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AUTOMATIC THEOREM PROVING: AN ATTEMPT TO IMPROVE READABILITY OF PROOFS GENERATED BY RESOLUTION

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AUTOMATIC THEOREM PROVING : AN ATTEMPT TO IMPROVE READABILITY OF PROOFS GENERATED BY RESOLUTION.*

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ABSTRACT

The resolution method to generate proofs by machine [CHANG & LEE 73] is widely used because it is a quick way to check whether a formula is a theorem. But the proofs it gives are hard to read. We set out criteria for readability of proofs, and then present a function which translates resolution proofs for the propositional logic into Natural Deduction proofs. The latter will satisfy the criteria.

RESUMO

A geração automática de provas de teoremas via o método de resolução é bastante utilizado, uma vez que é uma forma eficiente de verificar se uma dada fórmula é teorema. Mas as provas geradas por resolução são de leitura difícil. Nós estabelecemos critérios para a legibilidade de provas e apresentamos uma função que traduz provas em resolução para provas em /Dedução Natural/ Onde estas útimas satisfarão mais aos critérios estabelecidos.

KEY-WORDS: resolution, natural deduction, automatic translation, sequent calculus, intuitionism.

PALAVRAS-CHAVE: resolução, dedução natural, tradução automática, cálculo de sequentes, intuicionismo.

I. *Introduction*

The readability (understanding) of a proof in a given deductive system, depends on the following factors :

I - Whether an inference rule can convey more information than just that the consequent follows from the premisses.

II - Whether there are implicit translation rules for formulas as well as the explicit deduction rules.

III - The directness of the proof, that is, whether or not the theorem can be proved without using indirect means (e.g. reductio ad absurdum).

The first two items are related to the deductive system, while the last one is related to the proof's style, independent of the deductive system (taking into account the fact that there are deductive systems that can only provide indirect proofs, e.g., tableaux systems, resolution).

The resolution method usually applies syntactical transformations on the formula to be proved, for example the transformation of its negation into the conjunctive normal form. This language change then causes readability problems because implication and conjunction are replaced by disjunction and negation. Moreover, the transformation cannot be made in the resolution system itself and therefore it must be carried out by some other deductive system, or by a set of transformation rules without a deductive system characterization. All these facts lead us to conclude that :

- The resolution rule does not satisfy the first item of the readability criteria, for it makes it impossible to recapture the deductive role played by implication and conjunction.

- The transformation to the conjunctive normal form is not explicit in resolution, thus hiding important information about logical relationships of the parts of a proof. Moreover, it is not usually indicated which conjunctive normal form is associated to which clause in the proof, in particular whether to one of the axioms or to the negation of the formula to be proved.

- Proofs generated by resolution are indirect proofs, they are refutations of the theorem to be proved, while direct proofs are usually much more informative [STATHAN 77].

In spite of all this, the resolution method is largely used, and with good reason :

I - It has only one inference rule and only one goal to be achieved (the empty clause). The algorithm used does not raise questions such as which rule to use or which formula to demonstrate. These questions are very commonly posed in most other deductive systems.

II - Often we are not concerned with the proof, but only whether one exists and hence if a given formula belongs to a theory.

In order to solve the readability problem we propose (automatic) translation procedures that convert resolution proofs into Natural Deduction proofs, where we try to make explicit all the steps hidden by the resolution method. The proofs generated by these procedures will consequently have a greater degree of readability according to the criteria above. One of these procedures keeps the indirect character of the proof, while the other aims at the construction of a direct proof based on the refutation generated by resolution.

II. *The process for indirect proof construction.*

In order to be able to translate a resolution proof into a Natural Deduction proof one must be able to "write" the resolution rule in Natural Deduction. This is done in two steps :

1. We define a translation function that takes resolution proofs into sequent calculus proofs ;

2. We define a translation function that takes sequent calculus proofs into Natural Deduction proofs.

The translation from resolution to sequent calculus is based on the relationship between the resolution rule and the cut rule ; we shall use a version of sequent calculus where the antecedent as well as the consequent of a sequent are sets instead of sequences. We can think of this relationship as being based on the fact that a clause can be seen as a sequent with the antecedent formed by the set of atoms which are the negative literals in the clause, and with the consequent formed by the set of atoms which are the positive literals in the clause. The function "Ts" is defined so that it maps resolution deductions into sequent calculus deductions in such a way that the following result can be obtained [HAEUSLER 86] :

THEOREM I. Let Π be a refutation in resolution with a set of top-clauses Θ. Then $Ts(\Pi)$ is a deduction of the empty sequent with top-sequents $Cs(\Theta)$, in a sequent calculus which has the cut rule as the only inference rule, where Cs is the function that takes clauses into sequents in the way mentioned above.

The next step is the translation of sequent calculus deductions into Natural Deduction. One can observe that the top-sequents of deductions generated by function Ts are not necessarily initial sequents. Therefore the usual relationship between sequent calculus and Natural Deduction ([GENTZEN 35], [PRAWITZ 65], [ZUCKER 74] and [POTTINGER 77]) cannot be used. In order to solve this problem, we use the intuitive interpretation of sequents, where :

$$\Delta \Rightarrow \Gamma \equiv \bigwedge(\Delta) \rightarrow \bigvee(\Gamma) \text{ if } \Delta,\Gamma \neq \{\}$$

$$\Delta \Rightarrow \Gamma \equiv \bigvee(\Gamma) \text{ if } \Delta = \{\}$$

$$\Delta \Rightarrow \Gamma \equiv \neg\bigwedge(\Delta) \text{ if } \Gamma = \{\}.$$

Based on this sequent intuition, one can regard the cut rule as a derived rule in Natural Deduction (extended to include generalized conjunctions $\bigwedge$ and generalized disjunctions $\bigvee$). The cut rule in this new version has different configurations depending on the form of the antecedent or of the consequent. For each cut rule configuration the process of writing one derivation in Natural Deduction is called <u>partial translation</u>. The process of complete translation can now be defined in terms of a recursive function "Sn", where the basic step is the intuitive interpretation of sequents (applied to top-sequents) and where to each aplication of the cut rule α we build up a partial translation of α . The function corresponding to the intuitive interpretation of sequents is called "seqint". The following result can then be obtained [HAEUSLER 86] :

THEOREM II. Let Π be a deduction in sequent calculus containing only applications of the cut rule. Let Θ be the set of top-sequents of Π, and S the conclusion of Π, where S is not the empty sequent. Then Sn(Π) is a deduction of seqint(S) from the assumption set seqint(Θ).

As a corollary of theorem II we have that if Π has the empty sequent as conclusion, then Sn(Π) is a deduction of the absurd ($\perp$) in Natural Deduction.

Combining theorems I and II, one has an automatic method for translating a refutation generated by resolution into a proof of the absurd in Natural Deduction with the assumption set seqint(Cs(Θ)), where Θ is the set of top-clauses. Note that the functions Sn and Ts are recursive.

In order to have the whole process made explicit in Natural Deduction, one must solve the problems presented by the extra-resolution phases, which are :

1 - the transformation of the negation of the formula to be demonstrated into its equivalent conjunctive normal form ;

2 - the derivation of the formula as a theorem from a refutation of its negation.

In order to make explicit (in Natural Deduction) the conjunctive normal form transformation, we developed the function Tf [HAEUSLER86] which takes a formula A as argument and takes a deduction (in Natural Deduction) of the conjunctive normal form of A, from the hypothesis A, as value. This function is obtained by recursion on the degree of the formula.

A conjunctive normal form is viewed, in the context of this work, as a generalized conjunction of generalized disjunction of literals.

One cannot make the whole translation process explicit using only the functions Tf, Sn and Ts. The assumption set Θ of the deduction generated by "Sn $\circ$ Ts" is not necessarily made up of clauses , since each top-clause in the refutation is associated to an element of Θ by "seqint $\circ$ Cs". Therefore, to link up the deduction of the conjunctive normal form to the deduction associated to the refutation, it is necessary to write in Natural Deduction the relationship between Θ and Seqint(Cs(Θ)), where Θ is the set of literals of a generalized disjunction of literals. This is carried out by the function Cd [HAEUSLER86], which has as argument a set of literals Δ and as value a deduction (in Natural Deduction) of Seqint(Cs(Δ)) with $\bigvee(\Delta)$ as hypothesis. In this way one can connect the deduction generated by Tf with the deduction generated by "Sn $\circ$ Ts". The result of the process as a whole is therefore a proof of the absurd in Natural Deduction with the assumption set being the negation of the theorem. To connect the deduction generated by Tf, one must connect the conjunctive normal form (its own conclusion) with each proof generated by Cd. Note that the proof generated by Cd is a deduction of one of the hypothesis of the proof of the absurd. The process is then completed, inferring the theorem by means of an application of the classical rule of the absurd.

Example 1 : In order to demonstrate $(A \wedge B) \rightarrow A$, we first obtain the conjunctive normal form of its negation, i.e. :

$$A \wedge B \wedge \neg A$$

The only possible refutation in resolution is :

$$\frac{A \qquad \neg A}{(\ )}$$

The image of this refutation by "Sn ∘ Ts" is :

$$\frac{A \qquad \neg A}{\bot}$$

The image of $\neg((A \wedge B) \rightarrow A)$, by the function Tf is :

$$\Pi = \dfrac{\dfrac{\dfrac{\neg((A \wedge B) \rightarrow A) \qquad \dfrac{[A]}{(A \wedge B) \rightarrow A}}{\bot}}{\neg A} \qquad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{[\neg(A \wedge B)] \quad [A \wedge B]}{\bot}}{A}}{(A \wedge B) \rightarrow A} \qquad \neg((A \wedge B) \rightarrow A)}{\bot}}{A \wedge B}}{}}{\neg A \wedge A \wedge B}$$

In this specific case $Cd(\neg A) = \neg A$ and $Cd(A) = A$, and therefore the whole process results in :

$$\dfrac{\dfrac{\dfrac{\dfrac{[\neg((A \wedge B) \rightarrow A)]}{\Pi}}{\neg A \wedge A \wedge B}}{\neg A} \qquad \dfrac{\dfrac{\dfrac{[\neg((A \wedge B) \rightarrow A)]}{\Pi}}{\neg A \wedge A \wedge B}}{A}}{\dfrac{\bot}{(A \wedge B) \rightarrow A}}$$

III. *Construction of direct proofs*

We observe, in the example, that the resulting translation is not economic and simple, even though all the steps in the demonstration are made explicit, which represents a reasonable gain in readability. For example, the theorem demonstrated in Example 1 is simply the formula corresponding to conjunction elimination in Natural Deduction. In this section we describe a new translation process which generates direct proofs under certain conditions. A direct proof in the context of this paper is a proof which does not have as the last inference rule the classical rule of the absurd (i.e $(\neg A \rightarrow \bot) \vdash A$). It is characteristic of the proofs in resolution that each literal of the top-clauses of a refutation has a complementary literal. This characteristic is used to build up the direct proof because some of the literals complementary to those occurring in the top-clauses are the ones which form the formula whose negation is represented by the set of top-clauses.

It may also be observed that not all the literals of the theorem are negated when the negation of the theorem is transformed to the conjunctive normal form, for example those belonging to the antecedent of an implication. We can still note that the parts which are negated are exactly those parts that, once proved, make it possible to prove the theorem directly.

The deductive determinant (DD) of a formula is recursively defined as :

$DD(A) = \{A\}$ if A is a literal.

$DD(A \rightarrow B) = DD(B)$.

$DD(A \vee B) = DD(A) \cup DD(B)$.

$DD(A \wedge B) = DD(A) * DD(B)$.

$DD(\neg(A \vee B)) = DD(\neg A) * DD(\neg B)$.

$DD(\neg(A \wedge B)) = DD(\neg A) \cup DD(\neg B)$.

$DD(\neg(A \rightarrow B)) = DD(A) * DD(\neg B)$.

Where $\{A1,..,Ak\} * \{B1,...,Bm\} = \{A1 \wedge B1, A1 \wedge B2, ..., Ak \wedge Bm\}$.

The following result is obtained for DD [HAEUSLER 86] , and was the one which motivated the definition.

We say that a deduction is intuitionistic if it is an intuitionistically acceptable deduction.

THEOREM IV. Let Π be a proof of $A \in DD(F)$. Then we can build up a proof Σ of F having Π as sub-derivation, and Σ is such that, if Π is intuitionistic, then Σ is also intuitionistic.

Theorem IV, together with what was said above about complementary literals in refutations, gives us strong reason to believe that we can construct direct proofs from refutations in resolution. What we need now is a process which makes use of refutation in order to build up properly a proof of the literals which are part of the deductive determinant of the formula whose negation was refuted. It is possible to build up a method for a specific type of refutation and this method is codified by the function Trad-Pos, which has its semantic counterpart in the following result [HAEUSLER 86] :

THEOREM V. Let Π be a deduction of $\Rightarrow \bot$ in sequent calculus, such that, the only top-sequent which has the absurd is the sequent $\Delta \Rightarrow \bot$. Π has only occurences of the cut rule and of the negation rules. Then (Trad-Pos)(Π) is a deduction of the sequent $\Rightarrow \bigwedge(\Delta)$, with the same set of top-sequents (excluding the $\Delta \Rightarrow \bot$), which has only occurrences of the cut rule, negation rules and the rule for introduction of conjunction in the consequent (in it generalized version).

The direct method can roughly be described as :

1 - Translate the negation of the refuted formula to sequent

calculus, associating to the top-clauses, which represent deductive determinants, the sequent $\Delta \Rightarrow \bot$, where Δ is the set of literals complementary to the ones belonging to the clause. The top-clauses which do not represent deductive determinants are associated to top-sequents as in Ts. Therefore the function which performs this transformation is an extension of Ts, differing from this one only by the use of rules that manipulate negation.

2 - Make (Trad-Pos)(Π1) = Π2, where Π1 is the deduction obtained in step 1, if it is of the type made specific in theorem V.

3 - Translate Π2 into Natural Deduction, using an extension of Sn which takes into account the rules of the negation and the introduction of conjunction in the consequent. Call that extension Sn+.

4 - Use theorem IV to obtain the proof of the formula, since the proof generated in step 3 is a proof of the conjunction of the DD's of the formula.

The function Trad-Pos is such that, if (Sn+)(Π1) is intuitionistic then (Sn+)(Π2) is also, where (Trad-Pos)(Π1)=Π2. We should also notice that not all top-clauses in a refutation represent deductive determinants, as for example, in the derivation of an implication. In this case, in order to make it completely explicit, we proceed as in the indirect method. That is, we use Tf and Cd in order to prove the hypothesis associated with those clauses.

Therefore, with this process a direct proof is obtained which may or may not be intuitionist. In contrast with the indirect proof, only the parts that came from the antecedent of an implication make use of function Tf, that is, the deduction is build up from those literals which were resolved in the refutation and which belonged to the deductive determinant of the formula. If the refutation is obtained using only the antecedent of the implication, then the process of indirect construction already mentioned must be used. After this, we use the intuitionistic rule of the absurd (i.e $\perp \vdash A$) to deduce the consequent. Finally, we obtain the formula through an application of implication-introduction.

Example 2 : Let us consider the formula (A $\wedge$ B) -> A. Then DD((A $\wedge$ B) -> A) = {A}, and the sequent A => $\perp$ is associated to the clause $\neg$A. The function Cd $\circ$ Tf is applied to that part which is not a deductive determinant (A$\wedge$B in this case),and its value is the deduction A $\wedge$ B. Therefore, translating the refutation by Ts+, which is the extension of Ts mentioned in 1, we obtain :

$$\frac{\Rightarrow A \qquad A \Rightarrow \perp}{\Rightarrow \perp}$$

Applying Trad-Pos to this deduction the result is :

$$\Rightarrow A$$

Note : The idea of Trad-Pos is to follow downwards the deduction. At each application α of the cut rule which has a premiss with $\perp$ in its consequent we combine the result obtained up to that point (the process is recursive) with the deduction of the other premiss, which has the literal in its consequent as the cut-formula of α. Example 2 is the basis step of the recursion.

Therefore, the translation made by Sn+ give us the result :

$$A$$

Together with the deduction generated by Cd ∘ Tf, we obtain:

$$\frac{A \wedge B}{A}$$

Using the deductive determinant theorem we can construct the following derivation :

$$\frac{\dfrac{[A \wedge B]}{A}}{(A \wedge B) \rightarrow A}$$

In order to give an idea of the scope of this method we will prove $(A \wedge B) \rightarrow (A \vee C)$. The refutation in this case is the same as above, therefore the translation via (Sn+) ∘ (Ts+) is also the same. The only difference from the previous case is that the deductive determinant of $(A \wedge B) \rightarrow (A \vee C)$ is $\{A,C\}$, and that the smaller sub-formula to which A belongs is $(A \vee C)$. Therefore the final translation process (via theorem IV) is :

$$\frac{\dfrac{\dfrac{[A \wedge B]}{A}}{A \vee C}}{(A \wedge B) \rightarrow (A \vee C)}$$

Other theorems have short and efficient deductions as in this example. The advantage of the direct method in comparison to the indirect one is therefore evident , since the former method generates deductions of a much more natural kind than the latter.

IV. *Conclusion*

Both methods proposed in this paper convert resolution proofs into proofs which undoubtedly are clearer to read. The indirect proof method works for any refutation, but it does not produce proofs which are as clear as the ones produced by the direct proof method. The latter, however, does not work for all refutations (Theorem V).

We would like to suggest the following poinis for futher study.

1 - The scope of the direct translation method can possibly be extended by the discovery of weaker conditions on refutations.

2 - We could search for new resolution strategies that would provide refutations of the appropriate kind for the application of the direct method.

3 - We could use an "improved" direct method in order to obtain a characterization of theorems which have intuitionistic proofs. Note that the direct method presented above already gives a partial characterization.

4 - An optmization of resolution could possibly be achieved through the existing connection between clauses and sequents.

5 - These proccesses should be extended to first-order logic This is done in [HAEUSLER 86] for functions Sn+, Ts+, and Trad-Pos, together with a respective extension of the notion of deductive determinat. Basically the new problems involve Skolem functions.

If these problems can be solved, then we believe that an automatic partial program generator could be constructed. We would specify a language where terms represent programs, and try to demonstrate intuitionistically a formula of the type $\exists x P(x)$, where P represents the problem to be solved by the program we wish to construct.

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