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FINITE ELEMENT SOLUTION OF 3D VISCOUS FLOW PROBLEMS USING NONSTANDARD DEGREES OF FREEDOM

by

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ABSTRACT

Two velocity-pressure finite element methods for solving incompressible flow problems in three-dimension space are presented. The velocity is defined by means of a new type of degree of freedom called parametrized. Though nonconforming in velocity, optimal convergence results are proven to hold for both methods.

KEY-WORDS: Convergence, finite elements, incompressible, inf-sup condition, mixed methods, nonconforming, parametrized degrees of freedom, pressure, Stokes problem, three-dimension space, velocity, viscous flow.

RESUMO

Propõe-se métodos de elementos finitos mistos tetraédricos de tipo velocidade-pressão, para resolver as equações de Navier - Stokes incompressível. O método, que é uma generalização do elemento bidimensional com velocidade quadrática completa contínua e pressão constante por triângulo, produz convergência linear nas normas habituais para esse problema. Nesta versão, usa-se os mesmos tipos de aproximação nos tetraedros, só que para a velocidade introduz-se um conceito importante: o de graus de liberdade parametrados, permitindo obter os mesmos resultados de convergência, apesar da sua não continuidade. Demonstra-se também que com o acréscimo da função bolha de cada elemento para a velocidade, além do uso da pressão linear descontínua por tetraedro, obtém-se um elemento de convergência quadrática.

PALAVRAS-CHAVES: Condição inf-sup, convergência, elementos finitos, espaço de três dimensões, graus de liberdade, incompressível, métodos mistos, não conforme, parametrado, pressão, problema de Stokes.

CONTENTS

0. Notation

Let Ω be a bounded open set of $\mathbb{R}^n$, $n=2$ or 3, with boundary Γ. $\underset{\sim}{x}$ denotes the set of n space variables $(x_1, x_2, \ldots, x_n)$ and $d\underset{\sim}{x} = dx_1 dx_2 \ldots dx_n$.

For every set $\omega \subset \Omega$, we denote by $(.,)_{0,\omega}$ and $\|.\|_{0,\omega}$, respectively, the inner product and norm of $L^2(\omega)$, namely,

$$(f,g)_{0,\omega} = \int_\omega fg\, d\omega \quad \text{and} \quad \|f\|_{0,\omega} = (f,f)^{1/2}, \qquad f,g \in L^2(\omega).$$

$L^2_o(\Omega)$ denotes the subspace of $L^2(\Omega)$ of functions f, such that $\int_\Omega f d\underset{\sim}{x} = 0$.

If D is an open subset of Ω and $m \in \mathbb{N}$, $\|\ \|_{m,D}$ and $|.|_{m,D}$ denote respectively the usual norm and seminorm of the Sobolev space $H^m(D)$. We recall that $H^m(D)$ consists of those functions v, whose partial derivatives (taken in the weak sense) up to the m-th order, belong to $L^2(D)$.

More precisely, we write:

$$\|v\|_{m,D} = \{ \sum_{|\alpha| \le m} \int_D |\partial^\alpha v|^2 d\underset{\sim}{x} \}^{1/2}$$

$$|v|_{m,D} = \{ \sum_{|\alpha| = m} \int_D |\partial^\alpha v|^2 d\underset{\sim}{x} \}^{1/2}$$

where $\partial^\alpha v = \dfrac{\partial^{|\alpha|} v}{\partial \underset{\sim}{x}^\alpha}$, α being an integer multiindex,

$\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$, with $\alpha_i \ge 0$, and $|\alpha| = \sum_{i=1}^n \alpha_i$, and where $\partial \underset{\sim}{x}^\alpha$ stands for $\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \ldots \partial x_n^{\alpha_n}$.

The inner product of $H^m(D)$ will be denoted by $(.,.)$ which is given by:

$$(u,v)_{m,D} = \sum_{|\alpha|\leq m} \int_D \partial^\alpha u\, \partial^\alpha v\, d\underset{\sim}{x} .$$

Whenever D is Ω itself, we will drop the corresponding subscript in the above norms and inner products, and when m=0 we write $L^2(D)$ instead of $H^o(D)$.

$H_o^1(D)$ denotes the space of functions $v \in H^1(D)$ that vanish almost everywhere on Γ. We will equip $H_o^1(\Omega)$ with the norm $|v|_1, v \in H_o^1(\Omega)$.

We denote by $x \circ y$ the euclidean inner product of two vectors x and y of $\mathbb{R}^\ell$. ℓ will be either n, in case x and y are vectors of $\mathbb{R}^n$, or N^2 in case they are tensors of $\mathbb{R}^{n\times n}$. The norm associated with this inner product is denoted by $|.|$.

For every space V of functions defined over D, $\underset{\sim}{V}$ will denote the space of vector fields $\underset{\sim}{v}$, whose n components $v_1, v_2, \ldots, v_n$ belong to V. If $\|.\|$ denotes the norm of space V, the norm of $\underset{\sim}{V}$ is defined and denoted as follows:

$$\|\underset{\sim}{v}\| = |\underset{\sim}{z}| \text{ , where } \underset{\sim}{z} = (\|v_1\|, \|v_2\|, \ldots, \|v_n\|)$$

For every function or field y defined over an open set D, $y/_S$ denotes its restriction to a subset S of D.

Finally we denote by P_k the space of polynomials in n variables of degree less than or equal to k, defined over a given open subset of Ω. If ω is a subset of a linear manifold of $\mathbb{R}^n$ of dimension strictly less than n, then the corresponding space of polynomials is denoted by $P_k(\omega)$.

1. INTRODUCTION

In this work we study non standard mixed finite element methods, for solving three-dimensional incompressible Navier-Stokes equations. The elements, which are of the velocity - pressure type, can be viewed as optimal generalizations to the three-dimensioal case of two classical triangular elements. The latter are the continuous piecewise quadratic (P_2) triangular element with discontinuous piecewise constant pressure (P_0), introduced by FORTIN in [5], and its extension proposed by CROUZEIX & RAVIART in [4], in order to obtain second order approximations of both the velocity and the pressure, in the standard norms for this problem.

We will recall the precise definition of both elements in Section 2. For the moment, let us state the problem that we want to solve.

Given a flow region Ω of $\mathbb{R}^3$ with boundary Γ assumed to be polyhedric, we apply body forces $\underset{\sim}{f}$ to the fluid, whose viscosity is ν, and we want to find its velocity $\underset{\sim}{u}$ and pressure p up to an additive constant, both depending on the space variables $\underset{\sim}{x} = (x_1, x_2, x_3)$, such that:

(1.1) $$-\nu\Delta u + \sum_{j=1}^{3} u_j \frac{\partial \underset{\sim}{u}}{\partial x_j} + \underset{\sim}{\text{grad}}\ p = \underset{\sim}{f} \text{ in } \Omega$$

(1.2) $$\text{div}\ \underset{\sim}{u} = 0 \quad \text{in } \Omega$$

(1.3) $$\underset{\sim}{u} = \underset{\sim}{0} \quad \text{on } \Gamma$$

where $\Delta u = \text{div}(\underset{\sim}{\text{grad}}\ u) = \sum_{i=1}^{3} \frac{\partial^2 u}{\partial x_i^2}$

As usual in the finite element method, we write equations (1.1)~(1.3) in variational form, namely, we multily both sides of (1.1) by arbitrary (test) velocities $\underset{\sim}{v}\in \underset{\sim}{H}_o^1(\Omega)$ and (1.2) by arbitrary (test) pressures $q\in L_o^2(\Omega)$, and we integrate over Ω.

In so doing, we obtain the variational problem to solve, after integration by parts, namely, we wish to find $u\in H_o^1(\Omega)$ and $p\in L_o^2(\Omega)$ such that:

(1.4) $$\nu \int_\Omega \underset{\sim}{\mathrm{grad}}\ \underset{\sim}{u} \cdot \underset{\sim}{\mathrm{grad}}\ \underset{\sim}{v}\ d\underset{\sim}{x} + \int_\Omega (\underset{\sim}{u}\circ \mathrm{grad})\ \underset{\sim}{u} \circ \underset{\sim}{v}\ d\underset{\sim}{x} - \int_\Omega p\ \mathrm{div}\underset{\sim}{v} d\underset{\sim}{x} = \int_\Omega \underset{\sim}{f} \circ \underset{\sim}{v} d\underset{\sim}{x} \qquad \forall \underset{\sim}{v} \in \underset{\sim}{H}_o^1(\Omega)$$

and

(1.5) $$\int_\Omega q\ \mathit{div}\ \underset{\sim}{u}\ d\underset{\sim}{x} = 0 \qquad \forall q\in L_o^2(\Omega)$$

Without loss of the essential results of this paper, we will deal with finite element approximations of the following slow flow version of (1.1)~(1.3), namely, the Stokes problem below:

$$(\mathcal{P}) \quad \begin{cases} \text{Find } \underset{\sim}{u} \in \underset{\sim}{H}_o^1(\Omega) \text{ and } p\in L_o^2(\Omega) \text{ such that,} \\ \nu a(\underset{\sim}{u},\underset{\sim}{v}) - b(p,\underset{\sim}{v}) = L(\underset{\sim}{v}) \qquad \forall \underset{\sim}{v}\in \underset{\sim}{H}_o^1(\Omega) \\ b(q,\underset{\sim}{u}) = 0 \qquad \forall q\in L_o^2(\Omega) \end{cases}$$

where $a:\underset{\sim}{H}_o^1(\Omega)\times \underset{\sim}{H}_o^1(\Omega) \to \mathbb{R}$ is given by $a(\underset{\sim}{u},\underset{\sim}{v}) = \int_\Omega \underset{\sim}{\mathrm{grad}}\ \underset{\sim}{u}\circ \underset{\sim}{\mathrm{grad}}\ \underset{\sim}{v}\ d\underset{\sim}{x}$

$b:L_o^2(\Omega)\times \underset{\sim}{H}_o^1(\Omega) \to \mathbb{R}$ is given by $b(q,\underset{\sim}{v}) = \int_\Omega q\ \mathit{div}\ \underset{\sim}{v}\ d\underset{\sim}{x}$

and $L(\underset{\sim}{v}) = \int_\Omega \underset{\sim}{f}\circ \underset{\sim}{v}\ d\underset{\sim}{x}$, $\forall \underset{\sim}{v}\in \underset{\sim}{H}_o^1(\Omega)$, with $\underset{\sim}{f}\in \underset{\sim}{L}^2(\Omega)$

We will consider the finite element version of $(\mathcal{P})$ to be as follows:

We first given a partition $\mathcal{T}_h$ of Ω into a finite number of tetrahedrons with maximal edge length equal to h. We assume that $\mathcal{T}_h$ belongs to a regular family of partitions of Ω $\{\mathcal{T}_h\}_h$, that is to say, there exists a lower bound $\theta_o>0$ for all the angles of the faces of the tetrahedrons of $\mathcal{T}_h$, for all h..

Next we define the finite element approximate problem $(\mathcal{P}_h)$ by considerring only test velocities $\underset{\sim}{v}$ and pressures q, which belong to finite dimensional spaces $\underset{\sim}{V}_h$ and Q_h associated with $\mathcal{T}_h$, approximating $\underset{\sim}{H}_o^1(\Omega)$ and $L_o^2(\Omega)$ respectively. Then we search for $\underset{\sim}{u}_h\in\underset{\sim}{V}_h$ and $p_h\in Q_h$ instead of $\underset{\sim}{u}$ and p. More specifically we want to solve:

$$(\mathcal{P}_h)\quad\begin{cases}\text{Find } \underset{\sim}{u}_h\in\underset{\sim}{V}_h \text{ and } p_h\in Q_h \text{ such that}\\ \nu\, a_h(\underset{\sim}{u}_h,\underset{\sim}{v}) - b_h(p_h,\underset{\sim}{v}) = L(\underset{\sim}{v}) \qquad \forall \underset{\sim}{v}\in V_h\\ b_h(q,\underset{\sim}{u}_h)=0 \qquad \forall q\in Q_h\end{cases}$$

where a_h and b_h are continuous bilinear forms defined over $[\underset{\sim}{V}_h+\underset{\sim}{H}_o^1(\Omega)]\times[\underset{\sim}{V}_h+\underset{\sim}{H}_o^1(\Omega)]$ and $L_o^2(\Omega)\times[\underset{\sim}{V}_h+\underset{\sim}{H}_o^1(\Omega)]$, respectively givem by:

$$(1.6)\quad a_h(\underset{\sim}{u},\underset{\sim}{v}) = \sum_{K\in\mathcal{T}_h}\int_K \underset{\sim}{\text{grad}}\, \underset{\sim}{u}\circ \underset{\sim}{\text{grad}}\, \underset{\sim}{v}\, d\underset{\sim}{x}$$

(analogously we will write $a_h(u,v) = \sum_{K\in\mathcal{T}_h}\int_K \underset{\sim}{\text{grad}}\, u \circ \underset{\sim}{\text{grad}}\, v\, dx$ for $u,v\in V_h+H_o^1(\Omega)$)

$$(1.7)\quad b_h(q,\underset{\sim}{v}) = \sum_{K\in\mathcal{T}_h}\int_K q\cdot \text{div}\, \underset{\sim}{v}\, dx$$

With the above definitions we are allowing space $\underset{\sim}{V}_h$ to be nonconforming, i.e., if $v\in V_h$ we do have $v/_K\in H^1(K)$ $\forall K\in\mathcal{T}_h$, but we do not necessarily have $v \in H^1(\Omega)$, nor we have $v=0$ a.e. on Γ. In this case we must work with the discrete $H_o^1(\Omega)$-seminorm for $\underset{\sim}{V}_h$, given by:

(1.8) $\|\underset{\sim}{v}\|_h = [a_h(\underset{\sim}{v},\underset{\sim}{v})]^{1/2}$ (analogously $\|v\|_h = [a_h(v,v)]^{1/2}, v \in V_h$).

We will prove later on that $\|.\|_h$ is actually a norm for the chosen spaces V_h. Notice that whenever $\underset{\sim}{u}$, $\underset{\sim}{v} \in \underset{\sim}{H}_o^1(\Omega)$ we have:

$a_h(\underset{\sim}{u},\underset{\sim}{v})=a(\underset{\sim}{u},\underset{\sim}{v})$, $b_h(q,\underset{\sim}{v})=b(q,\underset{\sim}{v})$ $\forall q \in L_o^2(\Omega)$ and $\|\underset{\sim}{v}\|_h = |\underset{\sim}{v}|_1$.

In order to establish the well-posedness of $(\mathcal{P}_h)$ we must prove that spaces $\underset{\sim}{V}_h$ and Q_h satisfy the crucial inf-sup condition (also called the discrete LBB- condition) below (see e.g. [2]):

There exists $\beta > 0$ such that:

$$(1.9) \quad \inf_{q \in Q_h} \sup_{\underset{\sim}{v} \in \underset{\sim}{V}_h} \frac{b_h(q,\underset{\sim}{v})}{\|\underset{\sim}{v}\|_h \|q\|_o} \geq \beta$$

In this case, if β is independent of h, according to error bounds given in [9], $\underset{\sim}{u}_h$ and p_h satisfy:

$$(1.10) \quad \|\underset{\sim}{u}-\underset{\sim}{u}_h\|_h + \|p-p_h\|_o \leq C \left[\inf_{\underset{\sim}{v} \in \underset{\sim}{V}_h} \|\underset{\sim}{u}-\underset{\sim}{v}\|_h + \inf_{q \in Q_h} \|p-q\|_o + \sup_{\underset{\sim}{w} \in \underset{\sim}{V}_h} \frac{|E_h(\underset{\sim}{u},p,\underset{\sim}{w})|}{\|\underset{\sim}{w}\|_h} \right]$$

Where C is a constant independent of h(*), and

$$(1.11) \quad E_h(\underset{\sim}{u},p,\underset{\sim}{w}) = L(\underset{\sim}{w}) - \nu a_h(\underset{\sim}{u},\underset{\sim}{w}) + b_h(p,\underset{\sim}{w})$$

Notice that $E_h(\underset{\sim}{u},p,\underset{\sim}{w})$ vanish identically whenever $\underset{\sim}{w} \in \underset{\sim}{H}_o^1(\Omega)$.

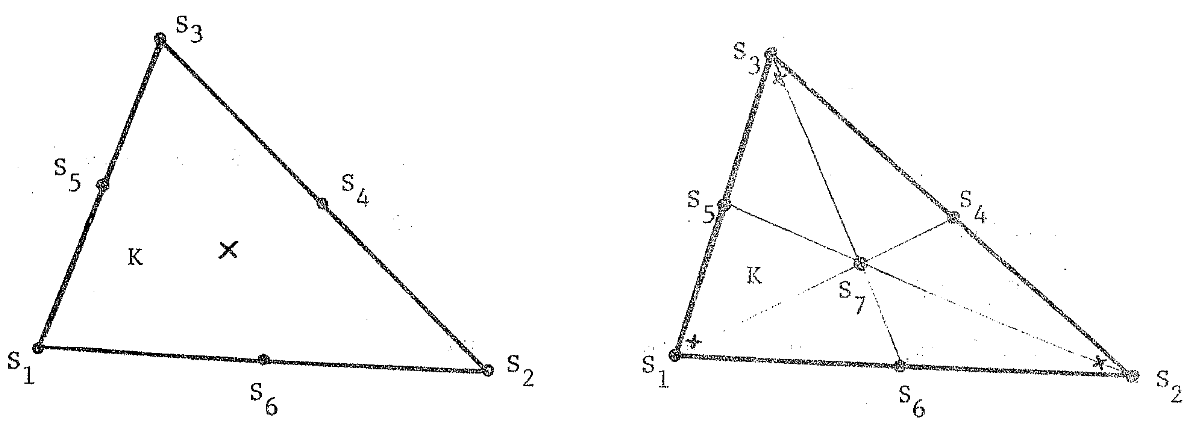
(*) The symbol C with or without subscripts will carry this meaning throughout the paper.

2. THE FINITE ELEMENT METHODS

Let us consider for a moment the natural two-dimensional versions of $(\mathcal{P})$ and $(\mathcal{P}_h)$. In the framework of these problems, FORTIN in [5] chose $V_h = V_h^o$ and $Q_h = Q_h^o$, where V_h^o is the set of continuous functions, whose restriction to every triangle K of $\mathcal{T}_h$ is a polynomial of P_2, associated with the following set of degrees of freedom (refer to Figure 2.1a)

a) Fortin's element

b) Crouzeix & Raviart's element



° node for velocity

× node for the pressure

The basic two-dimensional elements

Figure 2.1

- Functional values at the three vertices of K, S_1,S_2 and S_3 ;
- Functional values at the mid-points of the vertices of S_4, S_5 and S_6.

A degree of feedom will vanish if the associated node lies on Γ, the boundary of polygon Ω, and in this way we have $V_h \subset H_o^1(\Omega)$.

Q_h^o, in turn, is the space of functions q which are constant (P_o) over each triangle of $\mathcal{T}_h$, such that $\int_\Omega q d\underset{\sim}{x}=0$.

In [4] CROUZEIX & RAVIART considered an extension of the above element, associated with the spaces $V_h=V_h^1$ and $Q_h=Q_h^1$, as follows. A function $v \in V_h^1$ is still continuous and vanishes on Γ, i.e., $v \in H_o^1(\Omega)$, but its restriction to every $K \in \mathcal{T}_h$ is of the form $p+\alpha\psi$, where $p \in P_2, \alpha \in \mathbb{R}$ and ψ is the bubble-function of K, namely, $\psi=\lambda_1\lambda_2\lambda_3$, where λ_i is the bary - centric coordinate of K associated with vertex S_i (*) (refer to Figure 2.1). $v/_k$ is associated with the following set of degrees of freedom:

- Functional values at the three vertices of K, S_1, S_2 and S_3;
- Functional values at the mid-points of the edges of K, S_4, S_5 and S_6;
- Functional values at the barycenter S_7 of K.

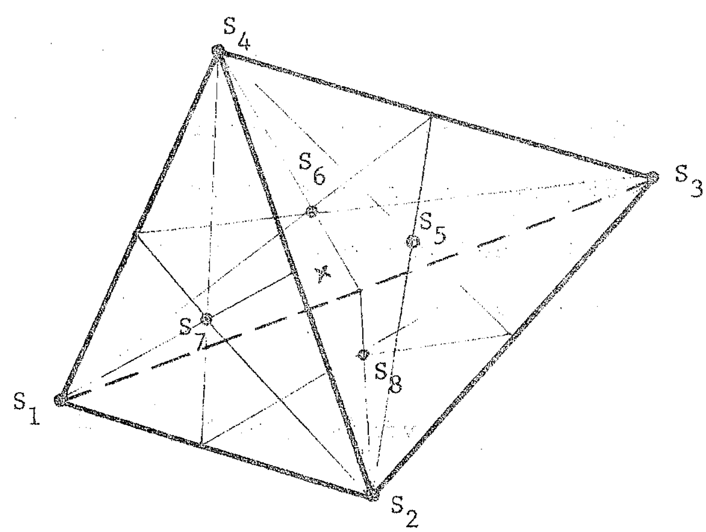
Any $q \in Q_h^1$ is a discontinuous function in general, whose restriction to every $K \in \mathcal{T}_h$ belongs to P_1. We assume that $q/_K$ is defined by its values at the vertices of K by convention, while we require $\int_\Omega q d\underset{\sim}{x}=0 \quad \forall q \in Q_h^1$.

Now if we denote by $\underset{\sim}{u}_h^i$ and p_h^i the solution $\underset{\sim}{u}_h$ and p_h to problem ($\mathcal{P}_h$), when $\underset{\sim}{V}_h$ and Q_h are respectively $\underset{\sim}{V}_h^i$ and Q_h^i, i=0 or 1, according to [4], the following error estimates hold:

$$(2.1) \quad \| \underset{\sim}{u}-\underset{\sim}{u}_h^i \|_h + \| p-p_h^i \|_o \leq Ch^{i+1} [|\underset{\sim}{u}|_{i+2} + |p|_{i+1}]$$

(*) λ_i is the polynomial of P_1 defined over K, such that $\lambda_i(S_j)$ equals one if i=j and zero otherwise.

We will introduce in this paper three - dimensional versions of both elements above, for which the same estimate (2.1) applies. Before doing it, let us recall that in [8] the author had proposed a tetrahedral version of FORTIN's element. The basic idea of this element was to choose besides the vertices, the barycenters of the faces of the tetrahedron as velocity nodes, instead of the mid-point of the edges, which are usually associated with the P_2 space in three dimensions. The main reason for this choice is the fact that in 3D flow problems, as far as the flux is concerned, the faces play the role of the edges in two dimensions, and not the edges themselves. Actually, the global error estimate (2.1) with i=0 also holds for this element with an eight node reduced quadratic velocity and a piecewise constant pressure (refer to Figure 2.2) , in spite of its nonconformity.



A first 3D version of the $P_2 \times P_0$ element

Figure 2.2

However, since one now works with incomplete quadratic functions for the velocity, it is not possible to extend it with the addition of one single velocity node, in order to obtain a second order element, like in the case of CROUZEIX & RAVIART's element. Actually, we conjecture that one cannot construct any simple second order version of this element. That is why we consider below a different 3D version of FORTIN's element as a starting point, for obtaining a second order element.

Let S_i, $i=1,2,3,4$ be the vertices of a tetrahedron K belonging to $\mathcal{T}_h$, and S_{ijk} be the barycenter of the face, whose vertices are S_i, S_j and S_k. Let also ℓ_{ij} be the edge of K whose ends are S_i and S_j (refer to Figure 2.3), and S_{ij} be its mid-point.

Now we define V_h^o to be the set of functions v, whose restriction to every $K \in \mathcal{T}_h$ belongs to P_2, $v_{/K}$ being defined by the following set of non standard degrees of freedom:

- The four functional values at the S_{ijk}'s
- The six linear combinations of the functional values at S_{ij} and the mean value over ℓ_{ij} denoted by $[v]_{ij}$, given by:

$$(2.2) \qquad [v]_{ij} = \frac{9}{5} \int_{\ell_{ij}} vds/length(\ell_{ij}) - \frac{4}{5} v(S_{ij})$$

We require coincidence of the latter type of degree of freedom for all the tetrahedrons of $\mathcal{T}_h$ for which ℓ_{ij} is a common edge, while we assign $[v]_{ij}=0$ whenever ℓ_{ij} lies on Γ. On the other hand we require continuity of any function $v \in V_h^o$ at the barycenters of the faces of the tetrahedrons of $\mathcal{T}_h$, and when one such point lies on Γ, the corresponding value of v is zero.

As one can easily verify, the above set of degrees of freedom is P_2-unisolvent, and the corresponding basis functions are given by:

- For the functional value at S_{ijk}, $1 \le i < j < k \le 4$:

$$(2.3) \qquad \phi^o_{ijk} = \frac{3}{7}\,[5\lambda_\ell^2 - 2\lambda_\ell - 1 + 10\,(\lambda_i\lambda_j + \lambda_i\lambda_k + \lambda_j\lambda_k)]\ ;$$

- For the linear combinations (2.2) related to ℓ_{ij}, $1 \le i < j \le 4$:

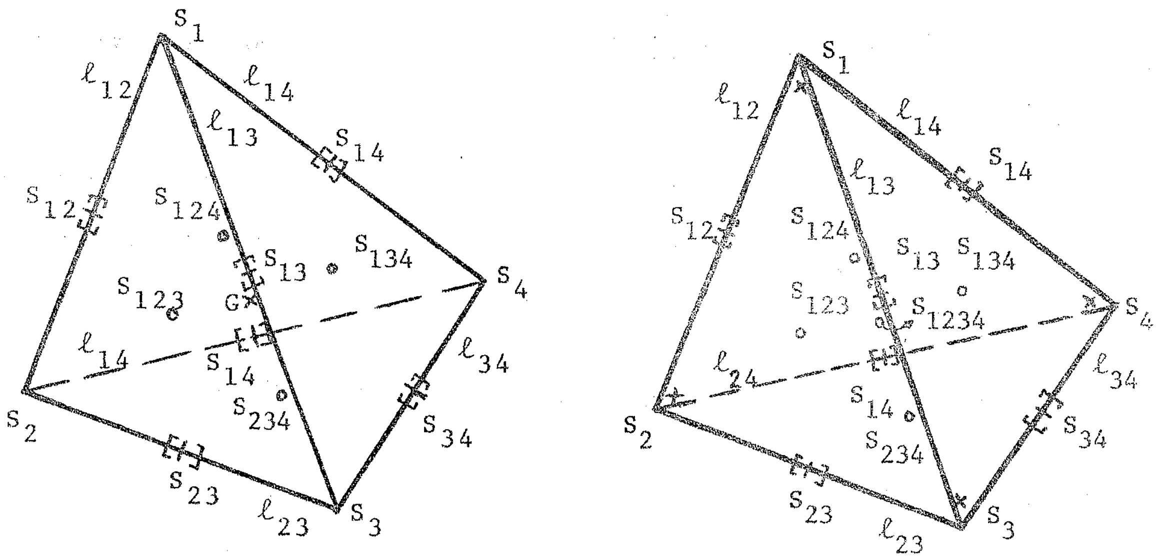
$$(2.4) \qquad \phi^o_{ij} = \frac{10}{21}\,[2(\lambda_i+\lambda_j) - (\lambda_k+\lambda_\ell) + \lambda_i\lambda_j + 10\lambda_k\lambda_\ell + 5(\lambda_i+\lambda_j)(\lambda_k+\lambda_\ell)]$$

where i,j,k and ℓ are all distinct, and λ_i is the barycentric coordinate associated with vertex S_i.

With the above definition, it is not possible to require that a function of V^o_h be continuous over the common face of two neighboring elements, and thus we have a nonconforming element. Actually, continuity is only guaranteed at the barycenter of this face. However, this is sufficient to verify that the expression (1.8) defines a norm for $\underset{\sim}{V}^o_h$.

Like in the two-dimensional case, the space of pressures Q^o_h that we associate with V^o_h, is defined to be the set of functions that are constant (P_o) over each K, $K \in \mathcal{T}_h$.

a) For spaces $\underset{\sim}{V}_h^o \times Q_h^o$ b) For spaces $\underset{\sim}{V}_h^1 \times Q_h^1$



pressure node (G is the barycenter of K): ×

velocity degrees of freedom: ∘ functional value

[|] linear combination (2.2)

The new three-dimensional elements

Figure 2.3

Now, in order to define a second order element, analogous to the CROUZEIX & RAVIART's element, we first change the pressure space into Q_h^1, namely, the set of functions q whose restriction to each $K \in \mathcal{T}_h$ belongs to P_1, without continuity requirements over interelement boundaries.

The associated space $V_h = V_h^1$ is defined in such a way that the restriction of every function $v \in V_h^1$ to $K \in \mathcal{T}_h$, belongs to a space P_2^1, namely, the direct sum $P_2 \oplus \{\psi\}$, where ψ is the bubble-function of tetrahedron K, that is to say, $\psi = \lambda_1 \lambda_2 \lambda_3 \lambda_4$.

One can verify that the set of ten degrees of freedom associated with the above defined space V_h^o, plus the functional value at the barycenter of tetrahedron K is P_2^1 - unisolvent, by calculating the corresponding basis functions. These are given by:

- For the functional value at S_{ijk}, $1\leq i<j<k\leq 4$:

$$(2.5) \quad \phi^1_{ijk} = \phi^o_{ijk} - \frac{528}{7}\psi \ ;$$

- For the linear combination (2.2) over ℓ_{ij}, $1\leq i<j\leq 4$

$$(2.6) \quad \phi^1_{ij} = \phi^o_{ij} + \frac{160}{7}\psi \ ;$$

- For the functional value at S_{1234}, the barycenter of K:

$$(2.7) \quad \phi^1_{1234} = 256\psi$$

For any function of V_h^1, these degrees of freedom vanish whenever ℓ_{ij} or S_{ijk} lies on Γ. We require the same coincidence of degrees of freedom over interelement boundaries, as for a function of V_h^o, by means of which one can check that the seminorm given by (1.8) is also a norm over $\underset{\sim}{V}_h^1$.

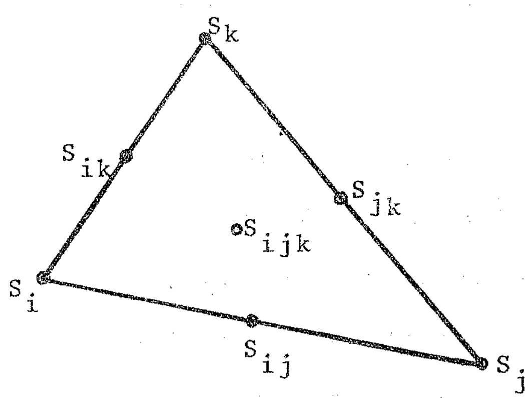
Clearly enough we have a nonconforming element also in this case. Moreover, the inclusion $V_h^o \subset V_h^1$ holds.

3 - BASIC PROPERTIES OF THE ELEMENTS

As it is well-known, in order to establish the convergence of $(\underset{\sim}{u}_h, p_h)$ to $(\underset{\sim}{u}, p)$ in $\underset{\sim}{H}_o^1(\Omega) \times L_o^2(\Omega)$, one must first prove the validity of the discrete inf-sup condition (1.9). Before doing this, we give three lemmas which characterize some basic properties of the finite element methods introduced in Section 2.

<u>Lemma 3.1</u>: Let K be a tetrahedron of $\mathcal{C}_h$, and w^n be a linear combination of the ϕ^n_{ij}'s and ϕ^n_{ijk}'s, given by (2.3), (2.4), (2.5) and (2.6), n=0 or 1. Then, if F_ℓ is the face of K opposite vertex S_ℓ, ℓ=1,2,3,4 the values of $\int_{F_\ell} \lambda_m w^n ds$ $\forall m$, m=1,2,3,4, depend only on the area of F_ℓ, and on the coefficients of the ϕ^n_{ij}'s and ϕ^n_{ijk}'s, with i,j and k distinct and non equal ℓ.

<u>Proof</u>: Since the restriction of w^1 over F_ℓ is a linear combination of the ϕ^o_{ij}'s and the ϕ^o_{ijk}'s, it suffices to prove the lemma for w^o. Referring to Figure 3.1, let us consider the following numerical quadrature formula over F_ℓ:



Face F_ℓ

Figure 3.1

$$(3.1) \quad \int_{F_\ell} p ds \cong \frac{area(F_\ell)}{60}\{3[p(S_i)+p(S_j)+p(S_k)]+8[p(S_{ij})+p(S_{ik})+p(S_{kj})] + 27p(S_{ikj})\}$$

where the approximation sign $\cong$ is to be replaced by the equality sign whenever $p \in P_3(F_\ell)$

If m=ℓ the lemma is trivial. Thus, without loss of generality, we take m=i and the lemma will be proved if we establish that

$$\int_{F_\ell} \lambda_i \phi^o_{i\ell}\, ds = \int_{F_\ell} \lambda_i \phi^o_{j\ell}\, ds = \int_{F_\ell} \lambda_i \phi^o_{ij\ell}\, ds = \int_{F_\ell} \lambda_i \phi^o_{jk\ell}\, ds = 0$$

Recalling (2.3) and (2.4), we have to compute the values of p at the seven quadrature points in four cases, namely:

<u>1st case</u> $p = \frac{21}{10} \lambda_i \phi^o_{i\ell} = \lambda_i[2\lambda_i - \lambda_k - \lambda_j - 5\lambda_i\lambda_k - 5\lambda_i\lambda_j + 10\lambda_k\lambda_j]$

We have $p(S_i)=2$; $p(S_{ij})=p(S_{ik}) = -3/8$, and $p(S_j)=p(S_k)$ $p(S_{kj})=p(S_{ijk}) = 0$.

<u>2th case</u> $p = \frac{21}{10} \lambda_i \phi^o_{j\ell} = \lambda_i[2\lambda_j - \lambda_k - \lambda_i + 10\lambda_i\lambda_k - 5\lambda_i\lambda_j - 5\lambda_j\lambda_k]$

We have $p(S_i) = 1$; $p(S_{ij}) = -3/8$; $p(S_{ik}) = 3/4$, and

$p(S_j) = p(S_k) = p(S_{kj}) = p(S_{ijk}) = 0$

<u>3rd case</u> $p = \frac{7}{3} \lambda_i \phi^o_{ij\ell} = \lambda_i(5\lambda_k^2 - 2\lambda_k - 1 + 10\lambda_i\lambda_j)$

We have $p(S_i) = -1$; $p(S_j) = p(S_k) = p(S_{jk}) = p(S_{ijk}) = 0$;

$p(S_{ij}) = 3/4$, and $p(S_{ik}) = -3/8$

<u>4th case:</u> $p = \frac{7}{3} \lambda_i \phi^o_{jk\ell} = \lambda_i(5\lambda_i^2 - 2\lambda_i - 1 + 10\lambda_j\lambda_k)$

We have $p(S_i)=2$; $p(S_j) = P(S_k) = p(S_{jk}) = p(S_{ijk}) = 0$, and $p(S_{ij}) = p(S_{ik}) = 3/8$.

In all the four cases, using (3.1), we conclude that $\int_{F_\ell} p\,ds = 0$, as required.

q.e.d.

Remark 3.1 : Lemma 3.1 basically states that, although a function $v \in V_h^n$ n=0 or 1, is not necessarily continuous over the common face F of two neighboring tetrahedrons K and K' of $\mathcal{T}_h$, we do have continuity in the following weak sense:

$$\int_F v/_K \ pds = \int_F v/_{K'} \ pds \qquad \forall p \in P_1(F) \qquad \square$$

Let now $W_h = V_h^1 + H_0^1(\Omega)$, where V_h^1 is the space defined in Section 2.

Lemma 3.2: Let $\{\mathcal{T}_h\}_h$ be a regular family of partitions of Ω into tetrahedrons, and z be a function of $H^{i+2}(\Omega) \cap H_0^1(\Omega)$, i = 0 or 1, ∂K denote the boundary of $K \in \mathcal{T}_h$, and $\frac{\partial u}{\partial n}$ denote the outer normal derivative on ∂K of a function u defined in K. Then we have:

$$(3.2) \quad \sum_{K \in \mathcal{T}_h} \int_{\partial K} \frac{\partial z}{\partial n_K} w \ ds \leq C \ h^{i+1} \ |z|_{i+2} \ \|w\|_h \qquad \forall w \in W_h.$$

Proof: Let $K \in \mathcal{T}_h$ and F be a face of K. We have:

$$\sum_{K \in \mathcal{T}_h} \int_{\partial K} \frac{\partial z}{\partial n_K} wds = \sum_{K \in \mathcal{T}_h} \sum_{F \subset \partial K} \int_F \frac{\partial z}{\partial n} w \ ds.$$

Now for every $K \in \ _h$, we define the projection operator for i=0 or 1.

$$\pi_F^i : H^{i+1}(K) \to P_i(F)$$

where F is a face of K.

Since $\pi_F^i(w)$ satisfies $\int_F \pi_F^i(w)p \ ds = \int_F w \ p \ ds \quad \forall p \in P_i(F)$, we clearly have, from the Trace Theorem:

(3.3) $\| \pi_F^i(w) \|_{L^2(F)} \leq C \| w \|_{L^2(F)} \leq \mathcal{C}(K) \| w \|_{1,K} \quad \forall w \in H^1(K)$

where $\mathcal{C}$ only depends on K.

Clearly enough, $\int_F pw\, ds = 0 \quad \forall p \in P_1(F)$, F being any face lying on Γ, for $w \in H_0^1(\Omega)$. Thus, using Lemma 3.1, and the assumption that the degrees of freedom of w attached to F vanish if $F \subset \Gamma$, for $w \in V_h^1$, we have:

$$\sum_{K \in \mathcal{C}_h} \sum_{F \subset \partial K} \int_F \pi_F^i\left(\frac{\partial u}{\partial n_K}\right) w\, ds = 0 \qquad \forall w \in W_h.$$

Indeed, if $F \not\subset \Gamma$, there are two tetrahedrons $K, K' \in \ \ _h$ for which F is common face, and we have:

$$\int_F \pi_F^i\left(\frac{\partial u}{\partial n_K}\right) w\, ds + \int_F \pi_F^i\left(\frac{\partial u}{\partial n_{K'}}\right) w\, ds = 0$$

which follows from the fact that $\underset{\sim}{n}_K + \underset{\sim}{n}_{K'} = \underset{\sim}{0}$ over F and from the regularity assumptions for u.

Thus we have $\forall w \in W_h$:

(3.4) $$\sum_{K \in \mathcal{C}_h} \int_{\partial K} \frac{\partial z}{\partial n_K} w\, ds = \sum_{K \in \mathcal{C}_h} \sum_{F \subset \partial K} \int_F \left[\frac{\partial z}{\partial n_K} - \pi_F^i\left(\frac{\partial z}{\partial u_K}\right)\right] w\, ds$$

Now we notice that $\pi_F^o(w) = \int_F w\,ds \,/\, \text{area}(F)$. This means that if $w \in V_h^1$, $\pi_F^o(w)$ only depends on the degrees of freedom attached to F, according to Lemma 3.1. Moreover, if $w \in H_0^1(\Omega)$, the Trace Theorem asserts that $\pi_F^o(w)$ coincides on both sides of F, the common face of two neighboring tetrahedrons of $\mathcal{C}_h$.

Thus we have:

$$\sum_{K\in\mathcal{T}_h}\ \sum_{F\subset\partial K}\int_F \left[\frac{\partial z}{\partial n_K} - \pi_F^i\left(\frac{\partial z}{\partial n_K}\right)\right]\pi_F^o(w) = 0 \quad \forall w\in W_h ,$$

which together with (3.4) yields:

$$(3.5)\quad \sum_{K\in\mathcal{T}_h}\int_{\partial K}\frac{\partial z}{\partial n_K}\, w\,ds = \sum_{K\in\mathcal{T}_h}\ \sum_{F\subset\partial K}\int_F \left[\frac{\partial z}{\partial n_K} - \pi_F^i\left(\frac{\partial z}{\partial n_K}\right)\right][w-\pi_F^o(w)]ds \quad \forall w\in W_h .$$

Let now $\hat{K}$ be the usual reference tetrahedron and $\mathcal{F}$ be the affine transformation which maps $\hat{K}$ onto K, i.e., if $\underset{\sim}{x} = \mathcal{F}(\hat{\underset{\sim}{x}})$ $\forall\hat{\underset{\sim}{x}}\in\hat{K}$, we have $K = \mathcal{F}(\hat{K})$.

We further define σ_F: $H^{i+1}(K)\times H^1(K)\rightarrow \mathbb{R}$, to be the bilinear form

$$(3.6)\qquad \sigma_F(y,w) = \int_F [y-\pi_F^i(y)][w - \pi_F^o(w)]\, ds,$$

where F is a given face of $K\in\mathcal{T}_h$

If we denote by $\hat{v}$ the function defined over $\hat{K}$ such that $v[\mathcal{F}(\hat{\underset{\sim}{x}})] = \hat{v}(\hat{\underset{\sim}{x}})$ for a given function v defined in K, we analogously define $\hat{\sigma}_F$: $H^{i+1}(\hat{K}) \times H^1(\hat{K})\rightarrow \mathbb{R}$ by:

$$\hat{\sigma}_F(\hat{y},\hat{w}) = \int_{\hat{F}} [\hat{y} - \hat{\pi}_{\hat{F}}^i(\hat{y})][\hat{w}-\hat{\pi}_{\hat{F}}^o(\hat{w})]d\hat{s},$$

where $\hat{F} = \mathcal{F}^{-1}(F)$ and $\hat{\pi}_{\hat{F}}^i$ is the orthogonal projection operator from $H^{i+1}(\hat{K})$ onto $P_i(\hat{F})$.

Noticing that $[\pi_F^i(z)](x) = [\pi_F^i(z)]\,[\mathcal{T}(\hat{x})] = [\hat{\pi}_F^i(\hat{z})](\hat{x})$, by standard arguments we have:

(3.7) $\sigma_F(y,w) \le Ch^2\,|\hat{\sigma}_F(\hat{y},\hat{w})| \quad \forall F \subset \partial K$ and $\forall K \in \mathcal{T}_h$.

On the other hand, applying (3.3) to $\hat{K}$, we have

$\|\hat{\Pi}_F^i(\hat{w})\|_{L^2(\hat{F})} \le \hat{C}\,\|\hat{w}\|_{1,\hat{K}}$, where $\hat{C}$ only depen-pends on $\hat{K}$

Thus, using the Schwartz inequality and again the Trace Theorem we get:

$$\hat{\sigma}_F(\hat{y},\hat{w}) \le \hat{C}\,\|\hat{y}\|_{1,\hat{K}} \qquad \forall \hat{y} \in H^{i+1}(\hat{K}) \text{ and } \forall \hat{w} \in H^1(\hat{K}).$$

Now, from the previous analysis we have:

$$\hat{\sigma}_F(\hat{y},\hat{w}) = 0 \qquad \forall \hat{y} \in H^{i+1}(\hat{K}) \text{ and } \forall \hat{w} \in P_o(\hat{K})$$

and

$$\sigma_F(\hat{y},\hat{w}) = 0 \qquad \forall \hat{y} \in P_i(\hat{K}) \text{ and } \forall \hat{w} \in H^1(\hat{K}),$$

with i = 0 or 1.

Thus from the Bramble-Hilbert Lemma in generalized form (see e.g. [3] Theorem 4.2.5), we conclude that

$$\hat{\sigma}_F(\hat{y},\hat{w}) \le \hat{C}\,\|\hat{\sigma}_F\|\,|\hat{y}|_{i+1,K}\,|\hat{w}|_{1,\hat{K}}$$

where

$$\|\hat{\sigma}_F\| = \sup_{\substack{\hat{y} \in H^{i+1}(\hat{K}) \\ \hat{w} \in H^1(\hat{K})}} \frac{\hat{\sigma}_F(\hat{y},\hat{w})}{\|\hat{y}\|_{i+1,\hat{K}}\,\|\hat{w}\|_{1,\hat{K}}}$$

Now, going back to element K, using standard estimates (see e.g.[7], and (3.7)) we obtain:

$$\sigma_F(y,w) \le Ch^{i+1}\,|y|_{i+1,K}\,|w|_{1,K}$$

Finally, if we set $y = \frac{\partial z}{\partial n_K}/_F$ $\forall F \subset \partial K$, $\forall K \in \mathcal{T}_h$, by summation over F and K we obtain (3.2). □ q.e.d.

Let us now introduce the linear operator $s_h: H_o^1(\Omega) \to V_h^i$, namely, the operator of orthogonal projection onto V_h^i, i=0 or 1, under the discrete H_o^1-inner product $a_h(.,.)$.

Lemma 3.3: Let $\{\mathcal{T}_h\}_h$ be a regular family of partitions of Ω, and e_h be the error of the projection $e_h = v - s_h v$, $v \in H_o^1(\Omega)$. Then the following estimate holds:

(3.8) $$\| e_h \|_0 \le Ch \| e_h \|_h$$

Proof: First we write:

$$\| e_h \|_0 = \sup_{f \in L^2(\Omega)} \frac{(f, e_h)_0}{\| f \|_0}$$

According to a classical result (see e.g. [3], p. 138) there exists a unique function $z \in M$, with $M = H^2(\Omega) \cap H_0^1(\Omega)$, such that $-\Delta z = f$. Moreover we have:

$$|z|_2 \le C \| f \|_0$$

Thus we can write

(3.9) $$\| e_h \|_0 = \sup_{z \in M} \frac{-(\Delta z, e_h)_0}{\| \Delta z \|_0} \le C \sup_{z \in M} \frac{|(\Delta z, e_h)_0|}{|z|_2}$$

On the other hand, from Green's formula, we have:

(3.10) $$|(\Delta z, e_h)| \le |(z, e_h)_h| + \left| \sum_{K \in \mathcal{T}_h} \int_{\partial K} \frac{\partial z}{\partial n_K} e_h \, ds \right|$$

Now applying Lemma 3.2 we obtain:

(3.11) $\sum_{K\in\mathcal{T}_h}\int_{\partial K}\frac{\partial z}{\partial u_K}e_h\,ds \le C_1 h\,|z|_2\,\|e_h\|_h.$

On the other hand we have:

$$(z,e_h)_h = (z-s_h z,e_h)_h \le \|z-s_h z\|_h\|e_h\|_h,$$

which, from well known results in approximation theory, yields:

(3.12) $(z,e_h)_h \le C_2 h|z|_2\|e_h\|_h.$

Taking (3.10), (3.11) and (3.12) to (3.9) we obtain (3.8). □

q.e.d.

4- THE INF-SUP CONDITION AND CONVERGENCE RESULTS

Let us first establish the existence and uniqueness of the approximate solution $(\underset{\sim}{u}_h,p_h)$ of $(\mathcal{P}_h)$ by proving the inf-sup condition (1.9), for the methods considered in Section 2.

Lemma 4.1 : Let $\{\mathcal{T}_h\}$ be a regular family of partitions of Ω. Then $\forall q_h^o\in Q_h^o$ one can find a vector field $\underset{\sim}{v}_h^o \in \underset{\sim}{V}_h^o$ such that

(4.1) $b_h(q_h^o, \underset{\sim}{v}_h^o) = \|q_h^o\|_0^2$

(4.2) $\|v_h^o\|_h \le C\,\|q_h^o\|_0$

Proof: Since $q_h^o\in L_o^2(\Omega)$, according to [6] Lemma 3.2, there exists $\underset{\sim}{v}^o\in \underset{\sim}{H}_o^1(\Omega)$ satisfying:

(4.3) $\operatorname{div}\underset{\sim}{v}^o = q_h^o$ in Ω

(4.4) $|\underset{\sim}{v}^o|_1 \le C'\|q_h^o\|_0$ in Ω, with C' independent of h and of q_h^o.

Let us now construct a field $\underset{\sim}{v}_h^o$ associated with $\underset{\sim}{v}^o$, in the following way: $\forall K \in \mathcal{T}_h$ we assign:

$$[v_h^o]_{ij} = [s_h v^o]_{ij} \qquad 1 \le i < j \le 4$$

and we set

$$(4.5) \qquad \int_{F_{ijk}} \underset{\sim}{v}_h^o ds = \int_{F_{ijk}} \underset{\sim}{v}^o ds \qquad 1 \le i < j < k \le 4$$

where F_{ijk} is the face of K whose vertices are S_i, S_j and S_k.

The existence and uniqueness of such $\underset{\sim}{v}_h^o$ is the direct consequence of the following identity, which applies to every $w \in V_h^i$, $i=0,1$:

$$\int_{F_{ijk}} w ds = \{5([w]_{ij}+[w]_{ik}+[w]_{kj})+27w(P_{ijk})\} \frac{\text{area}(F_{ijk})}{42}$$

Now, if we multiply both sides of (4.5) by $\underset{\sim}{n}_K$, and if we sum up over the four faces of K, using (4.3), we obtain:

$$\int_K \text{div } \underset{\sim}{v}_h^o \, d\underset{\sim}{x} = \int_K \text{div } \underset{\sim}{v}^o \, d\underset{\sim}{x} = \int_K q_h^o \, d\underset{\sim}{x}$$

(4.1) is then seen to be an immediate consequence of the above relation. On the other hand, by means of the same arguments as in [6] , Lemma 2.5 we have:

$$\| \underset{\sim}{v}_h^o \|_h \le C^* \, |\underset{\sim}{v}^o|_1 \, ,$$

from which we get (4.2), using (4.4). □

q.e.d.

Lemma 4.1 allows us to state the validity of the inf - sup condition (1.9) if $V_h = V_h^o$ and $Q_h = Q_h^o$, for in this case β is just $(C^*)^{-1}$. It is also the key to the proof that the same condition holds if $V_h=V_h^1$ and $Q_h=Q_h^1$, taking into account the following arguments and results.

Lemma 4.2: Let $V^1_{h_0}$ and Q^o_h be subspaces of V^1_h and Q^1_h, defined respectively by:

$$V^1_{h_o} = \{v \ / \ v \in V^1_h \ , \ v/_K \in H^1_o(K) \qquad \forall K \in \mathcal{T}_h\}$$

and

$$\tilde{Q}^o_h = \{q/q \in Q^1_h \text{ and } b_h(q,\underset{\sim}{v})=0 \quad \forall \underset{\sim}{v} \in \underset{\sim}{V}^1_{h_o}\}$$

Then we have $\tilde{Q}^o_h = Q^o_h$

Proof: Clearly enough $V^1_{h_o}$ is the space spanned by the bubble - functions of the elements of $\mathcal{T}_h$. Therefore we have $Q^o_h \subset \tilde{Q}^o_h$.

On the other hand, if $q_h \in Q^1_h$ but $q_h \notin Q^o_h$, the restriction of q_h over any $K \in \mathcal{T}_h$ is of the form $\sum_{i=1}^{3} \alpha^K_i \ x_i$, where α^K_i are given scalars.

Let $\underset{\sim}{v}$ be a nonzero vector field of $\underset{\sim}{V}^1_{h_o}$, and K be an element of $\mathcal{T}_h$ over which $\underset{\sim}{v}$ does not vanish identically. Since $\underset{\sim}{v}/_K = \underset{\sim}{c}^K \psi$, where ψ is the bubble-function of K and $\underset{\sim}{c}^K \in \mathbb{R}^3$ is such that $c^K_i \neq 0$ for some i, $1 \leq i \leq 3$, we have:

$$\int_K x_i \operatorname{div} \underset{\sim}{v}_h \ d\underset{\sim}{x} = \int_K x_i \sum_{j=1}^{3} c^K_j \frac{\partial \psi}{\partial x_j} d\underset{\sim}{x} = - c^K_i \int_K \psi d\underset{\sim}{x} \neq 0, \text{ since } \int_K \psi d\underset{\sim}{x} > 0.$$

Thus for every $q \in Q^1_h$ such that $q \notin Q^o_h$, it suffices to choose $\underset{\sim}{v}$ in such a way that $\underset{\sim}{c}^K = -\underset{\sim}{\alpha}^K$ for every $K \in \mathcal{T}_h$ to have $b_h(q,\underset{\sim}{v})>0$, which proves the Lemma. □ q.e.d.

Now we state two lemmas that lead to the proof of the inf-sup condition (1.9), and use Lemmas 4.1 and 4.2 as assumptions. The proofs of the former follow exactly the same arguments as those given by Stenberg in [10] in Lemma 3.2 and Theorem 3.1, respectively.

<u>Lemma 4.3 ([10] , Lemma 3.2)</u>: $\forall q_h \in Q_h^1$ there exists $\underset{\sim}{v}_h^1 \in \underset{\sim}{V}_h^1$ such that $\underset{\sim}{v}_h^1 \in \underset{\sim}{V}_{h_0}^1$, and:

(4.6) $\quad b_h(q_h, \underset{\sim}{v}_h^1) = b_h(q_h^1, \underset{\sim}{v}_h^1) \geq C_1 \| q_h^1 \|_0^2$

and

(4.7) $\quad \| \underset{\sim}{v}_h^1 \|_h \leq \| q_h^1 \|_0$

where q_h^1 is the orthogonal projection of q_h onto the orthogonal complement Q_h^o with respect to Q_h. □

<u>Lemma 4.4</u> Let $V_h = V_h^1$ and $Q_h = Q_h^1$. Then the inf-sup condition (1.9) holds with β independent of h.

<u>Proof:</u> This Lemma is the direct consequence of (4.1), (4.2), (4.6) and (4.7), and its proof follows the same arguments as in [10], Theorem 3.1. □

Summarizing we have:

<u>Theorem 4.1</u>: If we set $V_h = V_h^i$ and $Q_h = Q_h^i$, where V_h^i and Q_h^i are the spaces defined in Section 2, for i=0 or 1, then (1.9) holds with β independent of h. □

Now, in order to obtain an error estimate for our finite element methods using (1.10) and to establish corresponding convergence results, we first notice that:

(4.8) $\quad \inf_{\underset{\sim}{v} \in \underset{\sim}{V}_h^i} \| \underset{\sim}{u} - \underset{\sim}{v} \|_h \leq Ch^2 |\underset{\sim}{u}|_{i+2} \qquad i=0,1$

(4.9) $\quad \inf_{q \in Q_h^i} \| p-q \|_0 \leq Ch^{i+1} |p|_{i+1} \qquad i=0,1$

according to standard approximation results.

Thus, all that is left to do is to estimate the term $E_h(\underset{\sim}{u},p,\underset{\sim}{w})$, given bu (1.11)

Theorem 4.2: If $\underset{\sim}{u}\in\underset{\sim}{H}^{i+2}(\Omega)$ and $p\in H^{i+1}(\Omega)$, i=0 or 1, then (4.10)

$$E_h(\underset{\sim}{u},p,\underset{\sim}{w}) \le Ch^{i+1}[|\underset{\sim}{u}|_{i+2} + |p|_{i+1}] \, \| \underset{\sim}{w} \|_h$$

Proof: We have:

$$E_h(\underset{\sim}{u},p,\underset{\sim}{w}) = \sum_{K\in\mathcal{T}_h} \int_K [\nu \underset{\sim}{\mathrm{grad}}\, \underset{\sim}{u} \circ \underset{\sim}{\mathrm{grad}}\, \underset{\sim}{w} - p \,\mathrm{div}\, \underset{\sim}{w}] d\underset{\sim}{x} - \int_\Omega \underset{\sim}{f}\cdot\underset{\sim}{w}\, d\underset{\sim}{x}$$

Using Green's formula over each K we obtain

$$E_h(\underset{\sim}{u},p,\underset{\sim}{w}) = \sum_{K\in\mathcal{T}_h} \int_{\partial K} (\nu\frac{\partial u}{\partial n_K} + p\underset{\sim}{n}_K)\circ \underset{\sim}{w}\, ds$$

since $\underset{\sim}{u}$ and p satisfy $-\nu\Delta\underset{\sim}{u} + \underset{\sim}{\mathrm{grad}}\, p - \underset{\sim}{f} = \underset{\sim}{0}$.

According to the regularity assumptions on $\underset{\sim}{u}$ and p, we use the same arguments as in Lemma 3.2 to conclude that

$$E_h(\underset{\sim}{u},p,\underset{\sim}{w}) = \sum_{K\in\mathcal{T}_h} \sum_{F\subset\partial K} \int_F [\nu\frac{\partial \underset{\sim}{u}}{\partial n_K} + p\underset{\sim}{n}_K) - \Pi_F^i(\nu\frac{\partial \underset{\sim}{u}}{\partial n_K} + p\ \underset{\sim}{n}_K\ ]\circ[\underset{\sim}{w} - \Pi_F^o(\underset{\sim}{w})]ds$$

Finally, setting in (3.6) y successively equal to $\frac{\partial u_i}{\partial n_K}$ and pn_{K_i} over each F, and $w=w_i$, i=1,2,3, proceding exactly in the same way as in Lemma 3.2, and then summing up over i, we obtain (4.10) □

Finally we have the following error estimate and convergence result:

Theorem 4.3: If problem ($\mathcal{P}$) is approximated by ($\mathcal{P}_h$) by taking V_h and Q_h to be respectively spaces V_h^i and Q_h^i defined in Section 2, associated with a regular family of partitions $(\mathcal{T}_h)_h$ of Ω assumed to be polyhedric, then whenever the solution $(\underset{\sim}{u},p)$ of

($\mathcal{P}$) belongs to $H^{i+2}(\Omega) \times H^{i+1}(\Omega)$, the unique solution $(\underset{\sim}{u}_h, p_h)$ of ($\mathcal{P}_h$) denoted by (u_h^i, p_h^i) for $V_h = V_h^i$ and $Q_h = Q_h^i$, i=0 or 1, satisfies:

$$\| \underset{\sim}{u} - \underset{\sim}{u}_h^i \|_h + \| p - p_h^i \|_0 \leq Ch^{i+1} [|\underset{\sim}{u}|_{i+2} + |p|_{i+1}]. \quad \square$$

5. CONCLUDING REMARK

As one can infer from the results of this work, together with [8], the right 3D analogues of two-dimensional mixed simplicial finite element methods of the velocity-pressure type for the Stokes problem,with conforming velocity and a discontinuous pressure, are nonconforming in velocity. We intend to illustrate this fact with further examples in a forthcoming paper.

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