^S instead of the whole Q_h does not introduce any significant extra complexity from the computational point of view, as we are dealing with equal elements and the matrix corresponding to the term $(div\ \underset{\sim}{v}, p)$ does not need to be stored. Further details on this point are given the next section. □

5. NUMERICAL RESULTS

We have solved problem (P_h) for the following test-data: Ω is taken to be a unit cube with given flow over Γ, namely:

$$u_1 = x_2 x_3 (1-x_2)(1-x_3)$$

$$u_2 = u_3 = 0$$

f is taken to be zero and in this way the exact solution to (P) is given by:

$$\underset{\sim}{u} = (x_2 x_3 (1-x_2)(1-x_3), 0, 0) \;,$$

p being determined by $\underset{\sim}{\nabla} p = \Delta \underset{\sim}{u}$ in Ω with $\int_\Omega p d\underset{\sim}{x} = 0$

We have performed the computations with the well - known iterative algorithm of Uzawa type (see e.g. [4]) that we briefly recall:

Given $p_0 \in L_0^2(\Omega)$ compute for $n = 1,2,3,\dots,$ $\underset{\sim}{u}_n$ and p_n $s.t.$

$$\nu(\underset{\sim}{\nabla} \underset{\sim}{u}_n, \underset{\sim}{\nabla} \underset{\sim}{v}) = (div\ \underset{\sim}{v}, p_{n-1}) + (\underset{\sim}{f}, \underset{\sim}{v}) \qquad \forall \underset{\sim}{v} \in \underset{\sim}{H}_0^1(\Omega)$$

and

$$p_n = p_{n-1} - \rho\ div\ \underset{\sim}{u}_n \;,$$

where ρ is a real parameter satisfying $0<\rho<2/3$

It is well-known that with the above procedure one only needs to factorize the matrix associated with the term $\nu(\underset{\sim}{\nabla} u_i, \underset{\sim}{\nabla} v_i)$ $i=1,2$ or 3, in the discrete problem, and then solve at each iteration one linear system per component with this matrix.

As for the partition $\mathcal{T}_h$ described in Section 2 we have taken $M_1=M_2=M_3=M$ for four different values of M, and in this way the dimension of the above mentionned matrix is roughly $6M^3$. However, we have proceeded to an a priori nodal elimination, which reduces the dimension of this matrix to about $2M^3$. This elimination is performed in the following way:

First of all we regard the partition of Ω to be X_h rather than $\mathcal{T}_h$. Next, noticing that the nodes being centroids of the bases of elements of $\mathcal{T}_h$ are nodes lying in the interior of X_h, we see that they can be expressed only in terms of the five remaining nodes of this element. Thus if we replace these values in the linear equations associated with the outer nodes, we can eliminate the centroid nodes from the linear system.

All these operations can be easily executed at the level of the elements of X_h before assembling the whole matrix, which is definitively associated only with the vertices of the tetrahedrons of $\mathcal{T}_h$.

We give in Table 1 some representative results obtained for the velocities of our test problem. We have taken $\rho = 1/3$ and 100 iterations of the above algorithm, starting with $p_0 \equiv 0$.

M	Relative Average error of u_1	Absolute error of u_2 and u_3		Relative error of the flux over the midplane $x_1 = 1/2$	$\lvert div\ \underset{\sim}{u} \rvert$ after 100 iterations	
		Max.	Min.		Max.	Min.
3	0.1865	0,0121	0,0000	0,050	0,01743	0,00026
4	0.1048	0,0055	0,0000	0,142	0,01778	0,00015
5	0.0662	0.0028	0,0000	0,037	0,02110	0,00025
6	0.0474	0.0017	0,0000	0,029	0,01939	0,00005

Table 1
Numerical results for a test-problem (P_h).

<u>Remark 5.1:</u> The results in Table 1 are seemingly satisfactory if one compares them with similar tests using other methods (see e.g. [3]). Notice that, taking into account the amount of calculation involved in *3-D* problems, computations with fine grids are prohibitive.

<u>Remark 5.2</u>: The convergence of the pressure has not been observed. According to the analysis given in Section 4 this was to be expected. Nevertheless, near fixed points such as the origin, the pressure occilated only about 10% with respect to the maximum (resp.minimum) computed values for the four used meshes.

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