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by

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ABSTRACT

A family of simplicial finite element methods having the simplest possible structure, is introduced to solve biharmonic problems in $\mathbb{R}^n$, $n \geq 3$, using the primal variable. Athough the family is inspired in the MORLEY triangle for the two dimensional case, this element cannot be simply viewed as its member corresponding to the value $n=2$. On the other hand equivalent convergence results are proven to hold for this family of methods.

RESUMO

Uma família de métodos de elementos finitos com a estrutura mais simples possível para se resolver problemas biharmônicos em dimensão n, $n \geq 3$, é introduzida. Cada membro da família é construído com base em funções quadráticas por n-simplex, definidas com base em graus de liberdade não clássicos do tipo proposto pelo autor em artigos recentes. Embora se possa estabelecer uma analogia entre os membros dessa família e o elemento triangular quadrático devido a MORLEY, destinado à resolução do problema biharmônico em dimensão dois, este último não pode ser visto como o membro da família para o valor $n=2$. Resultados de convergência equivalentes aos que se aplicam àquele elemento são demonstrados.

KEY-WORDS: biharmonic, convergence, finite elements, nonconforming, parametrized degrees of freedom.

PALAVRAS-CHAVE: biharmônico, convergência, elementos finitos, graus de liberdade parametrados, não conforme.

CONTEÚDO

1 - INTRODUCTION

Consider as a model the Dirichlet problem for the biharmonic operator in an open set $\Omega\subset\mathbb{R}^n$ with a sufficiently smooth boundary Γ.

$$
(E)\quad \begin{cases} \text{Find } \phi \text{ such that} \\ \Delta^2\phi = f \text{ in } \Omega \\ \phi = \dfrac{\partial\phi}{\partial\nu} = 0 \quad \text{on } \Gamma, \text{ where} \end{cases}
$$

Δ is the laplacian operator, f is a given function and $\frac{\partial\phi}{\partial\nu} = \vec{\text{grad}}\phi.\vec{\nu}$, $\vec{\nu}$ denoting the unit outer normal vector with respect to Γ.

By introducing a real parameter $\sigma\in[0,1)$ and assuming $f\in L^2(\Omega)$, we may equivalently write equation (E) in the variational form:

$$
(P)\quad \begin{cases} \text{Find } \phi\in H_0^2(\Omega) \text{ such that} \\ a_\Omega(\phi,\psi) = \int_\Omega f\psi \qquad \forall\psi\in H_0^2(\Omega) \end{cases}
$$

where for an open set $D\subset\Omega$ we define

$$
a_D(\phi,\psi) = \sigma\int_D \Delta\phi\Delta\psi + (1-\sigma)\sum_{i,j=1}^{n}\int_D \frac{\partial^2\phi}{\partial x_i\partial x_j}\frac{\partial^2\psi}{\partial x_i\partial x_j}
$$

$H^m(D)$ denoting the Sobolev space for $m\in\mathbb{N}^+$, with the standard norm $\|.\|_{m,D}$ and seminorm $|.|_{m,D}$ as defined in the literature (see e,g,[1]), $H_0^2(\Omega)$ is defined by:

$$
H_0^2(\Omega) = \{\psi/\psi\in H^2(\Omega),\quad \psi = \frac{\partial\psi}{\partial\nu} = 0\}
$$

As it is well-known, $H_0^2(\Omega)$ can be normed by the seminorm $|\cdot|_{2,\Omega}$.

The finite element methods of solution that we consider in this work are to be placed in the following framework:

Let τ_h be a partition of Ω into n-simplices with maximal edge length equal to h. We assume that $\{\tau_h\}_h$ belongs to a regular family of partitions in the sense given in [2].

Let also V_h be a finite dimensional space associated with τ_h in a way to be specified later on. We assume that the restriction of every function of V_h to each n-simplex $K\epsilon\tau_h$ belongs to $H^2(K)$. By approximating the boundary conditions implicit in $H_0^2(\Omega)$ for the functions of V_h, the approximate problem to solve is:

$$(P_h)\qquad \begin{cases} \text{Find } \phi_h \epsilon V_h \text{ such that} \\ a_h(\phi_h,\psi_h) = \int_\Omega f\psi_h \qquad \forall \psi_h \epsilon V_h \end{cases}$$

where $a_h(\phi_h,\psi_h) = \sum_{K\epsilon\tau_h} a_K(\phi_h,\psi_h)$

Now if one wiskes to have $V_h \subset H_0^2(\Omega)$ $(a_h \equiv a_\Omega)$, V_h must consist in principle of functions of the C^1class. As it is well-known, even in the two-dimensional case, the construction of such spaces is difficult, and only C^1finite element methods, that is, based on functions of the C^1class, having a rather complicated structure or high number of degrees of freedom per element are known (see e.g.[2]). To the best of our knowledge, as far as three or higher dimensions are concerned, no C^1finite element methods have been proposed so far.

This justifies the use of nonconforming methods, that is, $V_h \not\subset H_0^2(\Omega)$, related to finite element with a possibly simple structure. In this case a function of V_h is of the C^1-class only at element level, but if certain minimum point-diffetentiability requirements are satisfied at interelement boundaires, one can generate convergent sequences of approximate solutions.

We refer to the work of LASCAUX & LESAINT[5] for the description and study of a number of nonconforming finite element methods for the biharmonic equation in $\mathbb{R}^2$. Among these, the simplest possible element, namely the MORLEY triangle [6], was considered. Since the methods that we study in this work have a close relation to it, we briefly recall below the definition of this element:

- The restriction of every function of V_h to a triangle of τ_h is a (complete) quadratic function;
- The degrees of freedom used to define a function of V_h in each triangle are:
  - Its values at the vertices;
  - The values of its outer normal derivative at the mid-points of the edges.
- The degrees of freedom of every function of V_h coincides for vertices or edges belonging to two or more elements;
- Function of V_h at a vertex $\underline{S}$ or its normal derivative at the mid-point of an edge $\underline{e}$ vanish wherever $\underline{S}$ or both ends of $\underline{e}$ belong to Γ.

The sequence $\{\phi_h\}_h$ of solutions of (P_h) computed with this element converges to ϕ in the discrete H^2-norm $||.||_{2,h}$ (see (5), Sec.2), with order h, provided that ϕ is sufficient smooth. For the proof we refer to [5] in case Ω is a polygon or to [7] in the general case.

The n-dimensional analogue of Morley's element for $n\geq 3$ to be presented in this work gives rise to equivalent convergen ce results, but it must be constructed with the help of special degrees of freedom called parametrized, first introduced in [8].

2. THE NEW ELEMENTS WITH MAIN PROPERTIES

In order to avoid non essential difficulties, we assume that Ω is a hyperpolyhedron of $\mathbb{R}^n$, $n\geq 3$. We call the (n-1)- faces of a simplex its n+1 faces of dimension n-1. We denote by λ_i the barycentric coordinate of a simplex related to vertex S_i and by F_i the (n-1)-face opposite to S_i, i=1,2,...,n-1. Let also G_i be the barycenter of F_i.

The family of finite elements, or yet the corresponding space V_h, is defined as follows:

(i) The restriction of a function $v \in V_h$ to every simplex of τ_h is a (complete) quadratic function.

(ii) The degrees of freedom used to define a funtion of V_h over each simplex are:

a) $D_i(v)$, the outer normal derivative with respect to F_i at G_i, i = 1,2,...,n+1;

b) $D_{ij}(v)$, a functional associated with the edge e_{ij} of the simplex with ends S_i and mid-point M_{ij}, $1 \le i < j \le n+1$, given by:

$$D_{ij}(v) = \mu v(M_{ij}) + (1-\mu) \int_{e_{ij}} v ds/\text{length}(e_{ij})$$

$\mu \in \mathbb{R}$ being a fixed parameter depending only on n.

(iii) The local degrees of freedom above of every function of V_h coincide for (n-1) faces or edges belonging to two or more elements of τ_h, respectively;

(iv) The degrees of freedom of both types above of every function of V_h vanish, whenever the corresponding (n-1)-face or edge lie on Γ.

Let P_2 be the $\binom{n+2}{2}$- dimensional space consisting of polynomials defined in an n-simplex, of degree less than or equal to two. In order to prove that the above set of $\binom{n+2}{2}$ degrees of freedom is P_2-unisolvent for a given choice of μ, it suffices to exhibit the corresponding basis functions. Before doing this however, we should take into consideration one of the basic conditions for convergence of our method, that will lead precisely to the determination of the value of μ.

Indeed, the gradient of a function of V_h should be continuous (resp. vanish) at the barycenter of every internal (resp. boundary) (n-1)-face of partition τ_h (*). Since the outer normal derivative already satisfies this requirement by construction, the element should be such that the tangential derivatives

(*) This is usually called the patch-test, for the case of a first order nonconformig method.

$\frac{\partial v}{\partial \tau_k}$ of $v \in V_h$ in mutually orthogonal directions $\vec{\tau}_k$ $k = 1,2,\ldots,n-1$ of the (n-1)-face, satisfy the same requirement at its barycenter. Taking into account that our element is nonconforming, this condition will be fullfilled if we establish that these (n-1) derivatives depend only on the degrees of freedom attached to the (n-1)-face under consideration. This will be a consequence of the following Lemma leading to the choice of μ.

Lemma 1 Let the $\binom{n}{2}$ parametrized degrees of freedom of type b) attached to an (n-1) - face F of a simplex vanish, for a quadratic function p. Then if $\{\vec{\tau}_k\}_{k=1}^{n-1}$ is an orthonormal set of directions in the hyperplane of F, and G is the barycenter of F, we have.

$$\frac{\partial p}{\partial \tau_k}(G) = 0 \quad \text{for } k = 1,2, \ldots,n-1, \text{ provided } \mu = 4-12/n.$$

Proof: First we notice that any polynominal $p \in P_2$ defined in an n-simplex is of the form:

$$p = \sum_{i=1}^{n+1} \alpha_i \lambda_i + \sum_{i=1}^{n} \sum_{j=i+1}^{n+1} \beta_{ij} \lambda_i \lambda_j \quad \text{where } \alpha_i, \beta_{ij} \in \mathbb{R} .$$

Without loss of generality we will prove the lemma for face F_{n+1}. Notice that the restriction r(p) of p over F_{n+1} is of form

$$r(p) = \sum_{i=1}^{n} \alpha_i \lambda_i + \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \beta_{ij} \lambda_i \lambda_j$$

Now, since $p/e_{ij} = \alpha_i \lambda_i + \alpha_j \lambda_j + \beta_{ij} \lambda_i \lambda_j$ we have

(1) $D_{ij}[p] = \frac{1}{2}(\alpha_i + \alpha_j) + (\frac{1}{6} + \frac{\mu}{12})\beta_{ij} = 0, \quad 1 \le i < j \le n$,

according to our assumptions.

Noticing that $\frac{\partial p}{\partial \tau_k}(G_{n+1}) = \frac{\partial [r(p)]}{\partial \tau_k}(G_{n+1})$, $k=1,2, \ldots, n-1$,

and that $\lambda_i(G_{n+1}) = 1/n$, $i=1,2, \ldots n$, we have:

$$(2) \quad \frac{\partial p}{\partial \tau_k}(G_{n+1}) = \sum_{i=1}^{n} \alpha_i \frac{\partial \lambda_i}{\partial \tau_k} + \frac{1}{n} \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \beta_{ij} \left(\frac{\partial \lambda_i}{\partial \tau_k} + \frac{\partial \lambda_j}{\partial \tau_k}\right)$$

Now we multiply both sides of (1) by $2(\frac{\partial \lambda_i}{\partial \tau_k} + \frac{\partial \lambda_j}{\partial \tau_k})$ and we sum up with respect to i and j.

Taking into account that $\frac{\partial \lambda_i}{\partial \tau_k} = - \sum_{\substack{j=1 \\ j \neq i}}^{n} \frac{\partial \lambda_j}{\partial \tau_k}$, we obtain:

$$(n-2) \sum_{i=1}^{n} \alpha_i \frac{\partial \lambda_i}{\partial \tau_k} + \left(\frac{1}{3} + \frac{\mu}{6}\right) \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \beta_{ij} \left(\frac{\partial \lambda_i}{\partial \tau_k} + \frac{\partial \lambda_j}{\partial \tau_k}\right) = 0$$

Finally, recalling (2), it is readily seen that $\frac{\partial p}{\partial \tau_k}(G_{n+1}) = 0$ for $k=1,2, \ldots, n-1$, if $\mu = 4-12/n$. q.e.d.

Now for the value of μ the canonical basis functions p_i associated with $D_i(.)$, $i=1,2, \ldots, n+1$ and p_{ij} associated with $D_{ij}(.)$, $1 \le i < j \le n+1$ are given by:

$$(3) \quad p_i = (\lambda_i - \frac{n}{2} \lambda_i^2)/\gamma_{ii}$$

$$(4) \quad p_{ij} = [-2(\lambda_i + \lambda_j) + n(\lambda_i + \lambda_j)^2 - 2 \sum_{\substack{k=1 \\ k \neq i,j}}^{n+1} p_k (\gamma_{ik} + \gamma_{jk})]/(n-2)$$

where $\gamma_{\ell m} = D_m(\lambda_\ell)$, $1 \le m, \ell \le n+1$.

Finally we can prove the following crucial result:

Lemma 2: Let $\|.\|_{2,h}$ be the seminorm of $H^2(\Omega) + V_h$ given by

(5) $$\|\psi\|_{2,h} = \{\sum_{K\in\tau_h} |\psi|^2_{2,K}\}^{1/2}$$

Then $\|.\|_{2,h}$ is a norm for the space $\tilde{V}_h$, namely the space defined exactly in the same manner as V_h, except that the normal derivatives at the barycenters of boundary (n-1)-faces do not necessarily vanish.

Proof. It suffices to established that for $\psi\in\tilde{V}_h$

$$\|\psi\|_{2,h} = 0 \Rightarrow \psi \equiv 0$$

In this case ψ is a linear function over each simplex of τ_h.

The continuity of $\binom{n}{2}$ linearly independent functionals of the form b) and of one normal derivative on every face of τ_h, implies that ψ is the same linear function in every simplex of τ_h.

Finally, since the same linearly independent functionals applied to ψ vanish on at least two distinct (n-1)-faces of Γ, we must have $\psi\equiv 0$. q.e.d.

As an immediate consequence of Lemma 2, form a_h is coercive over $\tilde{V}_h$ normed with $\|.\|_{2,h}$.

Since $V_h \subset \tilde{V}_h$, problem (P_h) admite a unique solution ϕ_h. Notice that $\|.\|_{2,h}$ is also a norm for $H^2_0(\Omega)$.

3. CONVERGENCE RESULTS

According to the celebrated Strang's inequality [10] for nonconforming methods applied to problem (P) approximated by (P_h), we have

$$(6)\quad \|\phi - \phi_h\|_{2,h} \le \frac{1}{1-\sigma}\{[(n-2)\sigma+2]\inf_{\psi_h\in V_h}\|\phi - \psi_h\|_{2,h} + \sup_{\psi_h\in V_h}\frac{|a_h(\phi,\ \psi_h) - \int_\Omega \psi_h|}{\|\psi_h\|_{2,h}}\}$$

Since

$$(7)\quad \inf_{\psi_h\in V_h}\|\phi-\psi_h\|_{2,h} \le Ch|\phi|_{3,\Omega}$$

if $\phi\in H^3(\Omega)$, according to standard approximation results [2][(*)], first order convergence of ϕ_h to ϕ in the $\|.\|_{2,h}$ norm will be demonstrated, if we prove the following estimate:

$$(8)\quad |a_h(\phi,\psi_h) - \int_\Omega f\psi_h| \le Ch[\|\phi\|_{3,\Omega} + \|\Delta\phi\|_{2,\Omega}]\|\psi_h\|_h \quad ,$$

which is known to hold for Morley's triangle [5], if Ω is a polygon.

In our case we have

$$(9)\quad a_h(\phi,\psi_h) - \int_\Omega f\psi_h = E_h^1(\phi,\psi_h) + E_h^2(\phi,\psi_h),\ \text{where}$$

$$(10)\quad E_h^1(\phi,\psi) = \sum_{K\in\tau_h} - \int_{\partial K}\frac{\partial\Delta\phi}{\partial\nu^K}\psi\, d\tau$$

and

(*) C denotes, as usual, any constant independent of h.

$$(11)\quad E_h^2(\phi\ ,\ \psi) = \sum_{K\in\tau_h} \{\sigma \int_{\partial K} \Delta\phi \frac{\partial\psi}{\partial\nu^K}\, d\tau + (1-\sigma)\int_{\partial K} [\frac{\partial^2\phi}{\partial\nu^K\partial\nu^K} \frac{\partial\psi}{\partial\nu^K} + \sum_{k=1}^{n-1} \frac{\partial^2\phi}{\partial\tau_k^K\ \partial\nu^K} \frac{\partial\psi}{\partial\tau_k{}^K}]\, d\tau\}$$

where $\frac{\partial .}{\partial\nu^K}$ denotes the outer normal derivative with respect to ∂K, the boundary of simplex K, and $\frac{\partial .}{\partial\tau_k^K}$ the derivative in the k-th orthonormal direction $\vec{\tau}_k^K$ of ∂K, the set $\{\vec{\tau}_k^K\}_{k=1}^{n-1}$ being defined face by face.

Thanks to Lemma 1, the bound

$$(11)\quad E_h^2(\phi\ ,\psi_h) \leq Ch\|\phi\|_{3,\Omega}$$

can be proven to hold using the same arguments as in [5] for Morley's element.

Therefore we confine ourselves here to proving the following

<u>Lemma 3</u>: If $\Delta\phi\in H^2(\Omega)$ we have:

$$(12)\quad E_h^1(\phi\ ,\ \psi) \leq Ch\|\Delta\phi\|_{2,\Omega}\|\psi\|_{2,h} \qquad \forall\psi\in V_h$$

<u>Proof</u>: Let F be an (n-1)-face of a simplex K and π_F be the operator

$$\pi_F:\ P_2 \rightarrow P_1(F)$$
$$p \mapsto \pi_F(p)$$

such that

$D_{ij}[\pi_F(p)] = D_{ij}(p)$ for n given pairs (i,j) $i\neq j$ associated with the indices of the vertices S_i of K belonging to F, $1\leq i\leq n+1$, where $P_1(F)$ denotes the space of linear functions defined on F.

Because of (iii) and (iv), we can write

$$\text{(13)}\quad E_h^1(\phi\ \psi_h) = -\sum_{K\in\tau_h}\ \sum_{F\subset\partial K}\int_F \frac{\partial\Delta\phi}{\partial\nu^K}[\psi_h - \pi_F(\psi_h)]d\tau \qquad \forall\psi_h\in V_h$$

Indeed, the assumption $\Delta\phi\in H^2(\Omega)$ implies the coincidence of $\frac{\partial\Delta\phi}{\partial\nu^K}\in L^2(F)$ on both sides of F, if F is an internal (n-1)-face, by the Trace Theorem.

Set now for a given $K\in\tau_h$, $L_K^F : H^1(K)\times P_2 \to \mathbb{R}$

$$L_K^F(\eta,\xi) = \int_F \eta[\xi-\pi_F(\xi)]d\tau \quad , F\subset\partial K.$$

Let $\hat{K}$ be the unit reference n-simplex (see e.g.[2]) such that $\mathcal{F}(\hat{K}) = K$, $\mathcal{F}$ being an affine invertible mapping from $\mathbb{R}^n$ onto $\mathbb{R}^n$.

Let $\hat{v}=v\circ\mathcal{F}$ for every function v defined in K, and define $\hat{L}: H^1(\hat{K})\times\hat{P}_2 \to \mathbb{R}$ by

$\hat{L}(\hat{\eta}, \hat{\xi}) = \int_{\hat{F}} \hat{\eta}[\hat{\xi}-\pi_{\hat{F}}(\hat{\xi})]\, d\hat{\tau}$, where $\hat{F} = \mathcal{F}^{-1}(F)$,

$\hat{P}_2$ being the space of polynomials of degree less than or equal to two defined in $\hat{K}$.

Noticing that $\pi_F(\xi) = \pi_{\hat{F}}(\hat{\xi})$ we have:

$$L_K^F(\eta,\xi) \le \frac{n(n-1)}{2}\,\text{meas}(F)\ \hat{L}(\hat{\eta},\hat{\xi}) \qquad \forall\eta\in H^1(K) \qquad \forall\xi\in P_2$$

or yet, following standard estimates:

$$\text{(14)}\quad L_K^F(\eta,\xi) \le C\ h^{n-1}\ \hat{L}(\hat{\eta},\hat{\xi}) \qquad \forall\eta\in H^1(K) \qquad \forall\xi\in P_2$$

On the other hand, since P_2 is a finite dimensional space, using the Trace Theorem we have:

$$\| L_K^F \| = \sup_{\substack{\eta \in H^1(K) \\ \xi \in P_2}} \frac{L_K^F(\eta,\xi)}{\|\eta\|_{1,K} \, \|\xi\|_{2,K}} < \infty \text{ and also } \|\hat{L}\| < \infty$$

Now we notice that $\hat{L}(\hat{\eta},\hat{\xi}) = 0$ whenever $\hat{\xi}$ is a linear function. Therefore, from the Bramble-Hilbert Lemma [2] there exists $C>0$ such that $\hat{L}(\hat{\eta},\hat{\xi}) \le C\|\hat{L}\| \, \|\hat{\eta}\|_{1,\hat{K}} |\hat{\xi}|_{2,\hat{K}} \quad \forall \hat{\eta} \in H^1(\hat{K}), \quad \forall \hat{\xi} \in \hat{P}_2$,

Using standard estimates we get

$$\|\hat{\eta}\|_{1,\hat{K}} \le Ch^{-n/2} \|\eta\|_{1,K} \quad \text{and} \quad |\hat{\xi}|_{2,\hat{K}} \le Ch^{-n/2+2} |\xi|_{2,K}$$

which yields, taking(14) into account:

$$L_K^F(\eta,\xi) \le Ch\|\eta\|_{1,K} |\xi|_{2,K} \qquad \forall \eta \in H^1(K), \quad \forall \xi \in P_2.$$

Thus setting $\eta = \vec{\text{grad}}\Delta\phi . \vec{\nu}_F^K$, $\vec{\nu}_F^K$ being the restriction of $\vec{\nu}^K$ to F, $\xi = \psi_h$, and summing up over $F \subset \partial K$ we get

$$\sum_{F\subset\partial K} \int_F \frac{\partial\Delta\phi}{\partial\nu^K} [\psi_h - \pi_F(\psi_h)] d\tau \le Ch\|\Delta\phi\|_{2,K} |\psi_h|_{2,K} \qquad \forall K \in \tau_h \quad ,$$

which by summation over $K \in \tau_h$ yields (12), realling (13). q.e.d.

Now taking into account (6)∿(12), we have:

<u>Theorem 1</u>: If the solution ϕ of (P) is such that $\phi \in H^3(\Omega)$ and $\Delta\phi \in H^2(\Omega)$ then the approximate solution ϕ_h of (P_h), when V_h is the space defined in Section 2, satisfies:

$$(15) \quad \|\phi - \phi_h\|_{2,h} \le Ch[\|\phi\|_{3,\Omega} + \|\Delta\phi\|_{2,\Omega}].$$

4. CONCLUDING REMARKS

1st) From (4) it is seen that Morley's element cannot be directly viewed as a member of this family for n=2, since such a member is not defined. However after some algebraic manipulations, this element becomes a member of the family. More specifically, we take $\mu=-2$ and we combine the basis functions, in such a way that the $D_{ij}(v)$'s equal to $[v(S_i)+ v(S_j)]/2$ for n=2, are transformed into functional values at the vertices of the triangle.

2nd) Parametrized degrees of freedom of type b) prove again in this work to be a powerful tool to define n-dimensional versions of conforming or nonconforming triangular finite element methods that work. This technique had already appeared to be useful for 3D fluid flow problems [8].

3rd) Error estimate (15) also applies to other biharmonic problems in a hyperpolyhedron Ω:
This is particularly the case of the following one:

$$(\tilde{E}) \begin{cases} \Delta^2\phi = f \quad \text{in } \Omega, \ f \in L^2(\Omega) \\ \phi = \Delta\phi = 0 \quad \text{on } \Gamma \end{cases}$$

Indeed, we still can write(E) in the equivalent variational form $(\tilde{P})$ obtained from(P) by replacing $H^2_0(\Omega)$ by the space $H^2(\Omega) \cap H^1_0(\Omega)$. We can also approximate (P) by $(\tilde{P}_h)$, where $(\tilde{P}_h)$ is obtained by replacing in (P_h), V_h by $\tilde{V}_h$.
Problem $(\tilde{P}_h)$ is still well posed and the whole convergence analysis given in this paper applies to its solution ϕ_h
Moreover if Ω is convex we have

$$\| \phi - \phi_h \|_{2,h} \leq Ch \; [\| \phi \|_{3,\Omega} + \| f \|_{0,\Omega}]$$

because in this case $\| \Delta\phi \|_{2,\Omega} \leq C \| \Delta^2 \phi \|_{0,\Omega} = C \| f \|_{0,\Omega}$, according to well-known results (see. e.g.[4]).

4th) The finite element method presented in this paper for n=3 or 4 is suitable for the solution of the following time dependent problem:

$$(\tilde{\tilde{E}}) \quad \begin{cases} \dfrac{\partial^2 \phi}{\partial t^2} - \Delta^2 \phi = f & \text{in } \Omega \times (0,T) \\ \phi = \dfrac{\partial \phi}{\partial \nu} = 0 & \text{on } \Gamma \times (0,T) \\ \phi(x,0) = \phi_0(x) & \text{in } \Omega \\ \dfrac{\partial \phi}{\partial t}(x,0) = \phi_1(x) & \text{in } \Omega \end{cases}$$

T being a given time, and f, ϕ_0 and ϕ_1 being given functions. More specifically we can partition the domain $\Omega \times (0,T)$ into tetrahedrons or 4-simplices, according to the number of space variables (two or three respectively), and then discretize $(\tilde{\tilde{E}})$ by means of the usual space-time finite element technique. Here the structure of the space-time test functions is the one described in this paper, and the initial conditions can be approximated in a straightforward way.

Notice that if there are two space variables (n=3) , equation $(\tilde{\tilde{E}})$ describes the vibrations of a clamped plate represented by Ω.

5th) Another possible application of the element presented in this work is the solution of the Stokes problem in $\mathbb{R}^3$ in potential vector formulation. Such a problem described in detail in [3] involves the biharmonic operator, but its approximate solution using the vector version of our element for n=3 cannot be studied as a trivial extension of the scalar case treated here. That is why it will be the subject of a forthcoming paper [9].

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