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A NEW THEORETICAL APPROACH TO RUNGE-KUTTA METHODS*

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ABSTRACT:

RK-methods are treated here as linear multistage methods within the author's concept of A-methods. As a consequence, the traditional difference in treatment of RK-methods and linear multistep methods vanishes; in particular, it is not necessary to consider elementary differentials. It is shown how the order of a RK-method depends on the error constants of its stages, and the conditions (nonlinear equations) for order p are generated from a very simple recursion.

KEYWORDS:

Integration of ordinary differential equations, Runge-Kutta Methods.

RESUMO:

Nesta monografia, os métodos de Runge-Kutta(RK) são tratados como métodos lineares compostos usando-se o conceito dos métodos-A. Assim, as diferenças no tratamento dos métodos de RK e dos métodos lineares de passo k desaparecem; em particular, não é mais necessário considerar "diferencias elementares" que complicam bastante a teoria clássica. É demonstrado como a ordem de um método de RK depende das constantes de erro de seus níveis, e as condições para uma determinada ordem de convergência p são obtidas para uma recursão simples.

PALAVRAS-CHAVE

Integração de equações diferenciais ordinárias, Métodos de Runge-Kutta.

1. Introduction

Multistage methods are effectively analyzed in their "A-method form", i. e. presented as

$$z_o=z_o(h);\ z_k=Az_{k-1}+h\ \hat{\phi}(x_{k-1},\ z_{k-1},\ z_k;\ h),\quad k=1(1)m_h$$

$$z_o\in \mathbb{R}^s,\ A\in \mathbb{R}(s,s),\ \hat{\phi}:\ [\alpha,\beta]\times\mathbb{R}^s\times\mathbb{R}^s\times[o,h_o]\to\mathbb{R}^s\ .$$

In earlier papers [1,2] , a general theory of such methods has been developed which proved quite useful for the analysis of linear cyclic methods, Nordsieck forms, error estimation and accumulation, error control, problems of order of convergence, etc. Essential parts of this theory are based on the assumption that all stages of the method have (at least) order p.

In this paper, we extend the theory by dropping this assumption. The technique presented here applies to any A-method with stages with different orders; however, we restrict this analysis to Runge-Kutta methods as the most important special case. They are treated here as <u>composite linear multistep methods</u> which leads to a theoretical approach, where the traditional difference in treatment between Runge-Kutta methods and linear multistep methods disappears. As a result, it is not necessary to consider elementary differentials (although they appear implicitly in the analysis), and the conditions for order are obtained, in terms of the error constants of the stages, from a very simple recursion (Theorem 4.6) which represents the main result of this paper.

Linear multistage methods with stages of different orders have also been considered by Cooper [5]. He obtains conditions for order p under the restriction that the difference in order of consistency of the stages must not exceed (p-1) which covers Runge-Kutta methods up to order 5.

2. Runge-Kutta Methods

2.1. Given the initial value problem

(2.1) $\quad Y' = f(x,Y) \; ; \; Y(x_o) = \eta_o \; ; \quad \eta_o \in \mathbb{R}^n$

Let $f: [\alpha,\beta] \times \mathbb{R}^n \to \mathbb{R}^n$ have $(p+1)$ continuously differentiable partial derivatives.

Define $Z_o := 0 \oplus \ldots \oplus 0 \oplus \eta_o \in \mathbb{R}^{(s+1)n}$, $\quad 0$: zero vector in $\mathbb{R}^n$,

(2.2) $\quad Z_{k+1} := Y_{k+a_1} \oplus \ldots \oplus Y_{k+a_{s+1}} \in \mathbb{R}^{(s+1)n}$

where $\quad Y_{k+a} := Y(x_k+ah) \in \mathbb{R}^n \; ; \; \mathbb{R}^{sn} := \mathbb{R}^n \times \ldots \times \mathbb{R}^n$

$\qquad u \oplus v := (u,v)^T \in \mathbb{R}^{2n}$ for $u,v \in \mathbb{R}^n$.

2.2. We consider one-step multistage integration methods of the form

$$y_o = y_o(h) \qquad \text{(usually } y_o = \eta_o)$$

$$\text{(2.3)} \quad \begin{aligned} y_{k+a_i} &= y_k + h \sum_{j=1}^{s} b_{ij} f(x_k + a_j h, y_{k+a_j}) , && i=1(1)s , \\ y_{k+a_{s+1}} &= y_k + h \sum_{j=1}^{s} b_j f(x_k + a_j h, y_{k+a_j}) && k=0,1, \ldots \end{aligned}$$

with

(2.4) $\quad a_i = \sum_{j=1}^{s} b_{ij} \quad , \; i=1(1)s.$

We assume throughout the normalization $a_{s+1}=1$ which can always be achieved by the substitution $h \leftarrow ch$ with appropriate $c>o$.

In correspondence to (2.2) we define vectors of approximations by

$$z_o(h) := 0 \oplus \dots \oplus 0 \oplus y_o(h)$$

$$z_{k+1} := y_{k+a_1} \oplus \dots \oplus y_{k+a_{s+1}}.$$

Then (2.3) can be represented by the A-method [2]:

(2.5) $\quad z_o = z_o(h); \; z_{k+1} = \underline{A} z_k + h \underline{K} \Phi(x_k, z_k, z_{k+1}; h) \; ; \quad k = o, 1, \dots$

where $\underline{A} := A \otimes I$, $\underline{K} = K \otimes I$, $I \in \mathbb{R}(n,n)$ identity,

$$\Phi(x_k, z_k, z_{k+1}; h) := f(x_k + a_1 h, y_{k+a_1}) \oplus \dots \oplus f(x_k + a_{s+1} h, y_{k+a_{s+1}})$$

and $\quad A \otimes B := (a_{ij}B) \in \mathbb{R}(mn, mn)$ for $A = (a_{ij}) \in \mathbb{R}(m,m)$, $B \in \mathbb{R}(n,n)$.

A and K are (s+1)x(s+1)-matrices

$$A = \begin{pmatrix} 0 & \dots & 0 & 1 \\ & \dots\dots & & \\ 0 & \dots & 0 & 1 \\ 0 & \dots & 0 & 1 \end{pmatrix} \quad ; \qquad K = \begin{pmatrix} b_{11} & \dots & b_{1,s} & 0 \\ & \dots\dots\dots & & \\ b_{s,1} & \dots & b_{s,s} & 0 \\ b_1 & \dots & b_s & 0 \end{pmatrix}$$

The matrix K is often called the <u>generating matrix</u> of (2.3), and we call A a <u>Runge-Kutta matrix</u>.

Usually the method is represented by the parameter scheme

a_1	b_{11}	b_{12}	...	$b_{1,s}$
a_2	b_{21}	b_{22}	...	$b_{2,s}$
$\vdots$	....			
a_s	$b_{s,1}$	$b_{s,2}$	...	$b_{s,s}$
a_{s+1}	b_1	b_2	...	b_s

a tableau which is conveniently represented by arrays in the form

$$\begin{array}{c|c} a & B \\ \hline a_{s+1} & b^T \end{array}$$

where $a := (a_1, \ldots, a_s)^T$; $b := (b_1, \ldots, b_s)^T$; $a_{s+1} \in \mathbb{R}$.
a_{s+1} is omitted in the normal case when $a_{s+1} = 1$.

2.3. Furthermore, we define the matrix $D \in \mathbb{R}(s,s)$

$$D := \operatorname{diag}(a_1, \ldots, a_s)$$

and the vectors

$$e := (1, \ldots, 1)^T \in \mathbb{R}^s$$
$$e^* := (1, \ldots, 1)^T \in \mathbb{R}^n.$$

Throughout this paper, we denote componentwise multiplication of two vectors $u, v \in \mathbb{R}^s$ by

$$u \cdot v := (u_1 v_1, \ldots, u_s v_s)^T ;$$

and

$$u^r := (u_1^r, \ldots, u_s^r)^T .$$

Sometimes it is convenient to write Du instead of $a \cdot u$, $a, u \in \mathbb{R}^s$.

2.4. <u>Definition</u>. Method (2.5) is a s-stage <u>Runge-Kutta method</u> (RK-method). It is <u>explicit</u> if $b_{ij} = 0$ for $j \geq i$, $i = 1(1)s$; otherwise it is <u>implicit</u>, in particular <u>semi-implicit</u> if $b_{ij} = 0$ for $j > i$ with $b_{ii} \neq 0$ for at least one index i.

In the implicit case, the method may have, in fact, (s+1) stages. We maintain the above definition of the stages for historical reasons but it would be more consistent to define the number of stages independent of the dimension s (similar to the proceeding in [3], sec.2.5.).

Methods of the semi-implicit type are frequently used to start linear multistep procedures. In such a case we have $a_1=0$ and a_i, $i=2(1)s+1$, are not known a priori but calculated in dependence of the problem and the required precision. Thus, a scheme of the following type would be useful

$$\begin{array}{c|ccc} 0 & 0 & 0 & 0 \\ a_2 & a_2/2 & a_2/2 & 0 \\ a_3 & a_2/2 & a_3/2 & (a_3-a_2)/2 \\ \hline a_4 & b_1 & b_2 & b_3 \end{array}$$

It has order $p=3$ if

$$\begin{aligned} b_1 &= (a_4-b_2-b_3) \\ b_2 &= a_4(3a_3-2)/6a_2(a_3-a_2) \\ b_3 &= a_4(2-3a_2)/6a_3(a_3-a_2). \end{aligned}$$

There is no reason why the last stage should not be implicit just as the other stages; however, the method then would not fall into the class (2.3). We shall come back to this point in paragraph 7.

2.5. Method (2.5) is a one-step procedure, but each of its stages represents a linear multistep formula (on a non-equidistant grid). Thus, it seems natural to analyze it within the concepts of <u>composite linear multistep methods</u>. This demands only simple, one-dimensional Taylor developments (mainly to obtain the error constants of the stages). Consideration of more complicated elementary differentials (which play an essential role in the classical analysis of RK-methods) can be avoided, they appear only implicitly in the analysis.

3. Basic Definitions

3.1. Definition [2]. The local discretization error $d(x_{k+1}) \in \mathbb{R}^{(s+1)n}$ of (2.5) at $x=x_{k+1}$ is defined by

$$(3.1) \qquad Z_{k+1} = \underline{A} Z_k + h\underline{K}\Phi(x_k, Z_k, Z_{k+1}; h) + hd(x_{k+1})$$

and the global error $q(x_k)$ at $x=x_k$ is $q(x_k) := (Z_k - z_k)$, $k=0,1,\ldots$

The components of $d(x_{k+1})$ are the local discretization errors of the stages of (2.3). By Taylor development of each stage we obtain

$$(3.2) \qquad hd(x_{k+1}) = \begin{pmatrix} c_{11} Y'h + c_{12} Y''h^2 + c_{13} Y''' h^3 + \ldots \\ c_{21} Y'h + c_{22} Y''h^2 + c_{23} Y''' h^3 + \ldots \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ c_{s,1} Y'h + c_{s,2} Y''h^2 + c_{s,3} Y''' h^3 + \ldots \\ \hat{c}_1 Y'h + \hat{c}_2 Y''h^2 + \hat{c}_3 Y''' h^3 + \ldots \end{pmatrix}$$

where $Y^{(i)} := \frac{d^i Y}{dx^i}(x_k) \in \mathbb{R}^n$ and

$$(3.3) \qquad c_{ij} = \frac{1}{j!} a_i^j - \frac{1}{(j-1)!} \sum_{\ell=1}^{s} b_{i\ell}\, a_\ell^{j-1} .$$

The c_{ij}, $j=1(1)(p+1)$, denote the error factors of the i-th stage, $i=1(1)s$, (we reserve the name "error constant" to the first non-vanishing error factor $c_{i,p+1}$ in a stage, divided by $\sum_{\ell=1}^{s} b_{i\ell}$).

Note that $c_{i1}=0$, $i=1(1)s$, (consistency) is equivalent to

$$(3.4) \qquad a = Be ,$$

which is satisfied due to (2.4).

In the last component of (3.2) we have

$$\hat{c}_j = \frac{1}{j!} a_{s+1}^j - \frac{1}{(j-1)!} \sum_{\ell=1}^{s} b_\ell a_\ell^{j-1}$$

$\hat{c}_j = o$, $j=1(1)p$, is equivalent to

(3.5) $\quad b^T a^{j-1} = \frac{1}{j}$, $j=1(1)p \quad$ (since $a_{s+1}=1$).

We define the last stage to have at least <u>order of consistency</u> p if (3.5) holds. (3.5) defines a <u>quadrature formula</u> of order p over the interval $[x_k, x_{k+1}]$ with (usually non-equidistant) grid points $(x_k+a_i h)$, $i=1(1)s+1$.

Defining vectors $c_j \in \mathbb{R}^s$ by

(3.6) $\quad c_j := (c_{1j}, c_{2j}, \ldots, c_{s,j})^T$

we obtain from (2.4) and (3.3):

(3.7)
$$c_1 = o$$
$$c_j = \frac{1}{j!} a^j - \frac{1}{(j-1)!} Ba^{j-1} \text{ , } j = 2(1)(p+1) \text{ .}$$

3.2. Equations (2.5) and (3.1) yield:

(3.8) $\quad q(x_{k+1}) = \underline{\mathbb{A}} q(x_k) + h \underline{\mathbb{K}} t(x_{k+1}) + hd(x_{k+1})$;

with

(3.9) $\quad t(x_{k+1}) = (t_{1,k+1}, \ldots, t_{s+1,k+1})^T$

$$t_{j,k+1} = f(x_k + a_j h, Y_{k+a_j}) - f(x_k + a_j h, y_{k+a_j}) \in \mathbb{R}^n$$

By induction from (3.8):

(3.1o) $\quad q(x_{k+1}) = \underline{\mathbb{A}}^{k+1} q(x_o) + h \sum_{\ell=o}^{k} \underline{\mathbb{A}}^{k-\ell} d(x_{\ell+1}) + h \sum_{\ell=o}^{k} \underline{\mathbb{A}}^{k-\ell} \underline{\mathbb{K}} t(x_{\ell+1})$;

$k=o,1,\ldots$

3.3. In what follows, we frequently use the notation $g(x_{k+1})=O(h^p)$ to indicate that $g(x_{k+1})=g(x_{k+1},h)\in\mathbb{R}^m$ "has order p (as $h\to o$)". This should be interpreted as

$$g(h) := \sup_{\alpha<x_{k+1}\leq\beta} |g(x_{k+1})| \leq Ch^p$$

for a constant $C>o$ and all sufficiently small $h>o$. Note that $g(x_{k+1})=O(h^p)$ implies $g(x_{\ell+1})=O(h^p)$, $\ell=0(1)k$.

In the next paragraph, we discuss the case of a single differential equation, i.e. $n=1$, and generalize the results later to $n>1$.

4. The General Order Condition for RK-Methods

4.1. With n=1, we consider the first s equations in (3.1o) separately from the last one in order to determine the matrix B such that for adequately chosen vector b the last component of $\dot{q}(x_{k+1})$ has order p (h→o).

Define $q_k, d_k, t_k \in \mathbb{R}^s$; $\hat{q}_k, \hat{d}_k, \hat{t}_k \in \mathbb{R}$ by

$$(4.1) \quad q(x_k) = \begin{pmatrix} q_k \\ \hat{q}_k \end{pmatrix} \; ; \quad d(x_k) = \begin{pmatrix} d_k \\ \hat{d}_k \end{pmatrix} \; ; \quad t(x_k) = \begin{pmatrix} t_k \\ \hat{t}_k \end{pmatrix}$$

Then we obtain for the first s components of (3.1o)

$$q_{k+1} = \hat{q}_o \, e + h \sum_{\ell=o}^{k-1} \hat{d}_{\ell+1} \, e + h d_{k+1} + h \sum_{\ell=o}^{k-1} (e \otimes b^T) t_{\ell+1} + hBt_{k+1}$$

and for the last component

$$\hat{q}_{k+1} = \hat{q}_o + h \sum_{\ell=o}^{k} \hat{d}_{\ell+1} + h \sum_{\ell=o}^{k} (b^T t_{\ell+1}) \; .$$

This proves the following theorem:

4.2. Theorem. Let the starting value $y_o(h) \in \mathbb{R}$ of method (2.3) have order p, and let its last stage have order of consistency p, i. e.

$$\hat{q}_o := Y(x_o) - y_o(h) = O(h^p); \quad \hat{d}_{k+1} = O(h^p) \; .$$

Iff

$$(4.2) \qquad (b^T t_{k+1}) = O(h^p) \; ,$$

then

$$(4.3) \qquad \hat{q}_{k+1} = O(h^p) \; ,$$

and

$$(4.4) \qquad q_{k+1} = hd_{k+1} + hBt_{k+1} + O(h^p) \; ,$$

i. e. the approximations obtained from the last stage have (global) errors of order p (as $h \to o$), and the (global) errors of the other stages are given by (4.4).

(4.2) represents the <u>general order condition for RK-methods</u>. In its present form, this condition is unhandy and difficult to apply. In what follows, we therefore try to express it in terms of b and B.

4.3. Let be
$$g_j(x_k+a_ih) := \frac{(-1)^{j+1}}{j!} f_{\underbrace{y\ldots y}_{j\text{-times}}}(x_k+a_ih,y)\Big|_{y=Y(x_k+a_ih)}$$
$$= g_j(x_k)+a_ihg_j'(x_k)+\frac{a_i^2}{2!}h^2g_j''(x_k)+\ldots+O(h^p)$$

(4.5) $$G_j := \operatorname{diag}\Big(g_j(x_k+a_1h),\ldots,g_j(x_k+a_sh)\Big)$$

(4.6) $$= g_j(x_k)I + hg_j'(x_k)D + \frac{1}{2!}h^2g_j''(x_k)D^2+\ldots+O(h^p).$$

Now observe that

$$-f(x_k+a_ih,y_{k+a_i}) = \sum_{j=o}^{p} g_j(x_k+a_ih)\left(Y_{k+a_i}-y_{k+a_i}\right)^j+O(h^{p+1})$$

where powers of j imply componentwise multiplication and $(Y_{k+a_i}-y_{k+a_i}) = O(h)$ (at least).

Then the Taylor development of t_{k+1} (i. e. the first s components of (3.9)) yields

(4.7a) $$t_{k+1} = G_1q_{k+1} + G_2q_{k+1}^2 + G_3q_{k+1}^3 + \ldots + O(h^p) .$$

Assume

(4.7b) $$q_{k+1} = hd_{k+1} + hBt_{k+1} + O(h^p)$$

where $hd_{k+1} = \sum_{i=2}^{p} c_i Y^{(i)} h^i + O(h^{p+1})$ (from (3.2)/(3.7)) ; then

$t_{k+1} = t_{k+1}(h)$ and $q_{k+1} = q_{k+1}(h)$ exist and are uniquely defined by (4.7a/b) in a neighbourhood of h=0 where

(4.8)
$$q_{k+1}(h) = r_2 h^2 + r_3 h^3 + \ldots + r_{p-1} h^{p-1} + O(h^p) \ .$$
$$t_{k+1}(h) = w_2 h^2 + w_3 h^3 + \ldots + w_{p-1} h^{p-1} + O(h^p) \ .$$

The general order condition (4.2) is thus equivalent to

(4.2)* $(b^T w_j) = o \ , \quad j = 2(1)p-1$.

$r_j, w_j \in \mathbb{R}^s$ can be recursively calculated due to the following theorem

4.4. <u>Theorem</u>. For $j=2(1)p-1$

(4.9a) $$r_j = c_j Y^{(j)} + Bw_{j-1} \ ; \quad w_1 := o$$

(4.9b) $$w_j = \sum_{\ell=o}^{j-2} D^\ell \Big(g_1^{(\ell)}(x_k) r_{j-\ell} + \sum_{\substack{i,k\geq 2 \\ i+k=j-\ell}} g_2^{(\ell)}(x_k)(r_i \cdot r_k) + \sum_{\substack{i,k,m\geq 2 \\ i+k+m=j-\ell}} g_3^{(\ell)}(x_k)(r_i \cdot r_k \cdot r_m) + \ldots \Big)$$

<u>Proof</u> by induction: (4.9a/b) hold for j=2. If they are true for $j \leq p-2$, then from (4.7b)

(4.1o) $$q_{k+1} = \sum_{i=2}^{j} r_i h^i + h^{j+1}(c_{j+1} Y^{(j+1)} + Bw_j) + O(h^{j+2}) \ .$$

Hence (4.9a) is true for (j+1). From (4.7a) we obtain with

$$s_{j+1} := \sum_{i=2}^{j+1} r_i h^i \in \mathbb{R}^s$$

$$t_{k+1} = G_1 s_{j+1} + G_2(s_{j+1})^2 + G_3(s_{j+1})^3 + \dots + O(h^{j+2})$$

$$= \sum_{i=2}^{j+1} h^i G_1 r_i + \sum_{\substack{i,k\geq 2\\ i+k\leq j+1}} h^{i+k} G_2(r_i \cdot r_k) +$$

$$+ \sum_{\substack{i,k,m\geq 2\\ i+k+m\leq j+1}} h^{i+k+m} G_3(r_i \cdot r_k \cdot r_m) + \dots + O(h^{j+2}) .$$

Considering (4.6), we obtain for the term with h^{j+1}:

$$w_{j+1} = \sum_{\ell=o}^{j-1} D^\ell \Big\{ g_1^{(\ell)}(x_k)\, r_{j-\ell+1} + \sum_{\substack{i,k\geq 2\\ i+k+\ell=j+1}} g_2^{(\ell)}(x_k)(r_i \cdot r_k) +$$

$$+ \sum_{\substack{i,k,m\geq 2\\ i+k+m+\ell=j+1}} g_3^{(\ell)}(x_k)(r_i \cdot r_k \cdot r_m) + \dots \Big\}$$

Hence (4.9b) holds for (j+1), and thus for all $j\leq p-1$.

4.5. Define the sums $w_j^* := \sum_{i=1}^{m_j} \alpha_{j\,i} \in \mathbb{R}^s$ recursively by

$$w_1^* = 0 \;; \quad r_j^* = c_j + Bw_{j-1}^* , \quad j=2(1)p-1 ,$$

(4.11)

$$w_j^* = \sum_{\ell=o}^{j-2} D^\ell \Big(r_{j-\ell}^* + \sum_{\substack{i,k\geq 2\\ i+k=j-\ell}} r_i^* \cdot r_k^* + \sum_{\substack{i,k,m\geq 2\\ i+k+m=j-\ell}} r_i^* \cdot r_k^* \cdot r_m^* + \dots \Big).$$

It is seen then from (4.9a/b) that $w_j = \sum_{i=1}^{m_j} \alpha_{j\,i}\, h_{j\,i}$,

where $h_{j\,i} \in \mathbb{R}$ are function values (elementary differentials) that depend on f. Hence, a sufficient condition for (4.2) is:

(4.12) $(b^T \alpha_{j\,i}) = 0, \quad i=1(1)m_j \;; \quad j=2(1)p-1$.

For $p \geq 5$, the set $\{h_{ji}\}$ contains function values which are equal (for example, $h_{44} = g_1'(x_k)g_1(x_k)Y''$ and $h_{46} = g_1(x_k)g_1'(x_k)Y''$) so that condition $b^T[BDc_2+DBc_2] = o$ could replace two of these in $(4.13b)^*$ below.

Thus, for n=1, (4.12) is not necessary for (4.2) which, of course, reflects the well-known fact that the case n=1 needs less conditions than n>1. It will be seen later, that all conditions (4.12) are necessary if n>1.

Resuming we have the following result:

4.6. Theorem. Let $w_j^* = \sum_{i=1}^{m_j} \alpha_{ji} \in \mathbb{R}^s$ be generated by (4.11). Then the conditions for a RK-method to have order p are given by

(4.13a) $\qquad b^T a^{i-1} = 1/i \;, \quad i=1(1)p \;, \quad$ (see (3.5))

(4.13b) $\qquad b^T \alpha_{ji} = o \;, \quad i=1(1)m_j \;;\; j=2(1)p-1 \;.$

This theorem represents a remarkably simple way to obtain the set of equations that defines a RK-method of order p.

4.7. Example. For p=5 we obtain from (4.11):

$$w_2^* = c_2 \;; \quad w_3^* = c_3 + Bc_2 + Dc_2$$

$$w_4^* = c_4+Bc_3+B^2c_2+BDc_2+Dc_3+DBc_2+D^2c_2+c_2^2 \;.$$

Hence, the following 17 equations must be satisfied

$(4.13a)^* \qquad b^T a^{i-1} = 1/i \;; \quad i=1(1)5$

$$(4.13b)^* \quad \begin{aligned} 0 &= b^T c_2 = b^T c_3 = b^T B c_2 = b^T D c_2 \\ &= b^T c_4 = b^T B c_3 = b^T B^2 c_2 = b^T B D c_2 \\ &= b^T D c_3 = b^T D B c_2 = b^T D^2 c_2 = b^T c_2^2 \ . \end{aligned}$$

The vectors c_j can be substituted by (3.7); using $(4.13a)^*$ we obtain

$$(4.14) \quad \begin{aligned} &b^T Ba = 1/6; \quad b^T Ba^2 = 1/12; \quad b^T B^2 a = 1/24; \quad b^T DBa = 1/8 \\ &b^T Ba^3 = 1/20; \quad b^T B^2 a^2 = 1/60; \quad b^T B^3 a = 1/120; \quad b^T BDBa = 1/40 \\ &b^T DBa^2 = 1/15; \quad b^T DB^2 a = 1/30; \quad b^T D^2 Ba = 1/10; \quad b^T (Ba)^2 = 1/20 \ . \end{aligned}$$

The representation $b^T c_2^2 = o$ shows that the last equation is, in particular, satisfied with $b_2 = 0$ and explicit stages of order 2, except in the first stage, i. e. $c_2 = (o, c_{22}, o, \dots, o)^T \in \mathbb{R}^s$, $c_{22} \neq o$.

The four first equations (4.14) define together with (4.13a) the conditions for order 4. The complete set of α_{ji} up to order p=8 is given in the appendix.

We did not impose conditions on the r_j^* which may be done in order to obtain RK methods of the Fehlberg type.

5. Notation for the case $n>1$

5.1. In order to generalize the results of paragraph 4 to the case $n>1$, the following notation will be needed.

$$\underline{B} := B \otimes I;\ \underline{D} := D \otimes I;\ I \in \mathbb{R}(n,n);\ \underline{e} := e \otimes e^*$$

$$(5.1)\qquad \underline{a} := \underline{Be};\ \underline{b} := b \otimes e^*;\ \underline{c}_j := c_j \otimes e^*;\ \underline{Y}^{(i)} := e \otimes Y^{(i)}$$

$$\underline{q}_k, \underline{d}_k, \underline{t}_k \in \mathbb{R}^{sn} \text{ and } \hat{q}_k, \hat{d}_k, \hat{t}_k \in \mathbb{R}^s \text{ as in (4.1).}$$

Vectors from $\mathbb{R}^{sn}$ will be denoted by bold face letters; other vectors are from $\mathbb{R}^n$ or $\mathbb{R}^s$.

Let L_r be a function from $\mathbb{R}^{rn}$ into $\mathbb{R}$ such that for $\alpha \in \mathbb{R}$ and each i: $L_r(b_1,\ldots,\alpha b_i+c_i,\ldots,b_r) = \alpha L_r(b_1,\ldots,b_r)+L_r(b_1,\ldots,c_i,\ldots,b_r)$. L_r then is a r-tensor (or r-linear form) on $\mathbb{R}^n$.

We shall consider r-tensors on $\mathbb{R}^n$ of the special form:

$$(5.2)\qquad L_r(a;\ z^1,\ldots,z^r) := \sum_{n_1=1}^{n} \cdots \sum_{n_r=1}^{n} g_{n_1\ldots n_r}(x_k+ah)\, z^{1,n_1}\cdots z^{r,n_r}$$

where $a \in \mathbb{R}$ is a parameter; $z^{j,i}$ is the i-th component of z^j and

$$(5.3)\qquad g_{n_1\ldots n_r}(x_k+ah) := \frac{(-1)^{r+1}}{r!}\,\frac{\partial^r}{\partial y_{n_1}\cdots\partial y_{n_r}} f(x,y)\Bigg|_{\substack{x=x_k+ah\\ y=Y(x_k+ah)}}$$

with $f: [\alpha,\beta] \times \mathbb{R}^n \to \mathbb{R}$.

For $f \in \mathbb{C}^{p+1}$, (5.3) permits a Taylor development at $x=x_k$, and we obtain

$$(5.4)\qquad L_r(a;\ z^1,\ldots,z^r) = \sum_{k=o}^{p-1} \frac{(ah)^k}{k!} M^{(k)}(z^1,\ldots,z^r) + O(h^p)$$

with r-tensors $M^{(k)}$, $k=o(1)p-1$, independent of a.

In this notation, a Taylor development at x_k of

$$t_{j,k+1} := f(x_k+a_jh, Y_{k+a_j}) - f(x_k+a_jh, y_{k+a_j})$$

takes the form

$$(5.5) \quad t_{j,k+1} = L_1(a_j;q_{j+1}) + L_2(a_j;q_{j+1},q_{j+1}) + L_3(a_j;q_{j+1},q_{j+1},q_{j+1}) + \ldots$$

with $q_{j+1} := \left(Y_{k+a_j} - y_{k+a_j}\right) \in \mathbb{R}^n$.

5.2. If, as in our case, $f: [\alpha,\beta] \times \mathbb{R}^n \to \mathbb{R}^n$, then we may by (5.2) associate a r-tensor $L_r^{(i)}$ to each component f^i, $i=1(1)n$, and assemble these tensors in the vector $u_r(a) \in \mathbb{R}^n$:

$$u_r(a) := \left(L_r^{(1)}(a; z^1,\ldots,z^r),\ldots,L_r^{(n)}(a; z^1,\ldots,z^r)\right)^T.$$

This may be done for different $a=a_j$ and $z^k=z_j^k \in \mathbb{R}^n$, $j=1(1)s$ (which will be related to the s stages of our integration method) yielding the vectors $u_r(a_j) \in \mathbb{R}^n$:

$$u_r(a_j) := \left(L_r^{(1)}(a_j; z_j^1,\ldots,z_j^r),\ldots,L_r^{(n)}(a_j;z_j^1,\ldots,z_j^r)\right)^T.$$

They finally define a vector $\underline{u}_r \in \mathbb{R}^{sn}$ by

$$(5.6) \quad \underline{u}_r(\underline{z}^1,\ldots,\underline{z}^r) := u_r(a_1) \oplus \ldots \oplus u_r(a_s)$$

where $\underline{z}^k := z_1^k \oplus \ldots \oplus z_s^k \in \mathbb{R}^{sn}$, $k=1(1)r$.

Applying (5.4) to each of the sn components of $\underline{u}_r$ yields

$$(5.7) \quad \underline{u}_r(\underline{z}^1,\ldots,\underline{z}^r) = \sum_{k=o}^{p-1} h^k \underline{D}^k \underline{v}_r^{(k)}(\underline{z}^1,\ldots,\underline{z}^r) + O(h^p)$$

$$(5.8)\quad \underline{v}_r^{(k)}(\underline{z}^1,\dots,\underline{z}^r) := N_1^{(k)} \oplus \dots \oplus N_s^{(k)} \in \mathbb{R}^{sn}$$

$$N_j^{(k)} := \frac{1}{k!}\left(M_1^{(k)}(z_j^1,\dots,z_j^r),\dots,M_n^{(k)}(z_j^1,\dots,z_j^r)\right)^T \in \mathbb{R}^n$$

with $M_i^{(k)}(z_j^1,\dots,z_j^r)$: r-tensors as in (5.4) associated to the component f^i of f.

5.3. The following rules of computation will be applied and are easily verified.

$$(5.9)\quad \underline{u}\cdot\underline{a} = \underline{a}\cdot\underline{u} = \underline{Du}\ ;\quad (\underline{a}\cdot\underline{u})\cdot\underline{v} = \underline{a}\cdot(\underline{u}\cdot\underline{v}) \qquad \text{(see sec.2.3.)}$$

For $\underline{a}$, $\underline{B}$ and $\underline{D}$ as defined in (5.1) we have

$$(5.10)\quad (\underline{Ba})\cdot\underline{v} = \underline{B}(\underline{a}\cdot\underline{v}) = \underline{BDv} \quad \text{if} \quad \underline{v} = e \otimes v\ ,\quad v \in \mathbb{R}^n.$$

For $\alpha \in \mathbb{R}$, $\underline{w} := w_1 e^* \oplus \dots \oplus w_s e^*$, $w_i \in \mathbb{R}$ and $\underline{v}_r^{(k)} \in \mathbb{R}^{sn}$ as in (5.8):

$$(5.11)\quad \underline{v}_r^{(k)}(\underline{z}^1,\dots,\alpha\underline{z}^i,\dots,\underline{z}^r) = \alpha\underline{v}_r^{(k)}(\underline{z}^1,\dots,\underline{z}^r)$$

$$(5.12)\quad \underline{v}_r^{(k)}(\underline{z}^1,\dots,\underline{w}\underline{z}^i,\dots,\underline{z}^r) = \underline{w}\underline{v}_r^{(k)}(\underline{z}^1,\dots,\underline{z}^r);\quad i=1(1)r.$$

For $y,z \in \mathbb{R}^m$; $\alpha,\beta \in \mathbb{R}$

(a) $u_1(\alpha y+\beta z) = \alpha u_1(y) + \beta u_1(z)$

(b) $u_2(\alpha y+\beta z,\ \alpha y+\beta z) = \alpha^2 u_2(y,y) + \alpha\beta u_2(y,z) + \alpha\beta u_2(z,y) + \beta^2 u_2(z,z)$

(5.13)

(c) $u_3(\alpha y+\beta z,\alpha y+\beta z,\alpha y+\beta z) =$

$= \alpha^3 u_3(y,y,y)+\alpha^2\beta u_3(y,y,z)+\alpha^2\beta u_3(y,z,y)+\alpha^2\beta u_3(z,y,y)+$

$+ \alpha\beta^2 u_3(y,z,z)+\alpha\beta^2 u_3(z,y,z)+\alpha\beta^2 u_3(z,z,y)+\beta^3 u_3(z,z,z).$

Sloppily speaking, $u_r(\alpha y+\beta z,\dots,\alpha y+\beta z)$ behave like $(\alpha y+\beta z)^r$.

6. Generalization to the case n>1

6.1. The general order condition (4.2) and equation (4.4) generalize to

(6.1) $\quad (b^T \otimes I)\underline{t}_{k+1} = O(h^p)$

(6.2) $\quad \underline{g}_{k+1} = h\underline{d}_{k+1} + h\underline{B}\underline{t}_{k+1} + O(h^p)$

where $\quad h\underline{d}_{k+1} = \sum_{i=2}^{p} \underline{c}_i \underline{Y}^{(i)} h^i + O(h^{p+1})$

and, in the notation of the previous paragraph, the Taylor development of $\underline{t}_{k+1}$ (i. e. the first sn components of (3.9)) yields (compare to (5.5))

(6.3) $\quad \underline{t}_{k+1} = \underline{u}_1(\underline{g}_{k+1}) + \underline{u}_2(\underline{g}_{k+1},\underline{g}_{k+1}) + \underline{u}_3(\underline{g}_{k+1},\underline{g}_{k+1},\underline{g}_{k+1}) + \ldots + O(h^p).$

$\underline{t}_{k+1} = \underline{t}_{k+1}(h)$ and $\underline{g}_{k+1} = \underline{g}_{k+1}(h)$ as given by (6.2) and (6.3) exist and are uniquely defined for $h \in [o,h_o]$, where

$$\underline{g}_{k+1}(h) = \underline{r}_2 h^2 + \underline{r}_3 h^3 + \ldots + \underline{r}_{p-1} h^{p-1} + O(h^p) \;,$$

$$\underline{t}_{k+1}(h) = \underline{w}_2 h^2 + \underline{w}_3 h^3 + \ldots + \underline{w}_{p-1} h^{p-1} + O(h^p) \;.$$

$\underline{r}_j, \underline{w}_j \in \mathbb{R}^{sn}$ can be recursively calculated due to the following generalization of theorem 4.4.

6.2. Theorem. For j=2(1)p-1

(6.4a) $\quad \underline{r}_j = \underline{c}_j \underline{Y}^{(j)} + \underline{B}\underline{w}_{j-1}$

(6.4b) $$\underline{w}_j = \sum_{\ell=0}^{j-1} \underline{D}^{\ell} \Big\{ \underline{v}_1^{(\ell)}(\underline{r}_{j-\ell}) + \sum_{\substack{i,k\geq 2\\ i+k=j-\ell}} \underline{v}_2^{(\ell)}(\underline{r}_i,\underline{r}_k) + \sum_{\substack{i,k,m\geq 2\\ i+k+m=j-\ell}} \underline{v}_3^{(\ell)}(\underline{r}_i,\underline{r}_k,\underline{r}_m) + \ldots \Big\}$$

Proof by induction: (6.4a/b) hold for j=2. If they are true for $j \le p-2$, then from (6.2)

$$\underline{q}_{k+1} = \sum_{i=2}^{j} \underline{r}_i h^i + h^{j+1}(\underline{c}_{j+1}\underline{Y}^{(j+1)} + \underline{B}\underline{w}_j) + O(h^{j+2}) .$$

Hence (6.4a) is true for (j+1). From (6.3) with $\underline{s}_{j+1} := \sum_{i=2}^{j+1} \underline{r}_i h^i$

$$\underline{t}_{k+1} = \underline{u}_1(\underline{s}_{j+1}) + \underline{u}_2(\underline{s}_{j+1},\underline{s}_{j+1}) + \underline{u}_3(\underline{s}_{j+1},\underline{s}_{j+1},\underline{s}_{j+1}) + \ldots + O(h^{j+2}) .$$

with (5.13):

$$= \sum_{i=2}^{j+1} h^i \, \underline{u}_1(\underline{r}_i) + \sum_{\substack{i,k\ge 2\\ i+k\le j+1}} h^{i+k}\, \underline{u}_2(\underline{r}_i,\underline{r}_k) +$$

$$+ \sum_{\substack{i,k,m\ge 2\\ i+k+m\le j+1}} h^{i+k+m}\, \underline{u}_3(\underline{r}_i,\underline{r}_k,\underline{r}_m) + \ldots + O(h^{j+2})$$

From (5.7), collecting the terms with h^{j+1}:

$$\underline{w}_{j+1} = \sum_{\ell=0}^{j-1} \underline{D}^{\ell} \Big\{ \underline{v}_1^{(\ell)}(\underline{r}_{j-\ell+1}) + \sum_{\substack{i,k\ge 2\\ i+k+\ell=j+1}} \underline{v}_2^{(\ell)}(\underline{r}_i,\underline{r}_k) +$$

$$+ \sum_{\substack{i,k,m\ge 2\\ i+k+m+\ell=j+1}} \underline{v}_3^{(\ell)}(\underline{r}_i,\underline{r}_k,\underline{r}_m) + \ldots + O(h^{j+2}) \Big\}$$

Hence (6.4b) holds for (j+1), and thus for all $j \le p-1$.

6.3. If $\underline{z}^i = e \otimes z^i$, $z^i \in \mathbb{R}^n$, $i=1(1)r$, it follows from (5.8) that

$$\underline{v}_r^{(k)}(\underline{z}^1,\ldots,\underline{z}^r) = e \otimes v^{(k)} , \quad v^{(k)} \in \mathbb{R}^n .$$

Using this and the relations (5.1o/11/12), it is seen (by induction) from (6.4a/b) that the sums $\underline{w}_j$ have the form

$$\underline{w}_j = \sum_{i=2}^{m_j} \left(\alpha_{ji}' \otimes e^*\right) \cdot \left(e \otimes h_{ji}'\right)$$

with $\alpha_{ji}' \in \mathbb{R}^s$ as defined in Sec. 4.5. and where $h_{ji}' \in \mathbb{R}^n$ are function values (elementary differentials) that depend on f. Their inspection yields that, in general, they are all different form each other.

Since $(b^T \otimes I)\underline{w}_j = \sum_{i=2}^{m_j} \left(b^T \alpha_{ji}'\right) h_{ji}'$, condition (6.1) is satisfied, for general f, if and only if the conditions (4.13b) hold. This generalizes theorem 4.6. to the case n>1.

7. Generalizations of RK-Methods

7.1. The last stage of method (2.3) is explicit, i. e. it does not contain $f_{k+a_{s+1}}$. This raises the question whether the theory presented here can be generalized to what may be called RK-methods with implicit last stage.
In view of our presentation of RK-methods as composite multistep procedures, such a generalization appears quite natural, and, in fact, it can be dealt with in a simple way: If the last stage is implicit we enlarge (2.3) by a repetition of the last stage. This increases the dimension s of a and B by one.

7.2. Example

$$
(7.1)\qquad
\begin{aligned}
y_o &= \eta_o\\
y_{j+2/3} &= y_j + \frac{h}{3}((2-3b_{22})f_j+3b_{22}f_{j+2/3})\\
y_{j+1/2} &= y_j + \frac{h}{16}(3f_j-3f_{j+2/3}+8f_{j+1/2})\\
y_{j+1} &= y_j + \frac{h}{6}(f_j+4f_{j+1/2}+f_{j+1}) \quad , \ j = o,1,\ldots
\end{aligned}
$$

Parameter scheme

o	o	o	o	
2/3	$(2/3-b_{22})$	b_{22}	o	
1/2	3/16	-3/16	1/2	
	1/6	o	4/6	1/6

Enlarged scheme of the form $\begin{array}{c|c} a & B \\ \hline & b^T \end{array}$

0	0	0	0	0
2/3	$(2/3-b_{22})$	b_{22}	0	0
1/2	3/16	-3/16	1/2	0
1	1/6	0	4/6	1/6
	1/6	0	4/6	1/6

It is easily verified that this scheme satisfies the equations

$$b^T a^{j-1} = \frac{1}{j}, \quad j=1(1)4,$$
$$b^T c_2 = o; \quad b^T c_3 = o; \quad b^T Dc_2 = o; \quad b^T Bc_2 \neq o$$

for all b_{22}. Hence, method (7.1) has order p=3.

7.3. It has been mentioned in paragraph 2 that methods of the above type (with $a_{s+1} \neq 1$) are widely used to start codes based on variable order linear multistep schemes. However, usually no attempt is made to optimize the order of such starting procedures.

In addition to this application, the interesting point of RK-methods with implicit last stage is their stability region. With $b_{22}=2/3$, for example, method (7.1) is A-stable; so is the method of order p=3 given by

$$\text{(7.2)} \qquad \begin{aligned} y_o &= \eta_o \\ \tilde{y}_{j+3/2} &= y_j + \frac{3h}{4}(f_j + f_{j+3/2}) \\ y_{j+1} &= y_j + \frac{h}{18}(7f_j - 4\tilde{f}_{j+3/2} + 15f_{j+1}) \end{aligned}$$

$j=o,1,\ldots$

7.4. It should also be observed that <u>one-step block-implicit methods</u> as discussed in [6] are special cases of RK-methods with implicit last stage. As long as all their stages have order (p-1) or p, they are easily constructed (see [2], pgs. 167-171) without the results of this paper, but our theory becomes relevant if the stages differ in order by more than one. Apparently, this case is not discussed in the literature.

7.5. Further generalizations of method (2.3) are <u>RK-type k-step methods</u> such as the following 2-step method with order p=3:

$$
\begin{aligned}
y_0 &= \eta_0 \; ; \; y_1 = \eta_1(h) \; ; \\
y_{j+1/2} &= y_j + \tfrac{1}{2} h f_j \\
\bar{y}_{j+1} &= y_j + h(-f_j + 2f_{j+1/2}) \\
y_{j+1} &= y_j + \frac{h}{60}(15f_{j+1} + 4\bar{f}_{j+1} + 16f_{j+1/2} + 28f_j - 3f_{j-1})
\end{aligned}
\tag{7.3}
$$

$$j=1,2,\ldots$$

It should be possible to generalize our theory to methods of this type, but things become more complicated for generalizations where the matrix A in (2.5) is <u>not</u> a Runge-Kutta matrix.

Conclusion

The alternative approach to RK-methods presented here is simple in its basic idea; the relative complexity in the case $n>1$ is merely notational. It should be possible to reduce this notational overhead by a "componentwise" presentation; however, we did not find a simple formalism for it. It is also not yet apparent whether this approach and, in particular, the recursion (4.11) can be helpful when solving the system (4.13a/b).

Apparently, the main point of this paper is not the generation of conditions for RK-methods of order p which are well-known [4], but the simplicity of the approach and the technique to handle methods with stages of different order. The transparence of the order equations may facilitate existence proofs and provide further insight in the structure of RK-methods. As a simple but interesting example (pointed out by one of the referees) consider the last equation of (4.13b)*. It shows that every RK-method of order 5 or greater has at least one weight b_i which is not positive.

The approach of this paper should also be seen in the wider context of general composite methods [3]; in a certain sense, it completes the author's efforts towards a unified treatment of numerical methods for the solution of (2.1).

The concept of A-methods, on which this treatment is based, was originally designed for linear cyclic methods where it furnished very satisfactory results [1;2]. Later, its scope of application widened up helping to solve quite a number of problems related to multistage methods [3]. Considering the results obtained so far, it becomes apparent that the A-method approach offers a very adequate frame for the analysis of O.D.E. methods, the potentiality of which does not yet seem exhausted.

This work profited considerably from the thorough suggestions of the referees.

APPENDIX: The set of vectors α_{ji} for $j=2(1)(p-1)$.

c_2

p=3

$c_3 \quad Bc_2 \quad Dc_2$

p=4

$c_4 \quad Bc_3 \quad B^2c_2 \quad BDc_2 \quad Dc_3 \quad DBc_2 \quad D^2c_2 \quad c_2^2$

p=5

$c_5 \quad Bc_4 \quad BDc_3 \quad BD^2c_2 \quad B^2c_3 \quad BDBc_2 \quad B^2Dc_2 \quad B^3c_2$

$Bc_2^2 \quad Dc_4 \quad DBc_3 \quad DBDc_2 \quad DB^2c_2 \quad D^2c_3 \quad D^2Bc_2 \quad D^3c_2$

$c_2 \cdot c_3 \quad c_2 \cdot (Bc_2) \quad Dc_2^2$

p=6

$c_6 \quad Bc_5 \quad B^2c_4 \quad B^2Dc_3 \quad B^2D^2c_2 \quad B^3c_3 \quad B^2DBc_2$

$B^3Dc_2 \quad B^4c_2 \quad B^2c_2^2 \quad BDc_4 \quad BDBc_3 \quad BDBDc_2 \quad BDB^2c_2$

$BD^2c_3 \quad BD^2Bc_2 \quad BD^3c_2 \quad BDc_2^2 \quad B(c_2 \cdot (Bc_2)) \quad B(c_2 \cdot c_3)$

$Dc_5 \quad DBc_4 \quad DBDc_3 \quad DBD^2c_2 \quad DB^2c_3 \quad DBDBc_2 \quad DB^2Dc_2$

$DB^3c_2 \quad DBc_2^2 \quad D^2c_4 \quad D^2Bc_3 \quad D^2B^2c_2 \quad D^2BDc_2 \quad D^3c_3$

$D^3Bc_2 \quad D^4c_2 \quad c_2 \cdot c_4 \quad c_2 \cdot (Bc_3) \quad c_2 \cdot (B^2c_2) \quad c_2 \cdot (BDc_2)$

$c_3^2 \quad c_3 \cdot (Bc_2) \quad (Bc_2)^2 \quad D(c_2 \cdot c_3) \quad D(c_2 \cdot Bc_2) \quad D^2c_2^2 \quad c_2^3$

p=7

$c_7 \quad Bc_6 \quad B^2c_5 \quad B^3c_4 \quad B^3Dc_3 \quad B^3D^2c_2 \quad B^4c_3 \quad B^4Dc_2$

$B^5c_2 \quad B^3c_2^2 \quad B^2Dc_4 \quad B^2DBc_3 \quad B^2DBDc_2 \quad B^2DB^2c_2 \quad B^3DBc_2$

$B^2D^2c_3 \quad B^2D^2Bc_2 \quad B^2D^3c_2 \quad B^2Dc_2^2 \quad B^2(c_2 \cdot (Bc_2)) \quad B^2(c_2 \cdot c_3)$

$BDc_5 \quad BDBc_4 \quad BDBDc_3 \quad BDBD^2c_2 \quad BDB^2c_3 \quad BDBDBc_2$

$BDB^2Dc_2 \quad BDB^3c_2 \quad BDBc_2^2 \quad BD^2c_4 \quad BD^2Bc_3$

$BD^2B^2c_2 \quad BD^2BDc_2 \quad BD^3c_3 \quad BD^3Bc_2 \quad BD^4c_2$

$B(c_2 \cdot c_4) \quad B(c_2 \cdot (Bc_3)) \quad B(c_2 \cdot (B^2c_2)) \quad B(c_2 \cdot (BDc_2))$

$Bc_3^2 \quad B(c_3 \cdot Bc_2) \quad B((Bc_2) \cdot (Bc_2)) \quad BD(c_2 \cdot c_3)$

$BD(c_2 \cdot (Bc_2)) \quad BD^2c_2^2 \quad Bc_2^3$

$Dc_6 \quad DBc_5 \quad DB^2c_4 \quad DB^2Dc_3 \quad DB^2D^2c_2 \quad BD^3c_3 \quad DB^2DBc_2$

$DB^3Dc_2 \quad DB^4c_2 \quad DB^2c_2^2 \quad DBDc_4 \quad DBDBc_3 \quad DBDBDc_2$

$DBDB^2c_2 \quad DBD^2c_3 \quad DBD^2Bc_2 \quad DBD^3c_2 \quad DBDc_2^2 \quad DB(c_2 \cdot (Bc_2))$

$DB(c_2 \cdot c_3)$

$D^2c_5 \quad D^2Bc_4 \quad D^2BDc_3 \quad D^2BD^2c_2 \quad D^2B^2c_3 \quad D^2BDBc_2$

$D^2B^2Dc_2 \quad D^2B^3c_2 \quad D^2Bc_2^2 \quad D^3c_4 \quad D^3Bc_3 \quad D^3B^2c_2 \quad D^3BDc_2$

$D^4c_3 \quad D^4Bc_2 \quad D^5c_2$

$c_2 \cdot c_5 \quad c_2 \cdot Bc_4 \quad c_2 \cdot (BDc_3) \quad c_2 \cdot (BD^2c_2) \quad c_2 \cdot (B^2c_3)$

$c_2 \cdot (BDBc_2) \quad c_2 \cdot (B^2Dc_2) \quad c_2 \cdot (B^3c_2) \quad c_2 \cdot (Bc_2^2) \quad c_3 \cdot c_4$

$c_3 \cdot (Bc_3) \quad c_3 \cdot (B^2c_2) \quad c_3 \cdot (BDc_2) \quad c_4 \cdot (Bc_2) \quad (Bc_2) \cdot (Bc_3)$

$(Bc_2) \cdot (B^2c_2) \quad (Bc_2) \cdot (BDc_2)$

$D(c_2 \cdot c_4) \quad D(c_2 \cdot (Bc_3)) \quad D(c_2 \cdot (B^2c_2)) \quad D(c_2 \cdot (BDc_2))$

$Dc_3^2 \quad D(c_3 \cdot (Bc_2)) \quad D(Bc_2) \cdot (Bc_2) \quad D^2(c_2 \cdot c_3)$

$D^2(c_2 \cdot (Bc_2)) \quad D^3(c_2 \cdot c_2) \quad c_2^2 \cdot c_3 \quad c_2^2 \cdot (Bc_2) \quad Dc_2^3$

p=8

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