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WHAT DO YOU MEAN BY " HERE IS A PROBLEM RELATED TO YOURS . . . " ?

Paulo A. S. Veloso

Departamento de Informática

PONTIFÍCIA UNIVERSIDADE CATÓLICA DO RIO DE JANEIRO
ARQUÊS DE SÃO VICENTE, 225 – CEP-22453
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Abstract

The basic question examined in this paper is : what should one understand by 'related' is assertions such as " Here is a problem related to yours ... ". Problems can be related in various ways, depending on the connexions between them. Each such connexion establishes relations between the solutions of the corresponding problems. Here, some precise formulations for these connexions and the corresponding relations are proposed and examined. A problem is defined as a mathematical structure, its solutions being functions satisfying the problem requirement. The connexions examined are reduction, homomorphism, analogy, and similarity. The aim is gaining a better understanding of these important notions by means of a precise investigation of their properties.

Key words : Problem solving, theory of problems, relations between problems, reduction, homomorphism, analogy, similarity, transfer of solution, behaviour of problems, heuristics.

Resumo

A questão básica examinada neste trabalho é : o que se deve entender por 'relacionado' em afirmativas como " Eis um problema relacionado com o seu ... ". Problemas podem se relacionar, uns com os outros, de diversas maneiras, dependendo das conexões entre eles, cada uma destas estabelecendo relações entre as soluções dos problemas correspondentes. Aqui, algumas formulações precisas para essas conexões e relações correspondentes são propostas e examinadas. Um problema é definido como uma estrutura matemática, suas soluções sendo funções que satisfazem a condição do problema. São examinadas as conexões redução, homomorfismo, analogia e similaridade, com o objetivo de se obter um melhor entendimento dessas noções importantes por meio de uma investigação precisa de suas propriedades.

Palavras chaves : Resolução de problemas, teoria de problemas, relações entre problemas, redução, homomorfismo, analogia, similaridade, transferência de soluções, comportamento de problemas, heurística.

Contents

1. Introduction

The basic question examined in this paper is : What should one understand by 'related' in assertions such as " Here is a problem related to yours... " ? This quotation from Polya [1971] refers to a powerful heuristical strategy in problem solving. The aim of this paper is proposing some appropriate precise formulations for this vague, but extremely useful, idea.

The structure of this paper is as follows. In the next section some ideas from Heuristics and Program Specification are employed to give a precise definition of a problem as a mathematical structure - consisting of domains of data and of results and a relation , the requirement, between them - and of solutions as certain functions. Then, in section 3, we examine reduction of a problem to another one - an archetypical case being the method of Cartesian coordinates - as a first relationship between problems, motivated by the desire to transfer solutions. Section 4, on the other hand, considers another kind of link between problems, motivated by the concept of homomorphism as a structure-preserving map. Then, in section 5 we put these concepts and results into perspective, paving the way for section 6, where analogy and similarity are introduced as generalisations of the previous concepts.

Our exam of these concepts follows a general pattern. First, a structural link is introduced, which induces a relation between the functions of the problems. Then, we introduce a concept of preservation of requirements, which establishes some behavioural relation between the solutions of the problems. The overall aim of this investigation is seeing when we can transfer information (mainly concerning solutions) about a problem to another related one.

2. Problems and solutions

We can say, with Polya [1971], that a problem cannot really be said to be completely specified until we have the following information : what are the given data, what are the expected results, and what is the problem condition, i. e. which results correctly match which data. For instance, in order to specify a problem, it is not enough to enunciate it just as " finding roots of polynomials "; one should be more definite about the kinds of polynomials : are they, say, quadratic or can they have arbitrary degree, do they have integral or real coefficients; also, what are the expected results, do we want integral roots or complex roots; and, finally , a polynomial in general has more than one root, do we want all of them, any one of them, or perhaps the smallest one is the one we want.

The basic structural ingredients of a problem are thus data, results and a condition or requirement connecting them. We capture this intuitive idea by defining a <u>problem</u> as [Veloso+Veloso 1981] a mathematical struture **P** = $\langle D, R, p \rangle$, where

D is a nonempty set, the **domain of data**;

R is a nonempty set, the **domain of results**;

p is a binary relation from D to R , the **problem requirement**.

This formulation of the concept of problem can be regarded as an abstraction arising from ideas related to program specification [Manna, 1974]. Accordingly, much as a problem is an abstraction of the input-output specification of a program, a solution is an abstraction of a program, namely the input-oputput transformation effected by it. Thus, we define a solution for a problem P to be a (total) function f from D to R assigning to each data d a result f(d) satisfying the problem requirement in that $\langle d,f(d)\rangle \in p$.

We have defined a problem as a mathematical structure. We may think of the behaviour of a problem as consisting of all its possible solutions. In order to state this more precisely we introduce the following definitions. Consider a problem P = $\langle D,R,p\rangle$, the function space of P is the set F(P) of all the functions from D to R ,and the solution space of P is the set S(P) of all its solutions, i. e. S(P) = { f:D--->R / $f \subseteq p$ }. Also, we call a problem P solvable iff it has at least one solution, i. e., S(P) ≠ ∅ .

As an example, consider the problem of " finding the center of a triangle in the plane ". We can formulate it, according to our definition, as follows. Its data domain D consists of all the triangles in the plane, its domain R of results consists of all the points in the plane, and the problem requirement is a relation p such that for all triangles d in D and all points r in R $\langle d,r\rangle$ is in p iff r is the center of d . A solution for this problem P is a function assigning to each triangle its center. Thus, we are considering the general problem of " finding centers of (all) triangles ", rather than any particular problem, such as that of " finding the center of this specific triangle ".

As another example we can formulate the problem of "finding a complex root of a polynomial with real coefficients and arbitrary degree " as follows. Its data domain E consists of all such polynomials, its result domain S is the set of complex numbers, and the problem requirement is the relation q defined by <p(x),c> is in q iff p(c) = 0 . A solution for this problem **Q** is a function g:E--->S, assigning to each polynomial one of its complex roots. Thus, a problem may very well have several solutions.

3. Reduction of problems

Reduction embodies the perhaps most intuitive idea of transfering solütions, namely the so-called "transform-solve-invert" technique [Eves 1983] . A very familiar example is provided by Descartes's analytic geometry : given an instance of a geometric problem, we first transform it into corresponding algebraic equations, these are then solved for a result which is subsequently translated back into a result for the original geometric problem instance.

In order to give a more precise description of the concepts underlying these intuitive ideas, let us consider two problems **P** = $\langle D,R,p \rangle$ and **Q** = $\langle E,S,q \rangle$. A <u>reduction link</u> from **P** to **Q** is a pair of functions, a **forward transform** a:D--->R , and a **backward transform** b:S--->R . By means of this pair of functions any function g:E--->S can be transformed into the composite b.g.a:D--->R . This assignment g ---> b.g.a defines a mapping **t** : F(**Q**) ---> F(**P**), called the induced <u>retrieval</u> from **Q** to **P** .

Notice that the above retrieval mapping is obtained by mere functional composition, with no regard to the problem requirements. We still have to examine when the basic aim of reduction, transfer of solutions, is actually achieved. For this purpose we introduce the following concept of preservation of requirements [Veloso 1980]

Consider a reduction link $\langle a,b\rangle$ from **P** to **Q** as above. We call such a link a **reduction** iff for all $d \in D$ and $s \in S$ if $\langle a(d),s\rangle \in q$ then $\langle d,b(s)\rangle \in p$. The next proposition shows that this concept of preservation of requirements is what we need in order to be able to transfer solutions.

Proposition. Consider a reduction $\langle a,b\rangle$ from **P** to **Q** with induced retrieval **t** . Let $f{:}D \dashrightarrow R$ and $g{:}E \dashrightarrow S$ be functions such that $f = \mathbf{t}(g)$; then, if $g \in S(\mathbf{Q})$ then $f \in S(\mathbf{P})$. In other words, $\mathbf{t}[S(\mathbf{Q})] \subseteq S(\mathbf{P})$. In particular, if **Q** is solvable then so is **P** .

Proof. Given $d \in D$,as $g \in S(\mathbf{Q})$, we have $\langle a(d),g[a(d)]\rangle \in q$, whence by preservation of requirements, $\langle d,b(g[a(d)])\rangle \in p$. Hence $f = b.g.a \in S(\mathbf{P})$ .QED

4. Morphisms between problems

Problems have been defined as (two-sorted) mathematical structures and from a mathematical point of view homomorphisms, as structure preserving maps, establish natural structural relationships among structures. So we now turn to examine morphisms between problems.

Consider problems **P** and **Q** as before. A <u>morphism link</u> from **P** to **Q** consists of a pair of functions , a **data map** i:D--->R and a **result map** j:R--->S . We have seen in the preceding section that a reduction link induces naturally a mapping between the function spaces of the problems involved. Now, a morphism link will induce naturally a relation between the function spaces, instead of a mapping, as made explicit in the next definition.

Consider a morphism link <i,j> from **P** to **Q** . We shall say that a function g:E--->S is <u>conjugate</u> to a function f:D--->R , denoted by f **k** g , iff g.i = j.f , i. e. the natural diagram involving these four functions commutes. Since conjugacy is a relation from F(**P**) to F(**Q**) , rather than a function, two natural questions concerning its domain and image, are examined in the next lemma.

Lemma. Consider a morphism link <i,j> from **P** to **Q** and let **k** be the induced conjugacy relation. If i:D--->E is injective then for each f ∈ F(**P**) there exists g ∈ F(**Q**) such that f **k** g . If j:R--->S is surjective then every g in F(Q) has a conjugate f in F(P)

Proof. Immediate from the fact that in this case i has a left inverse and j has a right inverse. QED

The relation of conjugacy concerns only the domains of the problems, as in the case of reduction link, and not their requirements. We now have to introduce appropriate notions for preservation of requirements. It will be seen that we have more than one such notion.

We call a morphism link <i,j> from **P** to **Q** a homomorphism from **P** to **Q** iff for all d in D and r in R if <d,r> ∈ p then <i(d),j(r)> ∈ q . The next proposition establishes transferal of solutions along conjugacy. We employ the following notation, which will be used often : given a relation m from A to B and subsets X of A and Y of B then

Xm = { b ∈ B / <x,b> ∈ m , for some x ∈ X } ,

mY = { a ∈ A / <a,y> ∈ m , for some y ∈ Y } .

Proposition. Consider a homomorphism <i,j> from to **Q** with a surjective data map i and let **k** be the induced conjugacy relation from F(**P**) to F(Q) . Let f k g , then if f ∈ S(**P**) then g ∈ S(Q) . In other words, S(**P**)k ⊆ S(Q) .

Proof. Given e in E, as i is onto, there exists d in D such that e = i(d) . Since f ∈ S(P) , <d,f(d)> ∈ p , whence, by the definition of homomorphism, <i(d),j[f(d)]> ∈ q . Thus, as f k g , we have <i(d),g[i(d)]> ∈ q , whence <e,g(e)> ∈ q . QED

Another somewhat natural candidate for an appropriate notion of preservation of requirements is obtained by reversing the implication in the definition of homomorphism, as follows.

A morphism link ⟨i,j⟩ from **P** to **Q** will be called a <u>conformism</u> from **P** to **Q** iff for all d in D and r in R , if ⟨i(d),j(r)⟩ ∈ q then ⟨i,j⟩ ∈ p . A conformism will also allow transferal of solutions, as made explicit in the next proposition, only in the reverse direction when compared with a homomorphism.

Proposition. Consider a conformism ⟨i,j⟩ from **P** to **Q** with induced conjugacy relation **k** . Let f **k** g , then if g ∈ S(**Q**) then f ∈ S(**P**) . In other words, **k**S(**Q**) ⊆ S(**P**) .

<u>Proof</u>. Given d in D , as g ∈ S(**Q**) , ⟨i(d),g[i(d)]⟩ ∈ q , whence, as f **k** g , ⟨i(d),j[f(d)]⟩ ∈ q , so ⟨d,f(d)⟩ ∈ p , by the definition of conformism. QED

A concept often found in the literature is the combination of homomorphism and conformism. A morphism link is a <u>strong homomorphism</u> iff it is both a homomorphism and a conformism.

Corollary. Consider a strong homomorphism ⟨i,j⟩ from **P** to **Q** with surjective data map i and let **k** be the induced conjugacy relation. If f **k** g then f ∈ S(**P**) iff g ∈ S(**Q**) . In other words, the conjugacy relation **k** refines the relation of equisolubility.

Putting the preceding results together we obtain the following theorem, which is stated so as to emphasise transferal of solvability.

Theorem. Consider a morphism link $\langle i,j \rangle$ from **P** to **Q**.

(a) If $\langle i,j \rangle$ is a homomorphism with bijective i then $j.f.i^{-1} \in S(Q)$ whenever $f \in S(P)$. In particular, if **P** is solvable then so is **Q**.

(b) If $\langle i,j \rangle$ is a conformism with surjective j then every g in $S(Q)$ is conjugate to some f in $S(P)$. In particular, if **Q** is solvable then so is **P**.

(c) If $\langle i,j \rangle$ is a strong homomorphism with bijective i then $f \in S(P)$ iff $j.f.i^{-1} \in S(Q)$. In particular, **P** is solvable iff **Q** is so.

5. General relations between problems

Let us briefly put into perspective the contents of the preceding sections. We have been considering a pair of problems P = <D,R,p> and Q = <E,S,q> related in some way. The way these problems are related is via some "structural link" l from P to Q. We have been considering structural links consisting of pairs of functions. Each such structural link induces a link L between the corresponding function spaces, i. e. L is a relation from F(P) to F(Q) , induced by the structural link alone, without considering the problem requirements. Then, we introduced some concept of preservation of requirements, which allows the relation L to induce a "behavioural link" B between the solution spaces. In this context we have been examining two kinds of questions, aiming at transferal of solutions.

1) Given f in F(P) can we find g in F(Q) so that f L g , or vice-versa ?
2) Assuming that f L g , from the fact that f is a solution for P (under what conditions) can we conclude that g is a solution for Q , or vice-versa ?

Each kind of link we have considered so far consists of a pair of functions, the difference lying in their "directions". Also, the various concepts of preservation of requirements resemble each other, appearing to be variations on a common theme. Our next step will be to examine this common generalisation. Since the difference between the links is due to the directions of the functional arrows, a natural way to get around this consists of replacing the functions by relations. For this purpose we will have to recall some concepts and notations pertaining to relations [Tarski 1941] .

Consider (binary) relations m from A to B and n from B to C . As usual we write xmy to mean $\langle x,y \rangle \in m$. The *converse* of m is the relation $\breve{m}$ from B to C defined by

$$\breve{m} = \{ \langle y,x \rangle \in B \times A \ / \ xmy \} .$$

The (*relative*) *product* of m by n is the relation m;n from A to C defined by

$$m;n = \{ \langle x,z \rangle \in A \times C \ / \ \text{for some } y \in B \ xmy \text{ and } ynz \} .$$

We also have a special relation, the *identity relation* on A , denoted by 1(A) . Some properties [Tarski 1947] of these concepts are given in the next lemma.

Lemma. With the above notation :

(a) the domain of m is all of A (Domm = A) iff $1(A) \subseteq m;\breve{m}$;

(b) the range of m is all of B (Imm = B) iff $1(B) \subseteq \breve{m};m$;

(c) m is a (partial) function from some subset of A into B iff $\breve{m};m \subseteq 1(B)$.

We recall that we can regard (the graph of) a function as a (special) relation; then the relative product f;g of two functions is (the graph of) their functional composition g.f . This enables us to restate in the language of relations the concepts of preservation of requirements previously introduced , which is the content of the next proposition.

Proposition. Consider problems **P** and **Q** as before.

(a) A reduction link $\langle a,b \rangle$ from **P** to **Q** is a reduction from **P** to **Q** iff $b;q;a \subseteq p$.

(b) A morphism link $\langle i,j \rangle$ from **P** to **Q** is

a homomorphism iff $p;j \subseteq i;q$, and

a conformism iff $i;q \subseteq p;j$.

Proof.

(a) We have $d\ p\ b(s)$ whenever $a(d)\ q\ s$ iff $d\ p\ b(s)$ whenever $d\ a;q\ s$ iff whenever $d\ a;q;b\ r$ then $d\ p\ r$ (since $b:S \text{---}{>} R$).

(b) For the first case, we have $i(d)\ q\ j(r)$ whenever $d\ p\ r$ iff $d\ i;q\ j(r)$ whenever $d\ p\ r$ iff whenever $d\ p;j\ s$ then $d\ i;q\ s$ (since $j:R \text{---}{>} S$) . The other case follows from the same kind of reasoning. QED

6. Analogy and similarity of problems

We can now introduce the generalised versions of our notions of "relatedness" between problems.

Consider problems $\mathbf{P} = \langle D,R,p\rangle$ and $\mathbf{Q} = \langle E,S,q\rangle$, as before. An <u>analogy link</u> from $\mathbf{P}$ to $\mathbf{Q}$ consists of two relations, a **data relation** v from D to E , and a **result relation** w from R to S .

An analogy link induces a relation between the function spaces of the problems involved. We call $g \in F(\mathbf{Q})$ <u>functionally analogous</u> to $f \in F(\mathbf{P})$, denoted by $f\ \mathbf{h}\ g$, iff $f;w \subseteq v;g$. One way to see that this definition is a reasonable generalisation of the preceding ones is by seeing whether it does specialise as expected. This is the content of the next lemma.

Lemma. Consider an analogy link $\langle v,w\rangle$ from $\mathbf{P}$ to $\mathbf{Q}$ as above.

(a) If v and w are functions then $\langle v,w\rangle$ is a morphism link, and in this case $f\ \mathbf{h}\ g$ iff $f\ \mathbf{k}\ g$.

(b) If v and $\tilde{w}$ are functions then $\langle v,\tilde{w}\rangle$ is a reduction link, and in this case $f\ \mathbf{h}\ g$ iff $f = \mathbf{t}(g)$.

<u>Proof</u>.

(a) We have $f\ \mathbf{h}\ g$ iff $f;w \subseteq v;g$ iff $f;w = v;g$ iff $w.f = g.v$ iff $f\ \mathbf{k}\ g$.

(b) If $f\ \mathbf{h}\ g$ then $f;w \subseteq v;g$, so $f;w = v;g$. Thus, $v;g;\tilde{w} = f;w;\tilde{w} \subseteq f$, so $v;g;\tilde{w} = f$, whence $f = \tilde{w}.g.v$. Conversely, if $f =$

$\breve{w}.g.v$ then $f = v;g;\breve{w}$, so $f;w = v;g;\breve{w};w \subseteq v;g$, whence f **h** g . QED

We now introduce our generalised version of preservation of requirements. An analogy link $\langle v,w \rangle$ from **P** to **Q** is called an <u>analogy</u> from **P** to **Q** iff $v;q \subseteq p;w$. Again we can see that this concept specialises to the previous ones as expected.

Proposition. Consider an analogy link $\langle v,w \rangle$ from **P** to **Q** .

(a) If both v and $\breve{w}$ are functions then $\langle v,w \rangle$ is an analogy iff $\langle v,\breve{w} \rangle$ is a reduction from **P** to **Q** .

(b) If both v and w are functions then $\langle v,w \rangle$ is an analogy iff it is a conformism from **P** to **Q** .

(c) If both $\breve{v}$ and $\breve{w}$ are functions then $\langle v,w \rangle$ is an analogy iff $\langle \breve{v},\breve{w} \rangle$ is a homomorphism from **Q** to **P** .

<u>Proof</u>.

(a) Assume that v and $\breve{w}$ are functions.

(==>) If $v;q \subseteq p;w$ then $v;q;\breve{w} \subseteq p;w;\breve{w} \subseteq p$.

(<==) If $v;q;\breve{w} \subseteq p$ then $v;q \subseteq v;q;\breve{w};w \subseteq p;w$.

(b) Immediate.

(c) Assume that $\breve{v}$ and $\breve{w}$ are functions.

(==>) First, as $\mathrm{Dom}\breve{v} = E$, $q;\breve{w} \subseteq \breve{v};v;q;\breve{w}$. But, since $\langle v,w \rangle$ is an analogy, $v;q \subseteq p;w$. Hence, $q;\breve{w} \subseteq \breve{v};p;w;\breve{w}$ $\subseteq \breve{v};p$, since $\breve{w}$ is a function.

(<==) First, since $\mathrm{Dom}\breve{w} = S$, $v;q \subseteq v;q;\breve{w};w$. But, since $\langle \breve{v},\breve{w} \rangle$ is a homomorphism, $q;\breve{w} \subseteq \breve{v};p$. Thus, $v;q \subseteq v;\breve{v};p;w$ $\subseteq p;w$, since $\breve{v}$ is a function . QED

These results suggest that the concept of analogy is an appropriate common generalisation of reduction, homomorphism and conformism. A result pertaining to analogy per se is the next one.

Theorem. Consider an analogy $\langle v,w \rangle$ from to **Q** such that $\tilde{w}$ is a partial function with image R . So, if $f\ \mathbf{h}\ g$ then $f \in S(P)$ whenever $g \in S(Q)$. In other words $\mathbf{h}S(Q) \subseteq S(P)$.

Proof. Consider $f\ \mathbf{h}\ g$, so $f;w \subseteq v;g$. Let $g \in S(Q)$, then $g \subseteq q$. Thus, $f;w \subseteq v;q$, whence $f;w \subseteq p;w$, as $\langle v,w \rangle$ is an analogy. So, $f;w;\tilde{w} \subseteq p;w;\tilde{w}$. Now, as $Im\tilde{w} = R$, $f \subseteq f;w;\tilde{w}$ and as $\tilde{w}$ is a function, $p;w;\tilde{w} \subseteq p$. Therefore $f \subseteq p$, i. e., $f \in S(P)$ .QED

Thus, we have seen that the concept of analogy is indeed a common generalisation of those introduced in the previous sections. The reduction link we have considered consists of functions a:D--->E and b:S--->R and might be called an "uncoupled link". However, in some contexts it may be of interest to employ more powerful versions of reduction link. One such possibility is the so-called "loosely coupled" link, with maps a:D--->E , as before , but b:ExS--->R, meaning that the backward transform can also use information about a data, in addition to a result, in translating back to the original problem [Veloso+Martins 1984].

In order to encompass the above cases of reduction as well as other ones we may consider a more general link between problems. As a tentative definition we suggest considering a <u>similarity link</u> from problem **P** to problem **Q** to be a quaternary relation $u \subseteq D\times R\times E\times S$. It is clear then that an analogy link $\langle v,w \rangle$ is just the special case when we have $\langle d,r,e,s \rangle$ in u iff $\langle d,e \rangle \in v$ and $\langle r,s \rangle \in w$. However, this notion of similarity, in part due to its generality, is still in a preliminary stage of development. In particular, it is not very clear what is a natural formulation for the concept of preservation of requirements in this case.

7. Conclusion

We have examined various possible ways to make precise the vague, but extremely fruitful, notion of "relatedness" between problems, which were precisely defined as mathematical (two-sorted) structures,consisting of domains of data and of result and a relation of requirement between them. The basic idea is that two problems are related when there is some structural link between them. This structural link then induces a relation between the corresponding function spaces. We then introduce the concept of a structural link preserving problem requirements, in order to see when the above relation yields one between the solution spaces of the problems.

We have examined first two functional links. The first one, yielding reductions, corresponds to the intuitive idea of transfering the solution of a problem to another one related to it; in this case we actually obtain a function mapping a solution space into the oher one. The second functional link examined, yielding homomorphism and variations thereof, corresponds to the idea of structural similarity. A common generalisation of these concepts was then introduced - analogy - via a relational link and the previous concepts were reformulated and generalised to the realm of relations, in lieu of functions. Finally, an even more general kind of link - similarity - was tentatively introduced.

The overall aim is gaining a better, and more precise, understanding of these notions, in order to establish more clearly their domains of applicability.

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