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# ON THE CONVERGENCE OF THE BILINEAR VELOCITY-CONSTANT PRESSURE FINITE ELEMENT METHOD IN VISCOUS FLOW *

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ABSTRACT

In this paper we show that the standard isoparametric $Q_1xQ_0$ velocity-pressure finite element method for solving viscous flow problems, leads to a fully and optimally convergent sequence of approximations, if an approxiate quadrangulation of the domain is used.

RESUMO

Este artigo trata do estudo da convergência do método de elementos finitos mistos de aproximação de problemas de escoamentos viscosos na formulação velocidade-pressão, com funções de forma bilineares isoparametricas ($Q_1$) constantes ($Q_0$) por quadrilätero, respectivamente. Sabe-se que no caso de partições de um retângulo em elementos retangulares, a convergência da velocidade só ocorre se a solução exata é regular e a da pressão simplesmente não ocorre em geral. Prova-se que, se uma malha especial de quadriläteros for usada, - a saber, a malha-mosaico-, resultados de convergência ótima da velocidade-pressão são obtidos, mediante hipóteses mínimas de regularidade da solução exata.

## 1. INTRODUCTION

We consider herein the numerical solution by the finite element method of two-dimensional problems related to incompressible media, and we take as a model the representative case of the Stokes problem.

A lot of research works on this subject have been carried out, as it is well-known to be an essential step towards solving nonlinear viscous flow problems (see e.g.[1] and [2]). In particular, the so-called $Q_1 x Q_0$ approximation, consisting of piecewise isoparametric bilinear velocity and constant pressure, appears to be very attractive, due to its simplicity and low computational cost, compared to other admissible methods. As a matter of fact, many authors have devoted themselves to the study of the mathematical properties of this method (see e.g. [3] and [4] ) and to its implementation (see e.g. [5] and [6], and reference therein).

Actually, the $Q_1 x Q_0$ method is known to be efficient in many cases. However, as far as existence and convergence of its velocity-pressure solutions are concerned, some difficulties arrise, such as the so-called "checker-board" pressure phenomenon [5]. Another drawback of this discretization has been recently pointed out in [7].

A very clear and elucidating analysis of these questions has been given by PITKÄRANTA in [4], where proofs for the following convergence results can be found. If the domain is a rectangle which is partitioned into a uniform MxN rectangular grid with M and N even, an O(h) convergence of the velocity in the $H^1_0$-Sobolev norm is guaranteed, provided the exact solution is smooth, whereas the $L^2$-norm convergence of the pressure can only be expected if some kind of smothing technique is used. At this point it is important to stress the fact that all the above mentioned limitations are related to the use of a specific rectangular grid. As we show in this paper, another construction of quadrangular grids can completely overcome this difficulty.

The construction of this partition, which we call the the *tile-grid*, applying to rectangular domains (or a suitable mapping of it) will be described later on. As a matter of fact it has been recently proposed in [8], where it is shown that the existence and the uniqueness of a pressure field for the corresponding approximate problems is guaranteed. The purpose of the present paper is to prove that actually *for this partition, the optimal rate $H_0^1 \times L^2$-convergence of the pair velocity-pressure is attained, with minimal regularity assumptions for the exact solution.*

Before doing this, we would like to recall, that the essential difficulty for proving such results, lies in deriving the so-called LBB (Ladyzhenskaya-Babuska-Brezzi) condition, in connection with the variational formulation (P) of the Stokes problem, as follows:

Let $\Omega$ be a bounded domain of $\mathbb{R}^2$ with boundary $\partial\Omega$. We wish to find a velocity field $u = (u_1, u_2)$ with $u_i \in H_0^1(\Omega)$, and a pressure $p \in L_0^2(\Omega)$, with

$$L_0^2(\Omega) = \{q / q \in L^2(\Omega), \int_\Omega q\,dx = 0\},$$

such that:

$$(P) \quad \begin{cases} \nu(\text{grad } u, \text{grad } v)_o + (p, \text{div } v)_o = (f, v)_o & \forall\, v \in H_o^1(\Omega) \times H_o^1(\Omega) \\ (q, \text{div } u)_o = 0 & \forall q \in L_o^2(\Omega) \end{cases}$$

for given external forces $f = (f_1, f_2)$, $f_i \in L^2(\Omega)$, and viscosity $\nu$, where $(.,.)_o$ denotes the standard inner product of $L^2(\Omega)$.

Let ${}_h$ be a partition of $\Omega$ into finite elements of maximal edge length h, and $V_h$ and $Q_h$ be finite element subspaces of $H_o^1(\Omega)$ and $L_o^2(\Omega)$ respectively. We define the approximate problem of (P) to be.

$$(P_h) \quad \begin{cases} \text{Find } (u_h, p_h) \in (V_h \times V_h) \times Q_h \quad \text{such that} \\ \nu\, (\text{grad } u_h,\ \text{grad } v_h)_o + (p_h,\ \text{div } v_h) = (f, v_h) & \forall v_h \in V_h \times V_h \\ (q_h,\ \text{div } u_h)_o = 0 & \forall q_h \in Q_h \end{cases}$$

For the existence and uniqueness of $(u_h, p_h)$, the LBB-condition requires in turn the existence of a strictly positive constant $\beta(h)$ such that:

$$(1.1)\qquad \begin{cases} \forall q_h \in Q_h,\ \exists v_h \in V_h \times V_h \quad \text{such that} \\ \dfrac{(\operatorname{div} v_h,\ q_h)}{|v_h|_1} \geq \beta\ \|q_h\|_o \end{cases}$$

where $\|\cdot\|_m$ and $|\cdot|_m$ denote the standard norm and semi-norm of the Sobolev space $[H^m(\Omega)]^n$, $n\geq 1$, $m\geq 0$, respectively.

Moreover, if this condition holds with $\beta$ independent of $h$, and if the best $V_h$- and $Q_h$- approximations of functions of $H^k(\Omega)$ and $H^{k-1}(\Omega)$, in the $H^1(\Omega)$ and $L^2(\Omega)$ norms are of order $k-1$, according to standard results [8] we have:

$$(1.2)\qquad |u_h-u|_1 + \|p_h-p\|_o \leq C_h^{k-1}\ [\|u\|_k + \|p\|_{k-1}]$$

## 2 - THE TILE-GRID AND ITS PROPERTIES

### 2.1. Definition

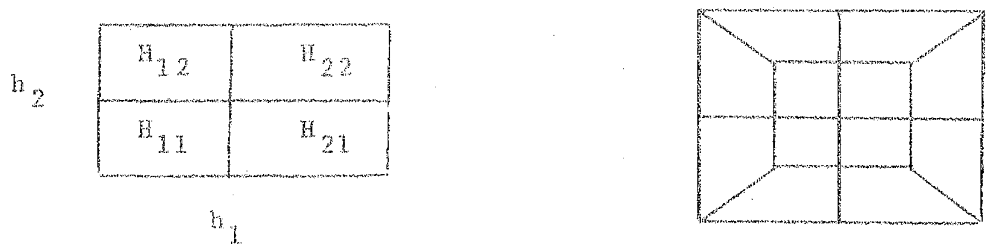
Let $\Omega$ be a rectangle, and $\{\chi_h\}_h$ be a family of non-necessarily uniform partitions $\chi_h$ of $\Omega$ into rectangles. Let $H\in\chi_h$ and assume that the edges of $H$ are parallel to the edges of $\Omega$ respectively, which for simplicity are in turn assumed to coincide with the $x_1$ and $x_2$ axes respectively. Let also $h_1$ and $h_2$ be the corresponding edge lengths of $H$ and denote by $\mathbf{\nu}_i$ the unit vector of axis $x_i$, $i=1,2$.

As usual, we further assume that there exists a strictly positive constant $c$ independent of $h$, such that:

$$(2.1)\qquad \min\{\min(\frac{h_1}{h_2}, \frac{h_2}{h_1})\} \geq c \qquad \forall h.$$

Now, partition $\mathcal{C}_h$ giving rise to the tile-grid, is construted in the following way:

We first subdivide each H, $H \in \chi_h$, into four equal rectangles, say $H_{11}, H_{12}, H_{21}$ and $H_{22}$, as shown in Figure 1a, thereby generating a new partition $\mathscr{J}_h$ of $\Omega$ into rectangles. Finally, each rectangle of $\mathscr{J}_h$ is subdivided into three quadrilaterals in the way shown in Figure 1b. Such quadrilaterals, whose common vertex is chosen to be the center of the rectangle of $\mathscr{J}_h$, are precisely the elements of $\mathscr{C}_h$, that we denote by K.




1a. An element $H \in \chi_h$ and four elements of $\mathscr{J}_h$

1b. A set of elements of $\mathscr{C}_h$ forming a macroelement of $\chi_h$

An illustration of the construction of the tile-grid

Figure 1

In this way, a set of twelve elements of $\mathscr{C}_h$ forming the pattern above, gives rise to an element of $\chi_h$, which we call a macroelement.

We define velocity fields of $V_h x V_h$ as fields of $H_o^1(\Omega)$, whose restriction to each $K \in \mathscr{C}_h$ is the mapping of a bilinear field defined on a unit square $\hat{K}$, by the bilinear field which maps in turn $\hat{K}$ onto K. We define a pressure of $Q_h$ as constant functions over each K, $K \in \mathscr{C}_h$, but not necessarily continuous along element interfaces.

## 2.2. LBB-Condition

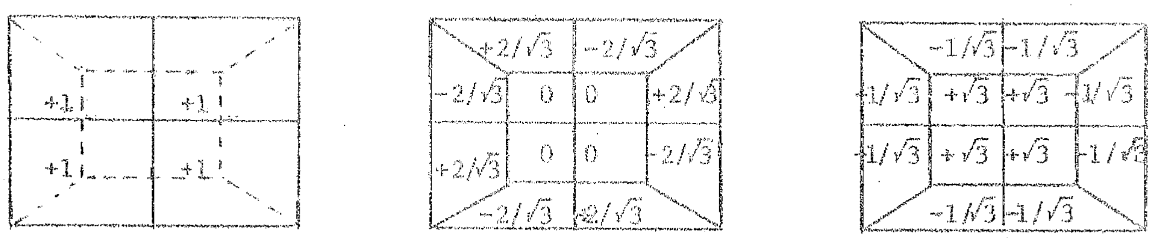
Let $\mathscr{C}$ be the space of those pressures that are constant over $\Omega$.

According to the definition above, we can say that the pressure subspace $Q_h \circledast \mathscr{C}$ can be spanned by the following set of orthogonal basis functions, with respect to the $L^2$ inner product.

$$\bigcup_{H \in \chi_h} \{\eta^H_{ij}\}^{1\le j\le 4}_{1\le i\le 3},$$

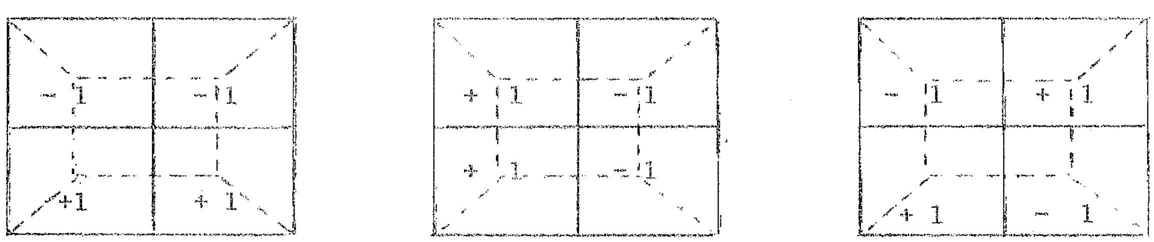
where $\eta^H_{ij}(x)$ equals zero if $x \notin H$ $\forall i,j$, and whose values over H are defined according to the self-explanatory figures below (refer also to Figure 1)

For j=1




i = 1     i = 2     i = 3

For i=1




j = 2     j = 3     j = 4

For $2\le i\le 3$ and $2\le j\le 4$ we set:

$$\eta^H_{ij} = \eta^H_{i1}\,\eta^H_{1j}\ .$$

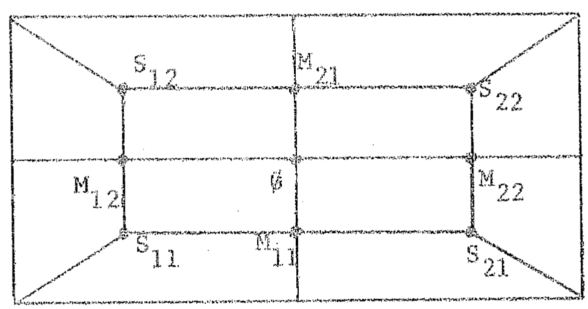
Let now $q_h \in Q_h$, $H \in X_h$, and $\xi_{ij}$ be the component of $q_h$ with respect to the basis function $\eta^H_{ij}$, that is to say:

$$q_h = \sum_{H \in X_h} \sum_{i=1}^{3} \sum_{j=1}^{4} \xi_{ij} \eta^H_{ij}$$

We further set:

$$(2.2) \qquad q_{ij} = \sum_{H \in X_h} \xi_{ij} \eta^H_{ij}, \qquad 1 \le i \le 3, \quad 1 \le j \le 4.$$

Next we refer to Figure 2 for the definition of points $\emptyset$, $S_{\ell k}$ and $M_{\ell k}$, $1 \le k, \ell \le 2$.




Inner nodes associated with
macroelements H
Figure 2.

Now we define fields $w_k(q_k)$ belonging to $V_h x V_h$ in terms of $q_h$, $k=2, 3, \ldots, 8$, in the following way: For every $H \in X_h$ set

$$\begin{cases} w_2(P) = w_3(P) = 0, \text{ where } P \text{ is any node of } H \text{ different of } \emptyset \\ w_2(\emptyset) = 2h_2 \, \xi_{12} \, \nu_2 \\ w_3(\emptyset) = 2h_1 \, \xi_{13} \, \nu_1 \end{cases}$$

$$\begin{cases} w_4(P) = 0, \ P \text{ being any node of } H \text{ different of } M_{ij}, 1 \le i, j \le 2 \\ w_4(M_{ij}) = (-1)^{i+1} h_j \xi_{14} \nu_j / 2 \end{cases}$$

$$\begin{cases} w_k(P) = 0, \ P \text{ being any node of } H \text{ different of } S_{ij}, \ 1 \le i, j \le 2 \\ w_k(S_{ij}) = \sqrt{3}\left(\xi_{2,k-4} \dfrac{d_{ij}}{6} + \xi_{3,k-4} \dfrac{d'_{ij}}{4}\right) \eta^H_{1,k-4}(S_{ij}), \quad k=5,6,7,8, \end{cases}$$

where $d_{ij}$ is the oriented diagonal of $H_{ij}$, defined by $2\,\vec{\partial} S_{ij}$, and $d'_{ij}$ is the other diagonal of $H_{ij}$, oriented in such a way that it corresponds to a rotation of $d_{ij}$ centered at $S_{ij}$, of an angle less than $\pi$, in the counter-clockwise sense.

Recalling (2.2), it is then straighforward to verify that (cf.[10]):

(2.3) $(\operatorname{div} w_k, q_{2,j-4}+q_{3,j-4})_{0,H} = \delta_{kj}(\| q_{2,j-4}\|^2_{0,H} + \| q_{3,j-4}\|^2_{0,H})$ $5\le k,j\le 8$,

(2.4) $(\operatorname{div} w_k, q_{1j})_{0,H} = \delta_{kj} \| q_{1j}\|^2_{0,H}$ $2\le k,j\le 4$, and

(2.5) $(\operatorname{div} w_k, q_{1j})_{0,H} = 0$, $5\le k\le 8$, $2\le j\le 4$,

where $(.,.)_{0,H}$ denotes the standard inner product of $L^2(H)$ and $\|\cdot\|_{0,H}$ the corresponding norm.

Moreover, as one can easily check, we have:

$$(\operatorname{div}(w_2+w_3+w_4), \sum_{i=2}^{3}\sum_{j=1}^{4} q_{ij})_{0,H} = -\sqrt{3}(\xi_{12}\,\xi_{32}+\xi_{13}\,\xi_{33}-\frac{2}{3}\xi_{14}\,\xi_{34})h_1\,h_2$$

which implies:

(2.6) $$(\operatorname{div}(w_2+w_3+w_4), \sum_{i=j}^{3}\sum_{j=1}^{4} q_{ij})_{0,H} \ge -\frac{\sqrt{3}}{2}[\sum_{i=1}^{2}\sum_{j=1}^{4}\| q_{ij}\|^2_{0,H} - \| q_{11}\|^2_{0,H}$$

Let us now set: $v_h^1 = \sum_{k=2}^{8} w_k$, and define $q_h^1$ to be the pressure of $Q_h$ such that $q_h^1 = \sum_{i=j}^{3}\sum_{j=1}^{4} q_{ij} - q_{11}$.

In so doing, the relations (2.3)∿(2.6) allow us to conclude immediately that:

(2.7) $(\operatorname{div} v_h^1, q_h^1)_0 \ge \beta_1 \| q_h^1\|^2_0$, with $\beta_1 = \frac{2-\sqrt{3}}{2}$.

On the other hand, one can easily prove the existence of a constant $C_1$ independent of h such that:

(2.8) $|v_h^1|_1 \le C_1 \| q_h^1 \|_0$.

Let now $Q_h^o$ be the space of such pressures of $Q_h$, that are constant over each rectangle of $\chi_h$, and $Q_h^1$ be such that $Q_h = Q_h^o \oplus Q_h^1$.

We then have:

Lemma 1: If $Q_h$ is replaced by $Q_h^1$, the LBB-condition (1.1) holds with $\beta$ independent of h, namely, with $\beta=\beta_1/C_1$.

Proof: It is an immediate consequence of (2.7) and (2.8). □

Now, combining the results of [11], Lemma 2.5, and [4], Lemma 3.2, one can prove the following:

Lemma 2: There exists constants $\beta_o$ and $C_o$ independent of h, with $\beta_o > 0$, such that:

$$\forall q_h \in Q_h^0 \qquad \exists\, v_h^0 \in V_h \times V_h \qquad \text{such that}$$

(2.9) $$(\operatorname{div} v_h^0 , q_h)_0 \geq \beta_0 \, \| q_h \|_0^2 \quad \text{and}$$

(2.10) $$| v_h^0 |_1 \leq C_0 \, \| q_h \|_0 . \qquad \square$$

Lemmas 1 and 2 provide all the necessary tools for proving our main result, namely:

Lemma 3: The LBB-condition (1.1) holds for the spaces $V_h$ and $Q_h$ defined above, with $\beta$ independent of h.

Proof: Given $q_h \in Q_h$ we write $q_h = q_h^o + q_h^1$ with $q_h^\ell \in Q_h^\ell$, $\ell=0,1$. The field $v_h$ defined by

$$v_h = v_h^0 + \theta \; v_h^1 \qquad \text{with} \qquad \theta = \frac{5}{2} \; \frac{C_0^2}{\beta_0 \beta_1}$$

satisfies (1.1) with $$\beta = \frac{1}{2\sqrt{2}\beta_0} \; \frac{\min(\beta_0^2, C_0^2)}{\max(C_0, \theta C_1)} ,$$

which is a consequence of (2.7) ∿ (2.10) and of the same arguments encountered in [12], Section 4 □

Now using Lemma 3 and standard approximation results for finite element subspaces we conclude with:

Theorema 1: Let $\Omega$ be a rectangle partitioned into rectangular macroelements of maximal edge length $h$, which consist of quadrilaterals forming the tile-grid of Figure 1. Then for a regular family of such partition in the sense (2.1), the error estimate (1.2) holds with $k=2$. □

## 3. APPLICATION TO LINEAR INCOMPRESSIBLE ELASTICITY

We would like to conclude by giving a short account on the application of the finite element method considered in this paper, to the following infinitesimal elasticity problem:

Let $\Omega$ be an elastic body whose boundary $\Gamma$ has a fixed portion $\Gamma_o$, $\Gamma_o \neq \Gamma$. Letting also

$$V = \{v/\ v \in H^1(\Omega),\ v=0 \text{ on } \Gamma_o\},$$

we wish to find a displacement vector field $\mathbf{u} \in V \times V$ and a pressure $p \in L^2(\Omega)$ such that:

$$(P')\quad \begin{cases} 2\mu\,(E(\mathbf{u}), E(\mathbf{v}))_0 + (p, \operatorname{div} \mathbf{v})_0 = (\mathbf{f}, \mathbf{v})_0 + (\mathbf{g}, \mathbf{v})_{0,\Gamma_1} & \forall\, \mathbf{v} \in V \times V \\ (q, \operatorname{div} \mathbf{u})_0 = 0 & \forall\, q \in L^2(\Omega), \end{cases}$$

where $\mathbf{f} \in L^2(\Omega) \times L^2(\Omega)$, $\mathbf{g} \in L^2(\Gamma_1) \times L^2(\Gamma_1)$, with $\overline{\Gamma}_1 x\, \overline{\Gamma}_o = \Gamma$,

$$E = \{\varepsilon_{ij}\}, \quad \varepsilon_{ij}(\mathbf{u}) = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right), \quad 1 \le i, j \le 2,$$

and $\mu$ is the second Lamé's coefficient

If $\Omega$ is a rectangle we can use the tile-grid and define the corresponding approximate problem (with some care so as to make the intersection of $\overline{\Gamma}_o$ and $\overline{\Gamma}_1$ coincide with suitable vertices of quadrilaterals of $\mathcal{C}_h$), namely:

$$(P'_h)\quad \begin{cases} \text{Find } (u_h, p_h) \in (V'_h \times V'_h) \times Q'_h \quad \text{such that} \\ 2\mu\,(E(u_h), E(v_h))_0 + (p_h, \operatorname{div} v_h)_0 = (\mathbf{f}, v_h)_0 + (\mathbf{g}, v_h)_{0,\Gamma_1} \quad \forall v \in V'_h \times V'_h \\ (q, \operatorname{div} u_h)_0 = 0 \qquad \forall q_h \in Q'_h, \end{cases}$$

where $V_h'$ and $Q_h'$ are constructed as prescribed in Section 2.1 for $V_h$ and $Q_h$, the only difference between them being the fact that $V_h'$ and $Q_h'$ are now subspaces of $V$ and $L^2(\Omega)$ respectively, instead of $H_0^1(\Omega)$, and $L_0^2(\Omega)$.

In so doing, the result of Lemma 1 remains unchanged for $V_h'$ and $Q_h'$. However, as for Lemma 2, like in [12], in general one can only prove the validity of (2.9) with $\beta_o = O(h^{2-s})$, where $s$ stands for the maximum index of the Sobolev space $H^s(\Omega)$, in which lies every solution $z$ to the following mixed Dirichlet-Neumann problem (see e.g. [12]).

$$\begin{cases} -\Delta z = f & \text{in } \Omega \\ z = 0 & \text{on } \Gamma_0 \\ \dfrac{\partial z}{\partial n} = 0 & \text{on } \Gamma_1 \end{cases}$$

$f$ being an arbitrary function of $L^2(\Omega)$.

Notice that $1.5-\varepsilon \le s \le 2$, where $\varepsilon$ is an arbitrary small positive real number. In this way Lemma 3 also holds, but now we have $\beta = O(h^{2-s})$ which implies the validity of the following error estimates for $(P')-(P_h')$: If $u \in H^d(\Omega) \times H^d(\Omega)$ and $p \in H^{d-1}(\Omega)$, $1<d\le 2$ we have:

$$(3.1) \quad |u-u_h|_1 \le C\, [h^{s+d-3} \, \|u\|_d + h^{d-1} \, \|p\|_{d-1}]$$

$$(3.2) \quad \|p-p_h\|_0 \le C\, [h^{2s+d-5} \, \|u\|_d + h^{s+d-3} \, \|p\|_{d-1}]$$

For more details on this case we refer to [12]. Let us here just mention that in some important particular cases one can prove the validity of the LBB-condition with $\beta_h$ independent of $h$ even with $\Gamma_o \ne \Gamma$ by a direct method, as follows:

First we note that, from Lemmas 1 and 2, like in Lemma 3, we can easily conclude that $\forall q_h \in Q_h$ $\exists v_h \in V_h' \times V_h'$ such that:

$$(\mathrm{div}\, v_h\,,\, q_h)_0 \geq \beta_0' \, \| q_h \|_0^2$$

$$|v_h|_1 \leq C_0' \, \| q_h \|_0 \qquad \text{with} \quad \beta_0' > 0.$$

Thus if we find $\beta_1' > 0$ such that for every constant pressure c we can find $v_h^* \in V_h' \times V_h'$ such that

$$(3.3) \qquad (\mathrm{div}\, v_h^*\,,\, c)_0 \geq \beta_1' \, \| c \|_0^2 \qquad \beta_1' \text{, independent of h.}$$

$$(3.4) \qquad |v_h^*|_1 \leq C_1' \, \| c \|_0 \, ,$$

then, using the same arguments as in Lemma 3, we can prove the validity of the discrete LBB-condition (1.1), in the case $Q_h$ is a subspace of $L^2(\Omega)$ instead of $L_0^2(\Omega)$, (where $V_h'$ is replaces $V_h$).

Actually, in some cases such vector field $v_h^*$ exists and can be constructed in a straighforward way. Among these, we mention the case where the intersection of $\Gamma_o$ and $\Gamma_1$ is a set of two points not being vertices of $\Omega$, with one coincident coordinate, say $x_i = b$, $i=1$ or 2.

Indeed, in this case the segment whose extremities are these two points subdivides $\Omega$ into two subdomains, say $\Omega_0$ and $\Omega_1$, with areas respectively equal to $S_0$ and $S_1$. Let $\Gamma_i$ be a part of the boundary of $\Omega_i$, $i=0,1$.

Without loss of generality, assume that $\forall x \in \Gamma_1$ we have $x_i \geq b$.

Now we define:

$$v_h^* = c \; \frac{\mu_i (x_i - b)}{S_1} \, (S_0 + S_1) \qquad \text{if} \quad x_i \geq b$$

$$v_h^* = 0 \qquad \text{if} \quad x_i < b.$$

We then have $v_h^* \in V_h'$ and that (3.3) and (3.4) hold with

$$\beta_1' = 1 \quad \text{and} \quad C_1' = \sqrt{1 + S_o / S_1},$$

which gives optimal convergence results in (3.1) and (3.2), namely $s=2$, assuming that $u \in H^2(\Omega) \times H^2(\Omega)$ and $p \in H^1(\Omega)$.

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