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by

Paulo A. S. Veloso

Departamento de Informática

PONTIFÍCIA UNIVERSIDADE CATÓLICA DO RIO DE JANEIRO
RUA MARQUÊS DE SÃO VICENTE, 225 – CEP-22453
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C O N T E N T S

ABSTRACT

This paper shows that many properties of linear time-invariant systems are due to the underlying group structure even if not commutative. Within a general time-systems framework some properties of additive time-varying systems are examined: attainability, observability, reachability, controllability, realizability and state-space construction. These are regarded from the viewpoint of mathematical foundations of general time-systems theory.

KEY WORDS

General systems theory, additivity, linearity, realization, state space.

RESUMO

Este trabalho mostra que muitas propriedades de sistemas lineares invariantes no tempo são devidas à estrutura de grupo subjacente mesmo no caso não comutativo. Dentro de uma teoria geral de sistemas são examinadas algumas propriedades de sistemas aditivos variantes com o tempo: atingibilidade, observabilidade, alcançabilidade, controlabilidade, realizabilidade bem como construção do espaço de estados. O ponto de vista é o de fundamentos matemáticos da teoria geral de sistemas.

PALAVRAS CHAVES

Teoria geral de sistemas, aditividade, linearidade, realizações, espaço de estados.

1. INTRODUCTION

In Windeknecht's general time-systems theory(1) a system is defined as a relation between input and output signals(2). Another distinct concept is that of realization, which involves internal states(3). The problems of analysis and synthesis can be described as that of correlating a system with a realization with the same input-output behavior. Thus non-anticipatory systems represent the behavior of deterministic realizations.

This paper employs this framework extended to time-varying systems in order to investigate the role played by linearity. The aim is showing that several properties usually attributed to linearity are actually due to a group structure, which does not have to be commutative(4).

This section ends with some notation for time functions. Then in section 2 we consider the notion of register as a factorization via a state space of an input-output transformation (its behavior). By means of restriction to the attainable subgroup and quotient by the (normal) unobservable subgroup, a reduction is obtained which is minimal among registers with the same additive behavior.

Here incorporation of state structure for distinct instants is made in three steps. Section 3 considers the internal transitions together with the reachable and controllable subgroups, which can be closely related. Then section 4 adds an output map to obtain a generalization of a linear sequential machine, which can be made completely observable. Section 5 merges registers with transitions to give realizations. Each one of the above concepts presents the property of response separation:

zero-input response + zero-state response

Section 6 deals with systems as relations and characterizes those with a behavior implementable by a register or by a realization, by constructing state spaces for them. We conclude with some comments on the results from the perspective of foundations of general time-systems theory.

In order to deal with discrete and continuous time simultaneously, we consider a fixed <u>time domain</u> $\mathcal{T} \neq \emptyset$ ordered by $\leqslant$. We shall also fix a non-empty subset J of T. Intuitively J consists of the times when the output is observed (not excluding the case $J=T$).

Given a function w with domain $L \subseteq T$ and instants $i \leq j$ in T we shall denote by w^j, w_i and w_i^j the restrictions of w to, respectively, $\{t \in L / t<j\}$, $\{t \in L / t \geq i\}$ and $\{t \in L / i \leq t < j\}$. This notation is extended naturally to sets of such functions.

2. REGISTERS

The idea of state consists in storing information about past behavior so that future outputs can be determined. If the output is to be observed form instant t onwards, the past input u^t can be regarded as used to set the present state. Notice that T is not assumed to have minimal elements.

An <u>additive register</u> (at time t) is a 7-tuple
$R = \langle I, Z, U, Y, Q(t), m^t, n_t \rangle$, where

- I and Z are groups (input and output alphabets);
- U is a subgroup of I^T (input space);
- Y is a subgroup of Z^T (output space);
- $Q(t)$ is a group (state space at time t);
- $m^t : U^T \to Q(t)$ is a homomorphism (memory map);
- $n^t : Q(t) \times U_t \to Y_t$ is a homomorphism (output map).

In order to simplify the treatment, we shall also assume the following closure property:

For all $t \in J$, and $u, v \in U$

$$u^t v \in U \tag{2.1}$$

where $u^t v = u^t \cup v_t$

An immediate consequence of (2.1) and addivity is the following special commutativity

$$u^t 0 + 0^t v = u^t v = 0^t v + u^t 0 \tag{2.2}$$

(Notice that our groups are <u>not</u> assumed to be commutative.)

By means of (2.2), one can easily get an equivalent definition of additive register, via the representation

$$n_t(q, u_t) = c_t q + d_t u_t \tag{2.3}$$

where $c_t : Q(t) \to Y_t$ and $d_t : U_t \to Y_t$ are homomorphisms.

A register may have redundant states in two ways. First consider the set of <u>attainable</u> states, defined by

$$Atn(t) = \{ q \in Q(t) \ / \ \exists v \in U \ q = m^t(u^t) \}.$$

Clearly, $Atn(t) = \mathrm{Im}\ m^t$ is a subgroup of $Q(t)$, and we may replace $Q(t)$ by $Atn(t)$.

Now, call q t-equivalent to q' iff for all $u \in U$ $n_t(q, u_t) =$

$= n_t(q', u_t)$. It is easy to see, using (2.3), that this is equivalent to $c_t q = c_t q'$.

Thus if we call $K(t)$ the set of all states t-equivalent to the zero state, we have

Lemma - $K(t) = Ker\ c_t$ is a normal subgroup of $Q(t)$.
Moreover, q is t-equivalent to q' iff $(q-q') \in K(t)$.

If we consider the natural projection $p_t: Q(t) \to Q(t)/K(t)$ we have

Proposition - Given an additive register R there exists a unique additive register $\bar{R} = <I, Z, U, Y, \bar{Q}(t), \bar{m}^t, \bar{n}_t>$, such that $\bar{Q}(t) = Atn(t)/K(t)$, $\bar{c}_t . p_t = p_t . c_t$ and $\bar{d}_t . p_t = d_t$.

This register $\bar{R}$ is the reduction of R, all its states being t-observable (i.e. for $q \neq q'$ in $\bar{Q}(t)$, $\bar{c}_t q \neq \bar{c}_t q'$) and t-attainable.

Define the (input-output) behavior of register R to be the input-output transformation $U \to Y_t$ given by $n_t[m^t(u^t), u_t]$.

The reduction $\bar{R}$ of R can be regarded as a minimal factorization of the behavior of R in the following sense.

Theorem - Given any t-register $\mathring{R}$ with the same behavior as R, $\bar{Q}(t)$ is a homomorphic image of the subgroup $Atn(t)$ of $Q(t)$ under a homomorphism h such that $\bar{c}_t . h = h . \mathring{c}_t$ and $\bar{d}_t . h = \mathring{d}_t$, besides $h . \mathring{m}^t = \bar{m}^t$.

Proof. Define $h: \mathring{A}tn(t) \to \bar{Q}(t)$ by $h\ [\mathring{m}^t(u^t)] =$

$= m^t(u^t) + K(t)$. Notice that h is well-defined and has the stated properties. QED

3. TRANSITIONS

Consider a linear continuous time-varying system, the trajectory of wich is described by a differential equation as

$$\dot{x}(t) = F(t).\, x(t) + G(t).u(t)$$

where $x(t) \in R^n$, $u(t) \in R^m$ and $F(.)$ and $G(.)$ are well-behaved and of appropriate dimensions.

Then, with $\Phi(t',t)$ denoting the state transition matrix we have (cf. e.g. (5), p. 71).

$$x(t_1) = A(t_1,t_0)\; x(t_0) + B(t_0,t_1)\; u|[t_0,t_1) \qquad (3.1)$$

where $A(t_1,t_0) = \Phi(t_1,t_0)$

and $B(t_1,t_0)\; u|[t_0,t_1) = \int_{t_0}^{t_1} \Phi(t_1,t)\; G(t)\; u(t)\; dt$

So, the state at time t_1 is the sum of two (time-dependent) linear functions : the zero input response and the zero state response.

Notice that here the state space is fixed, instead of having time-dependent dimension $n(t)$.

An <u>additive transition structure</u> (on J) is a quadruple

$\mathcal{T} = \langle I, U, Q, M\rangle$, where

. I and U are as in an additive register;
. for each $j \in J$ $Q(j)$ is a group (state space at j) ;
. for every $i \le j$ in J $M(j,i)$: $Q(i) \times U_i^j \to Q(j)$
is a (group) homomorphism (denoted simply by M when no confusion is likely), such that for every $u \in U$

(i) if $i=j$ then $M(q,u_i^j) = q$

(ii) for $t \in J$ with $i \le t \le j$, we have

$$M(q,u_i^j) = M\,[M(q,u_i^t),\; u_t^j]\,.$$

Similarly to (2.3) one can easily get an equivalent definition of additive transition structure, via the representation

$$M(j,i)(q,u_i^j) = A(j,i)\; q + B(j,i)\; u_i^j \qquad (3.2)$$

where $A(j,i) : Q(i) \rightarrow Q(j)$ and $B(j,i): U_i^j \rightarrow Q(j)$ are homomorphisms satisfying the analogs of (i) and (ii)

$$A(i,i)q = q \;; \qquad B(i,i)\; u_i^i = 0 \tag{3.3}$$

$$A(j,i)\; q = A(j,t)\,.\,A(t,i)q \tag{3.4}$$

$$B(j,i)\; u_i^j = A(j,t)\cdot B(t,i)\; u_i^t + B(j,t)\; u_t^j \tag{3.5}$$

Discrete analogs of the above are the response formulas for linear sequential machines (c.f. (6), for instance).

Consider the set of states <u>reachable</u> at time j from state 0 at time $i \leq j$ defined by

$$Rch(j,i) = \{q \in Q(j) / \;\exists\; u \in U\; M(0,u_i^j) = q\}$$

Some simple properties of reachability are:

- $Rch(j,i) = ImB(j,i)$ is a subgroup of $Q(j)$;
- for $i \leq j \leq k$ in J, $Rch(k,j)$ is a subgroup os $Rch(k,i)$.

Thus, the set of states reachable at time j from state 0 at some prior instant, which is the union of $Rch(j,i)$ for all $i \leq j$ in J, is a subgroup of $Q(j)$, as well.

Another important concept is controllability. The set of states at time i <u>controllable</u> to 0 at time $j \geq i$ is given by

$$Ctr(j,i) = \{q \in Q(i) \;/\; \exists u \in U \;\; M(q,u_i^j) = 0\}$$

Because of (3.2), we have

$$q \in Ctr(j,i) \text{ iff } A(j,i)q \in Rch(j,i)$$

whence we get

<u>Lemma</u> - With the above notation

(a) $Ctr(j,i) = A(j,i)^{-1}[\mathrm{Im}\; B(j,i)]$ is subgroup of $Q(i)$.

(b) for $i \leq j \leq k$ in J $Ctr(i,j)$ is a subgroup of $Ctr(k,i)$.

Thus, the set of all states at time i eventually controllable to zero is a subgroup of $Q(i)$.

A nice property is the controllability of all reachable states. The next result generalizes a usual condition for this to hold for a finite-dimensional linear time-invariant system (7)

Proposition - Let $i \le t \le t' \le j \le k$ in J and assume $Rch(k,i) \subseteq Rch(k,j)$. Then $Rch(t',t) \subseteq Ctr(k,t')$; in particular $Rch(j,i) \subseteq Ctr(k,j)$

Proof Clearly $Rch(k,i) = Rch(k,j)$. If $q \in Rch(t',t)$ then $A(k,t')q \in Rch(k,t) = Rch(k,t')$, whence by (3.6) $q \in Ctr(k,t')$ QED

4. MACHINES

Given an additive i-register R_i, one can use an additive transition function $M(j,i) : Q(i) \times U_i^j \to Q(j)$ to obtain an additive j-register R_j by setting $m^j(u^j) = M[m^i(u^i), u_i^j]$ and trying to take $n_j[m^j(u^j), u_j]$ to be the restriction $(n^i[m^i(u^i), u_i])_j$

However, returning to our example in section 3, we notice that frequently $y(t)$ is specified by a (time-varying) linear combination of $x(t)$ and $u(t)$. So, we have

$$y(t_1) = C(t_1,t_0)\ x(t_0) + D(t_1,t_0)\ u|[t_0,t_1]) \tag{4.1}$$

where $C(t_1,t_0)$ and $D(t_1,t_0)$ have expressions similar to those for $A(t_1,t_0)$ and $B(t_1,t_0)$.

So, the output can be obtained pointwise, which suggests the next definition.

An <u>additive machine</u> (on J) is a 7-tuple $M = \langle I, Z, U, Y, Q, M, N\rangle$ where

- $\langle I, U, Q, M\rangle$ is an additive transition structure;
- Z and Y are as for an additive register;
- for every $t \in J$ $N(t): Q(t) \times U(t) \to Z$ is a homomorphism (also denoted simply by N when safe).

An equivalent definition is easily seen to be obtained by adding to an additive transition structure a map given by

$$N\ [q,u(t)] = C(t)q + D(t)\ u(t) \tag{4.2}$$

where $C(t): Q(t) \to Z$ and $D(t): U(t) \to Z$ are homomorphisms.

For each $i \in J$ we have an induced output map $n_i: Q(i) \times U_i \to Y_i$ given by, for $j \geq i$ in J

$$n_i(q,u_i)(j) = N(j)[M(q,u_i^j),\ u(j)] \tag{4.3}$$

with similar expressions for c_i and d_i. Thus

$K(i) = \bigcap_{j \leq i} Ker[C(j)\ A(j,i)]$

for $j \geq i$ $\quad A(j,i)\ K(i)\ \underline{C}\ k(j)$

So, we have

<u>Theorem</u> - Given M there exists a unique machine (on J)
$\langle I, Z, U, Y, \bar{Q}, \bar{M}, \bar{N} \rangle$ such that for $j \geq i$ in J

$$\bar{A}(j,i)\, p_i = p_j A(j,i) \qquad \bar{B}(j,i) = p_j B(j,i)$$
$$\bar{C}(j)\, p_j = C(j) \qquad \bar{D}(j,i) = D(j,i)$$

This machine is completely observable at every $i \in J$ (i.e. for $q \neq q'$ in $\bar{Q}(i)$ $\bar{C}_i q \neq \bar{C}_i q'$).

5. REALIZATIONS

The idea of using a register together with a transition structure is now considered with instantaneous output as in the next definition.

Given $t \in J$, an <u>additive realization</u> (at t, on J) is an 8-tuple $P = \langle I, Z, U, Y, Q, m^t, M, N \rangle$, where

- $\langle I, Z, U, Y, Q, M, N \rangle$ is an additive machine (on J);
- $\langle I, U, Q(t), m^t, n_t \rangle$, with n_t given by (4.3), is an additive t-register.

Notice that for each $i \geq t$, P gives an additive i-register by (4.3). Thus, P can be transformed into a $\bar{P}$, which is completely observable and attainable for every $i \geq t$ in J, and has the same behavior, which is by (4.3), (4.2) and (3.2), for $j \geq t$

$$C(j)A(j,t)m^t(u^t) + C(j)B(j,t)u_t^j + D(j)u(j)$$

6. SYSTEMS

An <u>additive system</u> is a 5-tuple
$S = \langle I, Z, U, Y, S \rangle$, where

- I, Z, U, Y are as for the additive register;
- S is an additive relation from U to Y, i.e. a subgroup of $U \times Y$, with U as the domain of S.

Notice that S is a subgroup of a subdirect product of U by Y, proper or not depending on Im S.

Given an instant $t \in T$ we define the <u>t-section</u> of S by $S_t = \{(u, y_t) / (u, y) \in S\}$ and call S <u>t-functional</u> iff S_t is a function, or equivalently, iff $S_t: U \to Y_t$ is a homomorphism

Now, the behavior of an additive t-register R gives the t-section $R_t(u) = n_t[m^t(u^t), u_t]$ of a t-functional additive system. We shall now extend Nerode's construction to obtain the converse, using the following response separation property, which follows from (2.2)

$$\text{for all } u \in U \quad S_t(u) = S_t(u^t 0) + S_t(0^t u) \tag{5.1}$$

<u>Theorem</u> - Any t-section of a t-functional additive system S is the behavior of an additive t-register R, which is minimal.

<u>Proof</u>- As $Ker\ S_t(\_^t 0)$ is a normal subgroup of U^t we can take $Q(t)$ to be the corresponding quotient and use the natural projection $h: U^t \to Q(t)$ to define m^t and n_t as in the minimality theorem of section 2. *QED*

Now consider for $i \in J$ the relation

$$S^i = \{((u^i, u(i)), y(i)) / (u, y) \in S\}$$

We shall call S <u>*t-J-causal*</u> iff for each $i \geq t$ in J, S^i is a function, or equivalently, S^i is a homomorphism from $U^i \times U(i)$ into Z.

We can now characterize which systems have a realization.

<u>Theorem</u> - A t-section of an additive system S is the behavior of an additive t-J-realization iff S is t-J-causal.

<u>Proof</u> - If S is t-J-causal then for each $i \geq t$ S is i-functional and the preceding theorem gives an i-register R_i with $Q(i) = U^i / K(i)$ with $K(i) = Ker\ S_i(\_^i\ 0)$. So, we can define M and N by

$$N(u^i + K(i),\ v(i)) = S^i(u^i, v(i))$$
$$M(u^i + K(i),\ v_i^j) = u^i v^j + K(j)$$

The other direction followns from the preceding remarks. *QED*

Notice that these results have clear interpretations in terms of decompositions.

7. CONCLUSION

We have examined the role played by additivity as opposed to linearity with respect to some basic concepts and results of general time-systems theory. These results have been derived from simple facts about groups. In constrast to the commutative additive machines of (3) we did not assume commutativity, for whatever should commute does commute and we only need quotients by kernels of suitable homomorphisms.

Of course, linearity is a very useful; and frequently natural , assumption, especially in providing convenient representations. But we can at least get started simply with additivity, which is interesting from a foundations viewpoint.

We have made heavy usage of the closure property assumed for the input space. However this is not essential, we can replace it by relativization to the continuations of inputs $(C(u^t) = \{v_t / u^t v \in U\})$.

One point that deserves comments is the extension of Nerode's construction in section 6:

What we actually get is a phase space for each instant. These can then be merged into a single attainable and observable state space when some extra machinery is applicable.

Finally most definitions make sense with the word "additive" omitted or changed to, say, "continuous". This translates the concepts to another (concrete) category, where most proofs can be suitably adapted, in general.

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