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A B S T R A C T

Some logical properties of namable models for structured data (data structures and data types) are presented, mostly from a model-theoretic viewpoint. It is argued that "namability" captures the notions of accessibility and constructibility of structured data, the corresponding axiomatic theories serving as logical specifications for them. Related concepts are also discussed.

Key words: Namable models,
data structures,
data types,
axiomatic specification,
model theory,
Herbrand structures,
elementary classes,
initiality

RESUMO

Algumas propriedades lógicas de modelos nomeáveis para dados estruturados (estruturas e tipos de dados) são apresentadas, com ênfase em aspectos da teoria de modelos.

Argumenta-se que "nomeabilidade" capta as noções de acessibilidade e construbilidade de dados estruturados, as teorias axiomáticas correspondentes servindo como especificações lógicas para estes. Alguns conceitos relacionados também são discutidos.

Palavras chaves: Modelos nomeáveis,
estruturas de dados,
tipos de dados,
especificação axiomática,
teoria de modelos,
estruturas de Herbrand,
classes elementares,
inicialidade

C O N T E N T S

1. Introduction

The notions of ω-model, ω-consistency, ω-completeness arise quite naturally in the study of first-order theories of the natural numbers. Indeed, no matter how we try to describe them we are always confronted with (denumerable) non-standard models. Among the possible ways to rule out these unintended models there is a simple-minded one: every element of the intuitive intended model has a name for it in the language.
The consideration of namable models is also related to several important model theoretic techniques: diagrams, etc.

Namable models also arise in connection with computer programming, namely in the specification of structured data.
For a simple example consider

$$c \xrightarrow{f} a_1 \xrightarrow{f} \dots \xrightarrow{f} a_{n-1} \xrightarrow{f} d \circlearrowright f$$

Imagine that information is stored at each point a_i of this structure, the entry point of which is c. To gain acess to the information at a particular point one has to traverse the structure, starting at c and following the f-links. If c is denoted by a constant symbol then the accessible points are those that have a name f ... f(c).
Such structures of arbitrary length, possibly together with the infinite "limit"

$$c \xrightarrow{f} a_1 \xrightarrow{f} \dots \xrightarrow{f} a_n \xrightarrow{f} \dots \; d \circlearrowright f$$

form the class of singly-linked linear lists.

The basic problems here are, as in logic,

(i) give a formal description of the properties of such structures;

(ii) describe the class of structures given by a formal specification.

The main purpose of this paper is to present some properties of namable structures which seem to be useful for dealing with structured data. However, familiarity with programming is not essential, due to the manner it is organized.

Sections 2 to 7 present some general properties of namable structures, with a special case in section 8. The exposition is programming-independent although some issues are programming-motivated (especially in 4, 6, 7, 8). Section 9 elaborates on the relations to programming.

2. Namable Structures

Consider a first-order language L with sets C, F and R of constant, function and relation symbols, respectively. Let T denote the set of all variable-free terms of L. We always assume C non-empty, so that $T \neq \emptyset$.
Now consider a structure $\mathcal{A} = \langle A, C^{\mathcal{A}}, F^{\mathcal{A}}, R^{\mathcal{A}} \rangle$ for L. Each $t \varepsilon T$ denotes an element $t^{\mathcal{A}}$ in the domain, which defines a function $d : T \to A$ with image $T^{\mathcal{A}}$.
We shall call $\mathcal{A}$ namable (by T) iff $T^{\mathcal{A}} = A$, i.e. the denotation d is surjective. So, a T-structure is one in which every elements is denoted by a variable-free term, which is a name for it.

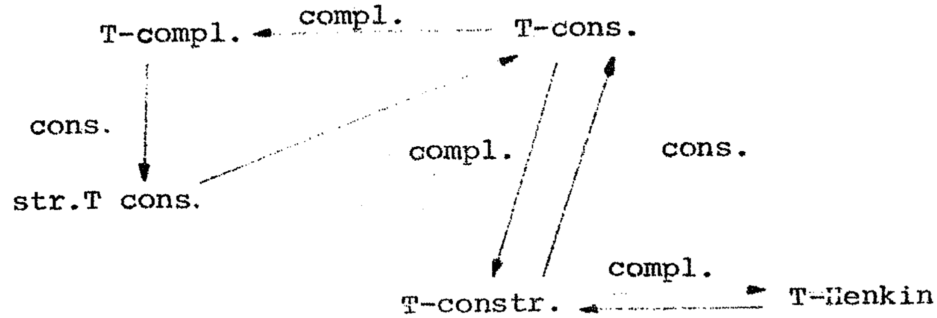
One advantage of considering T-structures is that assigning values to variables can be done within the language. (cf. |Shoenfield 67, p. 18)|.
In particular, for $\psi(v) \varepsilon L(v)$ (i.e. v is the only variable occurring free in $\psi(v)$)
$\mathcal{A} \models \exists v\ \psi(v)$ iff $\mathcal{A} \models \psi(t)$ for some $t \varepsilon T$
$\mathcal{A} \models \forall v\ \psi(v)$ iff $\mathcal{A} \models \{\psi(t)/t \varepsilon T\}$
In other words, for T-structures, the quantifiers can be replaced by infinitary connectives.
Another way of expressing namability (suggested by J. Malitz) is: a T-structure is one wich omits the one-type $\{\neg v = t/t \varepsilon T\}$.

3. Namable theories

Consider a theory Σ. When does Σ deserve to be called "namable"? Several answers are available, depending on what is meant.

A reasonable choice (suggested by the above case, of the theory of a T-structure is to require that $\forall v\psi(v) \in \Sigma$ whenever $\{\psi(t)/t\epsilon T\} \subseteq \Sigma$, for every $\psi(v) \in L(v)$. Such a theory Σ is called T-complete |Grzegorczyk 74, p. 301|.
On the other hand, call Σ T-satisfiable iff Σ has a T-model. A necessary condition for Σ to be T-satisfiable is that for every $\psi(v) \in L(v)$ with $\exists v\psi(v) \in \Sigma$ for some $t\epsilon T$ $\neg\psi(t) \notin \Sigma$. Such a Σ is called T-consistent. A variation is the case of T-constructive: for every $\psi(v) \in L(v)$ with $\exists v\psi(v) \in \Sigma$ for some $t\epsilon T$ $\psi(t) \in \Sigma$. |Grzegorczyk 74, p. 309|.
Incidentally, a sufficient condition for a countable consistent theory Σ to be T-satisfiable is T-completeness. This follows from Ehrenfeucht's omitting-types theorem (cf.e.g.| Shoenfield 67, p. 90, Chang - Keisler 73, p. 79|).
Still another variation is suggested by the "method of constants as witnesses". Call Σ T-Henkin iff for every $\psi(v) \in L(v)$ there exists $t\epsilon T$ such that $(\exists v\psi(v) \rightarrow \psi(t)) \in \Sigma$.
Finally, a condition, which is equivalent to T-satisfiability, is strong T-consistency |Henkin 64, p. 194|: Σ is strongly T-consistent iff for every finite $\{\psi_1(v), \ldots, \psi_n(v)\} \subseteq L(v)$ there exist $t_1, \ldots, t_n \epsilon T$ such that $(\exists v\psi_1(v) \wedge \neg\psi_1(t_1) \vee \ldots \vee \exists v\psi_n(v) \wedge \neg\psi_n(t_n)) \notin \Sigma$.

These notions coincide for complete consistent theories, but not in general, as displayed in the following diagram



The notion of T-completeness is related to a generalization of Carnap's ω-rule

$$\text{rule } T : \frac{\{\psi(t)/t\varepsilon T\}}{\forall v\psi(v)}$$

If this rule is added to the usual deductive calculus for first-order theories a complete one for countable languages is obtained. Thus the syntactical and semantical T-consequences of a set Γ of sentences coincide (cf. |da Costa 74, p. 445| and references therein).

4. Relations between T-structures

We shall now examine the familiar notions of substructure and homomorphism as they apply to T-structures. The extra constraints imposed by namability suggests some variations, which relate the sentences preserved.

Let $\mathcal{A}$ and $\mathcal{B}$ be structures for L.
First, consider the case $A \subseteq B$.
One says that $\mathcal{A}$ is a substructure of $\mathcal{B}$ iff the interpretations of the nonlogical symbols in $\mathcal{A}$ are the restrictions to A of the corresponding ones in $\mathcal{B}$. In this case we shall call $\mathcal{A}$ a strong substructure of $\mathcal{B}$ ($\mathcal{A} \subsetneq \mathcal{B}$). This notion is not very interesting for T-structures because if $\mathcal{B}$ is namable then $\mathcal{A} \subsetneq \mathcal{B}$ iff $\mathcal{A} = \mathcal{B}$.
We shall frequently use the weaker notion of weak substructure ($\mathcal{A} \sqsubseteq \mathcal{B}$) where for relation symbols $r \in R$ we require only $r^{\mathcal{A}} \subseteq r^{\mathcal{B}}$.
Call a function $h : A \to B$ a weak (resp. strong) homomorphism of $\mathcal{A}$ to $\mathcal{B}$ iff h is a homomorphism of the corresponding algebras (i.e. preserves C and F) and for every $r \in R$ $r^{\mathcal{A}} \subseteq h^{-1}\, r^{\mathcal{B}}$ (resp. $r^{\mathcal{A}} = h^{-1}\, r^{\mathcal{B}}$).

Of course, $\mathcal{A} \subsetneq \mathcal{B}$ (resp. $\mathcal{A} \sqsubseteq \mathcal{B}$) iff $A \subsetneq B$ and the natural inclusion $i : A \to B$ is a strong (resp. weak) homomorphism of $\mathcal{A}$ to $\mathcal{B}$.
The notion of weak homomorphism can be dispensed with. For $h : A \to B$ is a weak homomorphism of $\mathcal{A}$ to $\mathcal{B}$ iff there exists a weak extension $\mathcal{C} \sqsupseteq \mathcal{A}$ (with $C = A$) and a strong homomorphism $g : \mathcal{C} \to \mathcal{B}$ such that $h = g.i$, where i is the inclusion of $\mathcal{A}$ into $\mathcal{C}$.

Notice that a (weak or strong) homomorphism $h : \mathcal{A} \to \mathcal{B}$ must commute with the respective denotations $d^{\mathcal{A}} \cdot h = d^{\mathcal{B}}$. This places some restrictions on h.
If $\mathcal{A}$ is namable then h is unique. On the other hand, if $\mathcal{B}$ is namable then h is onto.
Thus, if $\mathcal{A}$ is a T-structure then so is $h(\mathcal{A})$. Conversely, if $h(\mathcal{A})$ is a T-structure then the strong substructure of $\mathcal{A}$ with

domain T^{A} has the same image under h.

In order to describe the formulas preserved by homomorphism it is useful to employ the concept of Γ-morphism from $\mathcal{A}$ to $\mathcal{B}$, namely a function $h : A \to B$ such that $\mathcal{B} \models \psi|h.\alpha|$ whenever $\mathcal{A} \models \psi|\alpha|$ for all $\alpha : V \to A$ and $\psi \varepsilon \Gamma$ |Shoenfield 67, p. 93, Lucas 76, p. 5|.
Thus, if $\mathcal{A} \sqsubseteq \mathcal{B}$ (resp. $\mathcal{A} \subseteq \mathcal{B}$) then the inclusion $i : A \to B$ is a Σ_1^{+} (resp. $\Sigma_1^{\neq}$)-morphism.
A strong (resp. weak) homomorphism $h : \mathcal{A} \to \mathcal{B}$ is a $\Sigma_1^{\neq}$ (resp. Σ_1^{+})-morphism; if h is onto then it is a $L^{\neq}$ (resp. L^{+})-morphism.
The superscript $\neq$ indicates "no negative occurrence of the equality sign with absolute interpretation (as identity)".

5. Axiomatizable T-classes

Consider a class $\mathbb{K}$ of T-structures, which is to be described, if possible, by their first-order properties.
Except in trivial cases, $\mathbb{K}$ cannot be elementary, in view of the compactness or the Löwenhein-Skolem-Tarski theorems.
So, we seek to axiomatize $\mathbb{K}$ by Σ in the sense that $\mathbb{K}$ consists exactly of the T-models of Σ, i.e. $\mathbb{K} = \text{Tmod}(\Sigma)$. Thus, $\mathbb{K}$ is T-axiomatizable iff $\mathbb{K} = \text{Tmod}(\text{Th}(\mathbb{K}))$.

In order to T-axiomatize $\mathbb{K}$ one may use the flexibility of arbitrary first-order sentences but this is not necessary.
Let $\mathbb{K}$ be the class of all T-models of Σ, and call $\mathbb{K}'$ the closure of $\text{Mod}(\Sigma)$ under extensions. Then $\mathbb{K}' = \text{Mod}(\Delta)$ for some set Δ of existential sentences (cf. e.g.|Chang-Keisler 73, p.139|).
But any T-model of Δ contains a model of Σ, so $\mathbb{K} = \text{Tmod}(\Delta)$.
Thus, existential axioms are enough.

6. N-structures

There is a generalization of T-structures worth mentioning, namely N-structures, where N is a non-empty subset of T.
An N-structure is one with the restriction of the denotation d to N onto.
These arise, for instance, in the case of the integers with zero, successor and predecessor.

Many concepts and results seen for T-structures generalize to N-structures by simple replacement of T by N. For instance, the N-satisfiable theories are the strongly N-consistent ones.

Of course, if $N \subseteq M$ any N-structure is an M-structure, in particular a T-structure.

An interesting case is that of N consisting of all terms involving only symbols from subsets E(entry points) of C and G (access operations) of F. When such a simple "canonical form" can be identified the task of writing down axioms that specify the intended structures can more confidently be carried out (cf. |Pequeno - Veloso 78|).

7. Herbrand structures

The set T of variable-free terms can be given a natural algebraic structure, making it into the algebra $\mathcal{T}$, freely generated by the constants |Grätzer 68, p. 162|.

Given a structure A for L the denotation d : T → A is the unique homomorphism of $\mathcal{T}$ into the algebra of A, which will be onto iff A is namable. Now, there is a natural way to interpret the relation symbols on T so that d becomes a strong homomorphism, namely $d^{-1}\, r^{A}$.
Call this structure $\mathcal{T}(A)$ the Herbrand structure induced by A, and notice that they are $L^{\neq}$-equivalent, if A is namable.

Thus, any T-structure is, up to a special quotient, a Herbrand structure. And Herbrand structures present the attractiveness of being specified by their positive diagrams |Chang-Keisler 73, p. 70|.

Also, Herbrand structures are a natural setting for definitions by recursion on the structure of the terms. So, we can have analogs of primitive recursion or more general recursions based on the relation of subterm, which is a well-founded partial order on T.

8. Initiality

Consider a clas $\mathbb{K}$ of T-structures. Assume that $\mathbb{K}$ can be T-axiomatized by Σ, so that $\mathbb{K} = \mathrm{Tmod}(\Sigma)$.

In some cases, we may be fortunate enough to have for each $A \varepsilon \mathbb{K}$ a simple set of "particularization axioms" Γ_A so that $\Sigma \cup \Gamma_A$ is a complete description of A, up to isomorphism.
In any case, each $A \varepsilon \mathbb{K}$ induces a structure $\mathcal{T}(A)$ on the domain T. It would be nice to have on this domain a structure having the properties shared by all A in $\mathbb{K}$. This would play a role similar to that of a free algebra.

To this end, make the algebra $\mathcal{T}$ into a T-structure $\mathcal{T}(\mathbb{K})$, where for each $r \varepsilon R$, $r^{\mathcal{T}(\mathbb{K})} = \bigcap \{ r^{\mathcal{T}(A)} / A \varepsilon \mathbb{K} \}$ (equivalently $\bar{t} \varepsilon r^{\mathcal{T}(\mathbb{K})}$, iff $r\bar{t}$ is a T-consequence of Σ).
Call this the natural structure of $\mathbb{K}$.

Then, for every $A \varepsilon \mathbb{K}$ there exists a unique weak homomorphism d of $\mathcal{T}(\mathbb{K})$ onto A (hence an L^+-morphism).

Unfortunately $\mathcal{T}(\mathbb{K})$ may fail to satisfy Σ. Assume L has two distinct constant terms c and t and a unary predicate symbol p. For Σ given by the axiom $p(c) \vee p(t)$, we get $p^{\mathcal{T}(\mathbb{K})} = \emptyset$. The same would happen were $\exists v\ p(v)$ the axiom. In either case $\mathcal{T}(\mathbb{K})$ is not in $\mathbb{K}$.
The basic reason for the above situation is that Σ does not specify well enough its atomic consequences.
Let us see when this happens.

Clearly, $\mathcal{T}(\mathbb{K}) \ \varepsilon\ \mathbb{K}$ iff $\bigcap \{ \mathcal{T}(A) / A \varepsilon \mathbb{K} \} \ \varepsilon\ \mathbb{K}$.
So, one can see that this happens iff the set of quantifier-free T- consequences of Σ is equivalent to a set of conditional sentences |Chang-Keisler 73, p. 14, 15|.

In to order to get a more syntactical condition, notice that Σ may be taken to be closed under T-consequence.

Let $\sigma\varepsilon\Sigma$, which may be assumed to be $\exists v\psi(v)$ in prenex normal form. If Σ is T-constructive then we have $\psi(t)\varepsilon\Sigma$ for some $t\varepsilon T$. Continuing we arrive at a quantifier-free instance of the matrix, which may be assumed in conjunctive normal form. So, it suffices to examine a disjunction of a positive sentence with a negative one, which reduces to the former in view of the L^+-morphism property. Thus, consider $\alpha = \alpha_1 v \ldots v\alpha_n$ with atomic α_j's. Then, $T(K) \models \alpha$ iff for some j $\alpha_j\varepsilon\Sigma$.

Hence, for a T-constructive $\Sigma = Th(K)$, $T(K)\varepsilon K$ iff Σ decides atomic disjunctions (i.e. we have some $\alpha_j\varepsilon\Sigma$ whenever $\alpha_1 v \ldots$ $\ldots v\alpha_n\varepsilon\Sigma$ with atomic α_i's).

9. Namable structures in programs

One way to specify a computer program P is to give input assertions $\phi(x)$, which give properties assumed about the input data, and output assertions $\psi(x, z)$, which state the required relations between input and output data.

To guarantee that P works as specified, one shows that for all input data satisfying $\phi(x)$: (i) P halts (termination); (ii) whenever P does halt then $\psi(x, z)$ is satisfied (partial correctness).
The latter proof reduces to proving $\phi'(x) \rightarrow \psi'(x, z)$ (transformed assections) within the theory of the data.
For instance, a program to compute the gcd of two integers is proven partially correct by using some properties of the integers (cf. e.g. |Manna 74, p.161; Lucchesi et al. 79, p.28|).
When the program manipulates structured data their properties must be formally specified for a proof to be possible.

The importance of T-structures in this context stems from the fact that both data structures and data types can be regarded as namable structures which can be conveniently T-axiomatized.
For a data structure a term indicates the sequence of access operations to be performed starting from the entry points. For a data type a term represents the sequence of transformation to be executed on the basic objects.
Furthermore, even though such structures are ultimately realized by simpler ones, it is conceptually convenient - both for program development and proving - to have some realization independent description for them (cf. abstract groups vs. groups of permutations, linear transformations vs. matrices). In this connection, the initial structure of section 8 is especially helpful.

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