PUC

Series: Monografias em Ciência da Computação
Nº 2/88

APPROXIMATION OF THE VECTOR POTENTIAL FOR VISCOUS INCOMPRESSIBLE FLOW VIA THE CONSTANT STRESS FINITE ELEMENT

by

Vitoriano Ruas

Departamento de Informática

PONTIFÍCIA UNIVERSIDADE CATÓLICA DO RIO DE JANEIRO

MARQUÊS DE SÃO VICENTE, 225 — CEP 22453

RIO DE JANEIRO — BRASIL

Series: Monografias em Ciência da Computação

Nº 2/88

APPROXIMATION OF THE VECTOR POTENTIAL FOR VISCOUS INCOMPRESSIBLE FLOW VIA THE CONSTANT STRESS FINITE ELEMENT

by

Vitoriano Ruas

Departamento de Informática

PUC/RJ - DEPARTAMENTO DE INFORMÁTICA

Series: Monografias em Ciência da Computação, nº 2/88
Editor: Paulo A. S. Veloso January, 1988.

APPROXIMATION OF THE VECTOR POTENTIAL FOR VISCOUS INCOMPRESSIBLE FLOW VIA THE CONSTANT STRESS FINITE ELEMENT *

By

Vitoriano Ruas

* This work has been partially sponsored by FINEP.

In charge of publications:

Rosane T.L. Castilho

Assessoria de Biblioteca, Documentação e Informação
PUC/RJ - Depto. de Informática
Rua Marquês de São Vicente, 225 - Gávea
22453 - Rio de Janeiro, RJ
BRASIL

ABSTRACT

A convergence analysis for a piecewise quadratic finite element method to solve vector biharmonic problems in $\mathbb{R}^3$ in primal variables is presented. The application to the equations of the vector potential for the flow of incompressible viscous fluids is focused. For simplicity, only the particular case of stationary stokesian flows is treated in detail, but it is showed that the method applies as well to more complex flows of such fluids.

KEY-WORDS: Biharmonic, constant stress, convergence, finite elements, incompressible, nonconforming, parametrized degrees of freedom, three-dimensional, vector potential, viscous flows.

RESUMO:

Este artigo trata da análise de convergência de um método de elementos finitos quadráticos para resolver problemas biharmônicos vetoriais em dimensão três. Visou-se com esse estudo a aplicação ao caso das equações do potencial vetor associado ao escoamento de um fluído viscoso incompressível. No interesse da concisão só o caso particular de escoamentos estacionários Stokesianos é considerado em detalhes, embora mostre-se que o método também se aplica a escoamentos mais complexos desse tipo de fluidos.

PALAVRAS-CHAVES: Biharmônico, convergência, elementos finitos , escoamentos viscosos, graus de liberdade parametrados, incompressível, não conforme, potencial vetor, tensões constantes, tridimensional.

CONTENTS

1. INTRODUCTION AND STATEMENT OF THE PROBLEM

The finite element method as a tool for solving the equations of viscous incompressible flow is nowadays a well established technique. Much progress has been made in the past two decades, mainly due to the decive conceptual contributions of some authors, most of which have concentrated their efforts on the primitive variable formulation. This is particularly true of the case of three-dimensional problems, for in the two-dimensional case both the stream function-vorticity and the pure stream function formulations are also commonly in use, though much less extensively.

The stream function formulation is particularly attractive to specialists, since only one scalar variable appears in the equations. However, the fact that it is a fourth order problem becomes a drawback, because good finite element methods of solution must necessarily have more complicated structures. The equivalent formulation in three dimension space is the one expressed in terms of the vector potential, which leads to vector biharmonic problems. Although the unknown is a vector field in this case, it can still be of practical interest, at least for low Reynolds number flows . This is because there is one unknown less than in the primitive variable case and, as we will recall later on, the three components of the vector potential only couple on the boundary of the flow region. The main problem of this formulation is again the lack of simple and efficient finite element methods of solution of 3D vector biharmonic problems.

In a recent paper [12] the author studied a new family of finite element methods to solve biharmonic problems in $\mathbb{R}^N$, $N \geq 3$. This family consists of nonconforming piecewise quadratic functions defined in N-simplices, and can be viewed as the N-dimensional version of the Morley triangle [8], since long known to be suitable for the solution of fourth order problems in $\mathbb{R}^2$. Similarly to that triangle, this family corresponds to the simplest possible finite element method to solve biharmonic problems in spaces of arbitrary dimension, but its construction requires the <u>use of parametrized degrees of freedom</u> introduced in [11].

In this paper we show that the member of this family for $N=3$ is perfectly adapted to the numerical solution of vector bi harmonic problems in $\mathbb{R}^3$, related to the equations of the vector potential for the flow of an incompressible viscous fluid, as long as they are expressed in terms of the sole vector potential.

Although the method also works for much more general situations, for the sake of simplicity a special emphasis is given to the case of stationary stokesian flows in a bounded domain Ω of $\mathbb{R}^3$, which is further assumed to be convex and to have a polyhedral boundary Γ. Moreover, we consider the case of a vector potential $\vec{\psi}$ whose tangencial components on Γ vanish.

Under the above assumptions the equations for $\vec{\psi}$ write (see e.g. [4])

$$(1.1) \quad \begin{cases} \left.\begin{array}{ll} \mu\Delta^2\vec{\psi} & = \operatorname{curl} \vec{f} \\ \operatorname{div}\vec{\psi} & = 0 \end{array}\right\} \text{ in } \Omega \\ \left.\begin{array}{ll} \vec{\psi}\wedge\vec{\nu} & = \vec{0} \\ \operatorname{curl}\vec{\psi}\wedge\vec{\nu} & = \vec{0} \end{array}\right\} \text{ on } \Gamma^{(1)} \end{cases}$$

where μ is the kinematic viscosity, $\vec{f}$ stands for given volumetric forces assumed to be in $H(\operatorname{curl},\Omega)$,

$$H(\operatorname{curl},\Omega) = \{\vec{g}/\vec{g}\in[L^2(\Omega)]^3 \ , \ \operatorname{curl}\vec{g} \in [L^2(\Omega)]^3\} \ ,$$

and $\vec{\nu} = (\nu_1, \nu_2, \nu_3)$ represents the unit outer normal vector with respect to Γ.

According to [4], system (1.1) has a unique solution and the velocity field $\vec{u}=\operatorname{curl}\vec{\psi}$, together with a hydrostatic pressure p uniquely defined up to an additive constant, satisfy the Stokes system, namely:

$$(1.2) \quad \begin{cases} \left.\begin{array}{l} -\mu\Delta\vec{u} + \operatorname{grad} p = \vec{f} \\ \operatorname{div}\vec{u} = 0 \end{array}\right\} \text{ in } \Omega \\ \vec{u} = \vec{0} \qquad \text{on } \Gamma \end{cases}$$

(1) As pointed out in [4] both conditions imply that $\operatorname{curl}\vec{\psi} = \vec{0}$ on Γ.

Incidentally, it may be convenient to rewrite system (1.1) in the following form:

$$
(1.3)\quad
\left.\begin{array}{ll}
\mu\Delta^2\vec{\psi} = \mathrm{curl}\,\vec{f} & \text{in } \Omega \\
\mathrm{div}\vec{\psi} = 0 & \\
\mathrm{div}\Delta\vec{\psi} = 0 & \\
\vec{\psi}\wedge\vec{\nu} = \vec{0} & \text{on } \Gamma \\
\mathrm{curl}\,\vec{\psi}\wedge\vec{\nu} = \vec{0} &
\end{array}\right.
$$

That (1.1) implies (1.3) is obvious. On the other hand, the equivalence of both formulations can be established as a consequence of the application of the divergence operator on both sides of the first equation of (1.3). Indeed, this yields a scalar biharmonic problem for $\mathrm{div}\vec{\psi}$ with homogeneous data and boundary conditions, which implies that $\mathrm{div}\vec{\psi}=0$ in the whole Ω.

An outline of the paper is as follows:

In Section 2 we study a variational form for system (1.3) to be used in connection with the finite element method of solution described in Section 3. In Section 4 we give a convergence analysis for this approximation method, and in Section 5 we briefly consider its extension to the case of the Navier-Stokes equations, among other problems. We conclude in Section 6 with some important remarks.

2. A VARIATIONAL FORM FOR POLYHEDRAL DOMAINS

First of all we note that in writing system (1.3) we implicitly admit that $\Delta\vec{\psi}\in[H^2(\Omega)]^3$, an assumption which will be made throughout the paper, together with $\vec{\psi}\in[H^3(\Omega)]^3$. Both are reasonable taking into account the assumed converxity of Ω (see [6] and [5]).

Let us now introduce the following space:

$$X = \{\vec{\phi}/\ \vec{\phi}\in[H^2(\Omega)]^3\ ;\ \vec{\phi}\wedge\vec{\nu}=\vec{0},\ \mathrm{curl}\vec{\phi}\wedge\vec{\nu}=\vec{0}\ \text{ and } \mathrm{div}\vec{\phi}=0\ \text{ on } \Gamma\}.$$

X can be normed by the canonical semi-norm of $H^2(\Omega)$, namely

$$(2.1) \qquad |\vec{\phi}|_{2,\Omega} = \Big(\sum_{i=1}^{3} \sum_{j=1}^{3} \int_{\Omega} \frac{\partial^2 \vec{\phi}}{\partial x_i \partial x_j} \cdot \frac{\partial^2 \vec{\phi}}{\partial x_i \partial x_j} \Big)^{\frac{1}{2}}$$

where the notation $\vec{x}.\vec{y}$ stands for the euclidean inner product of two vectors or tensors $\vec{x}$ and $\vec{y}$. Indeed, if $|\vec{\phi}|_{2,\Omega} = 0$ the field $\vec{\phi}$ is linear in Ω. Then the fact that Ω is bounded , together with $\vec{\phi} \wedge \vec{\nu} = \vec{0}$ on Γ, implies that $\vec{\phi} \equiv \vec{0}$.

Before giving the variational form to be used later on we recall below a more natural one already considered in [13].

Set $\forall \vec{\psi}, \vec{\phi} \in [H^2(D)]^3$,

$$(2.2) \qquad \bar{a}_D(\vec{\psi},\vec{\phi}) = \mu \int_D \text{grad curl}\vec{\psi} \cdot \text{grad curl}\vec{\psi} \; +$$

$$+ \mu \int_D \text{grad div}\vec{\psi} \cdot \text{grad div}\vec{\phi}$$

where D is any open subset of Ω with meas (D) $\neq 0$, and pose the problem:

$$(2.3) \qquad \begin{cases} \text{Find} \quad \vec{\psi} \in X \quad \text{such that} \\ \bar{a}_\Omega(\vec{\psi},\vec{\phi}) = (\text{curl}\vec{f},\vec{\phi}) \qquad \forall \vec{\phi} \in X \end{cases}$$

where the notation (.,.) stands for the usual inner product of $L^2(\Omega)$. The facts that (2.3) has a unique solution $\vec{\psi}$ and that $\vec{u}=\text{curl}\vec{\psi}$ is the velocity field that satisfies the Stokes system (1.2) have been established in [13]. Usual arguments based on Green's formulae, also allow to conclude that $\vec{\psi}$ solves system (1.3) or yet (1.1).

We shall now prove

<u>Lemma 2.1</u>: If the solution of (2.3) belongs to $[H^3(\Omega)]^3$ then it is also the unique solution of

$$(2.4) \qquad \begin{cases} \text{Find} \quad \vec{\psi} \in X \quad \text{such that} \\ a_\Omega(\vec{\psi} , \vec{\phi}) = (\text{curl}\vec{f} , \vec{\phi}) \qquad \forall \vec{\phi} \in X \end{cases}$$

where for $D \subseteq \Omega$, meas $(D)\neq 0$, and $\forall\vec{\psi},\vec{\phi}\in[H^2(D)]^3$,

$$(2.5)\quad a_D(\vec{\psi},\vec{\phi}) = \mu\sum_{i=1}^{3}\sum_{j=1}^{3}\int_D \frac{\partial^2\vec{\psi}}{\partial x_i\,\partial x_j}\cdot\frac{\partial^2\vec{\phi}}{\partial x_i\,\partial x_j}$$

Proof: The existence and uniqueness of a solution to (2.4) have been established in [13]. Therefore if suffices to prove the following result:

$\forall\vec{\phi}\in X$ and $\forall\vec{\psi}\in X\cap[H^3(\Omega)]^3$ we have:

$$a_\Omega(\vec{\psi},\vec{\phi}) = \bar{a}_\Omega(\vec{\psi},\vec{\phi})$$

First of all we assume that the cartesian axes are oriented in such a way that $\nu_i\neq 0$, $i=1,2,3$ almost everywhere on Γ. Next we note that

$$(2.6)\quad \bar{a}_\Omega(\vec{\psi}\,,\vec{\phi}) - a_\Omega(\vec{\psi}\,,\vec{\phi}) = \mu\sum_{1\leq j<i\leq 3} A_{ij}\quad,\qquad \text{where}$$

$$(2.7)\quad A_{ij}=\sum_{k=1}^{3}\int_\Omega\Big[\frac{\partial^2\psi_i}{\partial x_i\partial x_k}\cdot\frac{\partial^2\phi_j}{\partial x_j\,\partial x_k}+\frac{\partial^2\psi_j}{\partial x_j\,\partial x_k}\,\frac{\partial^2\phi_i}{\partial x_i\,\partial x_k}-\frac{\partial^2\psi_i}{\partial x_j\partial x_k}\,\frac{\partial^2\phi_j}{\partial x_i\,\partial x_k}-\frac{\partial^2\psi_j}{\partial x_i\,\partial x_k}\,\frac{\partial^2\phi_i}{\partial x_j\,\partial x_k}\Big]$$

Using integration by parts we obtain:

$$(2.8)\quad A_{ij} = \sum_{k=1}^{3}\Big\{\int\Big[\frac{\partial^2\psi_i}{\partial x_i\partial x_k}\,\nu_j-\frac{\partial^2\psi_i}{\partial x_j\,\partial x_k}\,\nu_i\Big]\frac{\partial\phi_j}{\partial x_k} - \int_\Gamma\Big[\frac{\partial^2\psi_j}{\partial x_i\partial x_k}\nu_j-\frac{\partial^2\psi_j}{\partial x_j\,\partial x_k}\,\nu_i\Big]\frac{\partial\phi_i}{\partial x_k}\Big\}$$

Now we note that since $\vec{\phi}\wedge\vec{\nu} = \vec{0}$ on Γ and $\phi_i\in H^2(\Omega)$, $i=1,2,3$, for any unit tangential directions $\vec{\tau}$ and $\vec{\sigma}$ with respect to Γ, we have $\partial\phi_\sigma/\partial\tau = 0$ a.e. on Γ, where $\phi_\sigma = \vec{\phi}.\vec{\sigma}$ and $\partial v/\partial\tau$ denotes the tangential derivative in the direction of $\vec{\tau}$ of a function $v\in H^{3/2}(\Gamma)$, defined almost everywhere on Γ. Thus if we take both $\vec{\sigma}$ and $\vec{\tau}$ equal to $s_{mn} = (s^1_{mn}, s^2_{mn}, s^3_{mn})$, $1 \le m < n \le 3$, where $s^i_{mn} = (\delta_{in}\nu_m - \delta_{im}\nu_n)/(\nu_m^2 + \nu_n^2)^{\frac{1}{2}}$, δ_{ij} being the Kronecker delta, since $\vec{\nu}$ is piecewise constant with respect to $\vec{s}_{mn}$, the following equality holds for any k and ℓ:

$$(2.9) \quad 2\nu_\ell \nu_k \left(\frac{\partial \phi_\ell}{\partial x_k} + \frac{\partial \phi_k}{\partial x_\ell} \right) = \nu_k^2 \frac{\partial \phi_\ell}{\partial x_\ell} + \nu_\ell^2 \frac{\partial \phi_k}{\partial x_k} \quad a.e \text{ on } \Gamma .$$

Taking into account that $\text{curl}\vec{\phi} = \vec{0}$ on Γ, we get:

$$(2.10) \quad \frac{\partial \phi_\ell}{\partial x_k} = \frac{\nu_k}{2\nu_\ell} \frac{\partial \phi_\ell}{\partial x_\ell} + \frac{\nu_\ell}{2\nu_k} \frac{\partial \phi}{\partial x_k} \quad a.e \text{ on } \Gamma.$$

Now taking (2.10) into (2.8) we obtain:

$$(2.11) \quad A_{ij} = \sum_{k=1}^{3} \int_\Gamma \left[\frac{\partial}{\partial s_{ji}} \left(\frac{\partial \psi_i}{\partial x_k} \right) \left(\frac{\nu_k}{2\nu_j} \frac{\partial \phi_j}{\partial x_j} + \frac{\nu_j}{2\nu_k} \frac{\partial \phi_k}{\partial x_k} \right) - \right.$$

$$\left. - \frac{\partial}{\partial s_{ji}} \left(\frac{\partial \psi_j}{\partial x_k} \right) \left(\frac{\nu_k}{\partial \nu_i} \frac{\partial \phi_i}{\partial x_i} + \frac{\nu_i}{2\nu_k} \frac{\partial \phi_k}{\partial x_k} \right) \right]$$

Set now in (2.10) $\vec{\phi} = \vec{\psi}$. Taking into account that grad $\psi_i \in [H^2(\Omega)]^3$, $i=1,2,3$ and $\text{curl}\vec{\psi} = \vec{0}$ on Γ, it is possible to differentiate both sides of (2.10) with respect to s_{ji} and to write:

$$(2.12) \quad \frac{\partial}{\partial s_{ji}} \frac{\partial \psi_\ell}{\partial x_k} = \frac{\nu_k}{2\nu_\ell} \frac{\partial}{\partial s_{ji}} \frac{\partial \psi_\ell}{\partial x_\ell} + \frac{\nu_\ell}{2\nu_k} \frac{\partial}{\partial s_{ji}} \frac{\partial \psi_k}{\partial x_k} ,$$

$\ell = i$ or j, on every plane face of Γ, using again the assumption that Ω is a polyhedron.

Now, if $k \ne i$ and $k \ne j$ the condition div $\vec{\phi}=0$ on Γ allows us to replace $\frac{\partial \phi_k}{\partial x_k}$ in (2.11) with $-\left(\frac{\partial \phi_i}{x_i} + \frac{\partial \phi_j}{\partial x_j}\right)$. Next we perform a a similar substitution in (2.12) for $\frac{\partial \psi_k}{\partial x_k}$, and we take the resulting expressions into the just modified relation (2.11), for k=1,2 and 3.

In so doing we obtain after some straightforward calculations:

$$(2.13)\qquad A_{ij} = \int_\Gamma \frac{1}{\nu_j\nu_i}\left[\frac{\partial}{\partial s_{ji}}\frac{\partial\psi_i}{\partial x_i}\frac{\partial\phi_j}{\partial x_j} - \frac{\partial}{\partial s_{ji}}\frac{\partial\psi_j}{\partial x_j}\frac{\partial\phi_i}{\partial x_i}\right]$$

On the other hand it can be easily verified that:

$$(2.14)\qquad \frac{\partial v}{\partial s_{13}} = \frac{\nu_3}{\nu_2}\frac{\partial v}{\partial s_{12}} + \frac{\nu_1}{\nu_2}\frac{\partial v}{\partial s_{23}}$$

for every sufficiently smooth function v defined on Γ. We may then modify (2.13) for A_{31} accordinglly.

We may further replace $\frac{\partial\psi_2}{\partial x_2}$ and $\frac{\partial\phi_2}{\partial x_2}$ in the expressions (2.13) for A_{21} and A_{32} by using the same argument as above with $k=2$.

After having performed these modifications, we sum up with respect to i and j thereby obtaining:

$$\sum_{1\le j<i\le 3} A_{ij} = 0.$$

Finally recalling (2.6) the lemma is proved, q.e.d

3. THE FINITE ELEMENT APPROXIMATION

For the sake of clearness we first briefly recall the scalar version of the finite element method studied in this paper. A more detailed description of such a version can be found in [12] .

Let then T_h be a tetrahedrization of Ω respecting the usual intersection rule for the finite element method and satisfying $\bigcup_{K\in T_h} \bar{K} = \bar{\Omega}$. We assume that T_h belongs to a quasiuniform family of tetrahedrizations of Ω in the usual sense, parametrized by h, the maximum diameter of a tetrahedron of T_h.

Let S_i, $i=1,2,3,4$ be the vertices of a tetrahedron $K\in T_h$, F_i be the face opposite to S_i and G_i be the barycenter of F_i.

We shall define a quadratic function ϕ over K by means of the following set of ten degrees of freedom:

- $D_i(\phi)$, the outer normal derivative of ϕ at G_i, $1\le i\le 4$,

- $D_{ij}(\phi)$, the mean value of ϕ along the edge S_iS_j, $1\le i<j\le 4$, that is

$$D_{ij}(\phi) = \int_{S_iS_j} \phi \; ds/\text{length}(S_iS_j)$$

If we denote by λ_i the barycentric coordinate of K with respect to S_i, then the basis functions associated with these degrees of freedom are respectively:

$$p_i = (\lambda_i - \frac{3}{2}\lambda_i^2)/\gamma_{ii} \quad , \quad 1\le i\le 4$$

$$p_{ij} = (\lambda_i+\lambda_j)^2 - 2(\lambda_i+\lambda_j) - 2\sum_{\substack{k=1\\ k\ne i,j}}^{4} p_k(\gamma_{ik}+\gamma_{jk}), \quad 1\le i<j\le 4 \quad ,$$

where $\gamma_{\ell m} = D_m(\lambda_\ell)$, $1\le m,\ell\le 4$.

Now we define V_h to be the space of functions ϕ defined in Ω, whose restriction to each tetrahedron of T_h is quadratic and that satisfy the following conditions.

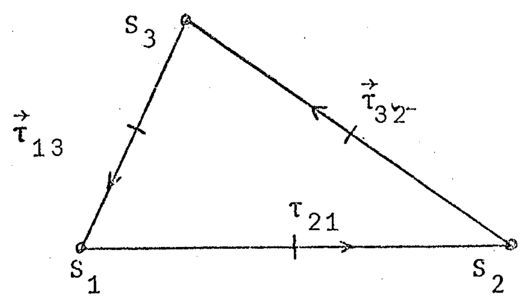
- The derivative of ϕ in a given normal direction to any face common to two tetrahedrons of T_h is continuous at the barycenter of this face.

- The mean values at a given edge of T_h of the restrictions of ϕ to the tetrahedrons intersecting at this edge coincide.

<u>Remark 3.1</u>: It is interesting to recall [12] that both conditions imply the continuity of gradϕ at the barycenter of the faces of T_h. This is because the tangential derivatives of ϕ at the barycenter of any face of a tetrahedron depend linearly on the mean values over the edges of this face. More specifically, it is easy to prove that if S_i,S_j and S_k are the vertices of $K\in T_h$ belonging to a given face with barycenter G then:

$$\frac{\partial\phi}{\partial\tau_{ij}}(G) = 2\,\frac{D_{ki}(\phi) - D_{kj}(\phi)}{\text{length}(S_iS_j)} \qquad i,j,k \text{ distinct}$$

where $\vec{\tau}_{ij}$ is the unit tangential vector of the face oriented from S_j to S_i. We refer to Figure 3.1 for an illustration. □



Tangential directions of face $S_1\ S_2\ S_3$
Figure 3.1

We are now ready to define a discrete analogue X_h of X based on the finite element space V_h. Noticing that a field of X satisfies three types of boundary conditions, we define X_h to be the space of fields $\vec{\phi}_h$ whose cartesian components belong to V_h and are such that their respective degrees of freedom attached to faces or edges lying on Γ are related in the following way:

(i) The discrete analogue of $\vec{\phi}\wedge\vec{\nu} = \vec{0}$ on Γ:

The mean value of any tangential component of $\vec{\phi}_h$ with respect to Γ over every boundary edge of T_h vanishes;

(ii) The discrete analogue of $\text{curl}\vec{\phi}\wedge\vec{\nu} = \vec{0}$ on Γ:

The normal derivative at the barycenter of a boundary face of T_h of any tangential component of $\vec{\phi}_h$ with respect to this face, is a linear combination of the mean values of the normal component of $\vec{\phi}_h$ over the three edges of this face. The coefficients of this linear combination depend only on the lenght of these three edges and on the tangential direction itself.

(iii) The discrete analogue of $\text{div}\vec{\phi} = 0$ on Γ:

The normal derivative of the normal component of $\vec{\phi}_h$ with respect to any boundary face of T_h vanishes at the barycenter of this face. □

It is important to stress the fact that properties (ii) are consistent with Remark 3.1 and property (i). For this reason property (ii) and property (iii) actually mean that at the barycen ter of boundary faces $\mathbf{curl}\vec{\phi}_h \wedge \vec{\nu} = \vec{0}$ and $\mathrm{div}\vec{\phi}_h = 0$, respectively. Moreover Remark 3.1 together with property (i) imply that at the same points $\mathbf{curl}\vec{\phi}_h . \vec{\nu} = \vec{0}$ too.

Now we equip $[V_h]^3$ with the discrete H^2-seminorm, namely:

$$(3.1) \qquad \| \vec{\phi}_h \|_{2,h} = [\sum_{K \in T_h} \sum_{i,j=1}^{3} \int_K \frac{\partial^2 \vec{\phi}_h}{\partial x_i \partial x_j} \cdot \frac{\partial^2 \vec{\phi}_h}{\partial x_i \partial x_j}]^{\frac{1}{2}}$$

It is actually possible to prove:

<u>Lemma 3.1</u>: The seminorm $\|.\|_{2,h}$ given by (3.1) is a norm for X_h.

<u>Proof</u>: If $\vec{\phi}_h \in X_h$ and $\| \vec{\phi}_h \|_{2,h} = 0$ then $\vec{\phi}_h$ is a linear field in every tetrahedron of $K \in T_h$. The continuity of the normal derivatives at the barycenter and of the mean values over the edges of every face of T_h for fields of $[V_h]^3$ implies that $\vec{\phi}_h$ is continuous (see e.g. [12]) over the whole Ω. Moreover, since three distinct mean values of the tangetial components of $\vec{\phi}_h$ with respect to the boundary faces of T_h vanish, we must have $\vec{\phi}_h \wedge \vec{\nu} = \vec{0}$ on Γ. The boundedness of Ω implies in turn that $\vec{\phi}_h = \vec{0}$ on Γ. q.e.d.

Now we pose the discrete analogue of problem (2.4) based on space X_h:

$$(3.2) \qquad \begin{cases} \text{Find } \vec{\psi}_h \in X_h \text{ such that} \\ a_h(\vec{\psi}_h, \vec{\phi}_h) = (\mathbf{curl}\vec{f}, \vec{\phi}_h) \qquad \forall \vec{\phi}_h \in X_h \end{cases}$$

where

$$(3.3) \qquad a_h(\vec{\psi}_h, \vec{\phi}_h) = \sum_{K \in T_h} a_K(\vec{\psi}_h, \vec{\phi}_h) \quad ,$$

for which we can prove:

<u>Theorem 3.1</u>. Problem (3.2) has a unique solution

<u>Proof</u>: According to Lemma 3.1 if suffices to establish the coerciveness of the bilinear form a_h over X_h with respect to norm $\| . \|_{2,h}$. Recalling (2.5) and (3.1) we have indeed:

$$a_h(\vec{\phi}_h,\vec{\phi}_h) \geq \mu \; \|\vec{\phi}_h\|^2_{2,h} \qquad \forall \vec{\phi}_h \in X_h. \quad \text{q.e.d}$$

Space X_h has appropriate approximation properties with respect to sufficiently smooth fields of X. More specifically, if $\vec{\phi} \in X \cap [H^3(\Omega)]^3$ we may define $\pi_h\vec{\phi}$, the field of X_h that interpolates $\vec{\phi}$ at all the degrees of freedom of this space, except for the normal derivatives of the tangential components of $\vec{\phi}$ at the barycenter of faces lying on Γ. This is because, according to Remark 3.1, these are necessarily linear combinations of three mean values of the normal component of $\pi_h\vec{\phi}$, that is of $\vec{\phi}$, as specified in property (ii) of X_h.

Nevertheless the following approximation property holds.

Theorem 3.2: If $\pi_h\vec{\phi} \in X_h$ is the interpolate of $\vec{\phi} \in X \cap [H^3(\Omega)]^3$ defined as above, then we have:

$$(3.3) \qquad \|\vec{\phi} - \pi_h\vec{\phi}\|_{2,h} \leq C \; h \; |\vec{\phi}|_{3,\Omega}$$

where $|.|_{3,\Omega}$ denotes the standard semi-norm of $H^3(\Omega)$ and C denotes a constant independent of h.

Proof: Let $\vec{\phi}_h$ be the $[V_h]^3$-interpolant of $\vec{\phi}$. Standard approximation results allow us to prove (see e.g. [2]):

$$(3.4) \qquad \|\vec{\phi} - \vec{\phi}_h\|_{2,h} \leq C \; h \; |\vec{\phi}|_{3,\Omega} \; .$$

Let us now estimate $\|\vec{\phi}_h - \pi_h\vec{\phi}\|_{2,h}$.

First we note that by construction all the degrees of freedom of the D_{ij}-type coincide for $\vec{\phi}_h$ and $\pi_h\vec{\phi}$. Moreover all the degrees of freedom of $\vec{\phi}_h$ and $\pi_h\vec{\phi}$ of the D_i-type coincide, as long as they are not attached to faces lying on Γ.

As for the normal derivative at the barycenter of a face of a tetrahedron $K \in T_h$ lying on Γ, we can assert the following:

For simplicity let us first assume that T_h is such that $\bar{K} \cap \Gamma$ contains at most one face of K, $\forall K \in T_h$. In this case we may choose the local numbering of the vertices of K to be such that S_1, S_2 and S_3 are the vertices of its face F_4^K lying on Γ. Let $\vec{\nu}_4$ be the unit outer normal vector with respect to F_4^K and G_4^K be the barycenter of this face. We further denote by D_i^K and D_{ij}^K the degrees of freedom of types D_i and D_{ij} respectively if related to tetrahedron K. Finally for a field $\vec{v}$ defined in K we set:

$$v_{ij} = \vec{v}.\ \vec{\tau}_{ij}\ ,\ i \neq j,\ \ and\ \ v_\nu = \vec{v}.\vec{\nu}_4\ .$$

In so doing we have:

$$(3.5)\qquad D_4^K(\pi_h \phi_\nu) = D_4^K(\phi_{h\nu}) = 0$$

This is due to the fact that $\mathrm{div}\vec{\phi}=0$ on Γ, which implies that $D_4^K(\phi_{h\nu}) = D_4^K(\phi_\nu) = 0$, and to property (iii) of X_h.

On the other hand, from property (ii) of X_h and Remark 3.1 we get:

$$(3.6)\qquad D_4^K(\pi_h \phi_{ij}) = 2\ \frac{D_{ik}^K(\phi_{h\nu}) - D_{jk}^K(\phi_{h\nu})}{\text{length } (S_i S_j)}\ ,\ i,j,k \text{ distinct,}$$

whereas

$$(3.7)\qquad D_4^K(\phi_{hij}) = D_4^K(\phi_{ij})$$

Notice that the latter value differs in principle from $D_4^K(\pi_h \phi_{ij})$.

Let then p_4^K be the basis function related to the normal derivative at G_4^K for a tetrahedron $K \in T_h^*$, where

$$T_h^* = \{K/K \in T_h \text{ and area } (\bar{K} \cap \Gamma) \neq 0\}.$$

Then we have $\forall K \in T_h^*$

$$(3.8)\quad (\vec{\phi}_h - \pi_h \vec{\phi})_{/K} = \{[D_4^K(\phi_\tau) - D_4^K(\pi_h \phi_\tau)]\ \vec{\tau}_K + [D_4^K(\phi_\sigma) - D_4^K(\pi_h \phi_\sigma)]\ \vec{\sigma}_K\}\ p_4^K,$$

$\vec{\tau}_K$ and $\vec{\sigma}_K$ being two unit orthogonal vectors in the plane of F_4^K, and for a field $\vec{v}$, $v_\tau = \vec{v}.\vec{\tau}_K$ and $v_\sigma = \vec{v}.\vec{\sigma}_K$.

From the assumption that $\{T_h\}_h$ is quasiuniform, we can readily establish that there exists a constant C such that:

$$(3.9)\qquad \|\pi_h\vec{\phi} - \vec{\phi}_h\|^2_{2,h} \leq C \sum_{K\in T_h^*} \{\|p_4^K\|^2_{2,K} \sum_{1\leq j<i\leq 3} |D_4^K(\phi_{ij}) - D_4^K(\pi_h\phi_{ij})|^2\}$$

The same assumption on $\{T_h\}_h$, together with standard arguments, allow us also to conclude that there exists another constant C for which

$$(3.10)\qquad \|p_4^K\|^2_{2,K} \leq C\ h \qquad \forall K\in T_h^* .$$

Let us now turn our attention to the other term in (3.9), that is, $|D_4^K(\phi_{ij}) - D_4^K(\pi_h\phi_{ij})|$

Recalling (3.6) and taking into account that $\mathrm{curl}\vec{\phi} = \vec{0}$ on Γ we have:

$$(3.11)\qquad D_4^K(\phi_{ij}) - D_4^K(\pi_h\phi_{ij}) = J_K(\phi_\nu)$$

where

$$(3.12)\qquad J_K(\phi_\nu) = 2\,\frac{D_{jk}^K(\phi_\nu) - D_{ik}(\phi_\nu)}{\mathrm{length}(S_iS_j)} + \frac{\partial\phi_\nu}{\partial\tau_{ij}}(G_4^K)$$

where i,j,k are distinct and J_K is a continuous linear functional defined on $H^3(\Omega)$, according to the Sobolev Embedding Theorem (see e.g. [1]).

Now let $\hat{K}$ be the usual reference tetrahedron and $\hat{S}_i$ be its vertex corresponding to S_i with $\hat{S}_4 = (0,0,0)$. If $\mathcal{F}$ is the linear mapping such that $\mathcal{F}(\hat{K}) = K$, we set $\hat{\phi}_\nu = \phi_\nu \circ \mathcal{F}$ and we define $\hat{J}: H^3(\hat{K}) \to \mathbb{R}$ to be the continuous linear functional given by:

$$(3.13) \quad \hat{J}(\hat{\phi}_\nu) = \sqrt{2}\,[\hat{D}_{jk}(\hat{\phi}_\nu) - \hat{D}_{ik}(\hat{\phi}_\nu)] + \frac{\partial\hat{\phi}_\nu(\hat{G})}{\partial\hat{\tau}_{ij}}, \quad i,j,k \text{ dis-}$$

tinct, where $\hat{D}_{ij}(\hat{v})$ is the mean value of a function $\hat{v}$ over $\hat{S}_i\hat{S}_j$, $\hat{G}_4 = \mathcal{F}^{-1}(G_4^K)$ and $\vec{\tau}_{ij}$ is the unit vector along $\hat{S}_j\hat{S}_i$.

In so doing we have:

$$(3.14) \qquad J(\phi_\nu) = \frac{\hat{J}(\hat{\phi}_\nu)}{\text{length }(S_iS_j)} \,.$$

Now we note that $\hat{J}(\hat{\phi}_\nu) = 0$ whenever $\hat{\phi}_\nu$ is a quadratic function. Thus from the Bramble-Hilbert Lemma (see e.g. [2]) we get:

$$(3.15) \qquad \hat{J}(\hat{\phi}_\nu) \leq C\, |\hat{\phi}_\nu|_{3,\hat{K}}$$

From standard estimates we further obtain

$$(3.16) \qquad |\hat{J}(\hat{\phi}_\nu)| \leq C\, h^{3/2} |\phi_\nu|_{3,K} \,.$$

Combining (3.14) and (3.16) we get

$$(3.17) \qquad |J(\phi_\nu)| \leq C\, h^{1/2} |\phi_\nu|_{3,K} \,.$$

Finally, taking into account (3.9)∿(3.12) and (3.17) we conclude that there is a constant C independent of h such that

$$\|\vec{\phi}_h - \pi_h\vec{\phi}\|_{2,h} \leq C\, h |\vec{\phi}|_{3,\Omega} \,,$$

which together with (3.4) implies (3.3).

Except for additional complications in the notation, the above analysis can be applied to the case where one or more tetrahedrons of T_h have two or three faces on Γ. Hence the Lemma is proved. q.e.d.

<u>Remark 3.2</u>: In purely viscous flow the stress rates are given in terms of the second order derivatives of the vector potential. This suggests that we refer to the present element as the <u>constant stress finite element</u>. □

4. CONVERGENCE ANALYSIS

Since $X_h \not\subset X$, according to well-known results (see e.g. [3]), the error in the approximation of $\vec{\psi}$ by $\vec{\psi}_h$ can be estimated using the inequality:

$$(4.1)\quad \| \vec{\psi} - \vec{\psi}_h \|_{2,h} \leq \frac{1}{\mu} \Big\{ \inf_{\vec{\phi}_h \in X_h} \| \vec{\psi} - \vec{\phi}_h \|_{2,h} + \sup_{\vec{\phi}_h \in X_h} \frac{|a_h(\vec{\psi},\vec{\phi}_h) - (\mathrm{curl}\vec{f}, \vec{\phi}_h)|}{\| \vec{\phi}_h \|_h} \Big\}$$

The first term on the right hand side can be estimated using Theorem 3.2, that is:

$$(4.2)\quad \inf_{\vec{\phi}_h \in X_h} \| \vec{\psi} - \vec{\phi}_h \|_{2,h} \leq C\, h |\vec{\psi}|_{3,\Omega} .$$

In order to estimate the second term we need some technical lemmas. For convenience, we summarize previous regularity assumptions by writing:

$$\vec{\psi} \in W = \{\vec{\phi} / \vec{\phi} \in X,\quad \vec{\phi} \in [H^3[\Omega])^3,\quad \Delta\vec{\phi} \in [H^2(\Omega)]^3\}.$$

<u>Lemma 4.1</u>: If the solution $\vec{\psi}$ of problem (2.4) belongs to W then $\forall \vec{\phi}_h \in X_h$

$$(4.3)\quad \frac{1}{\mu} [a_h(\vec{\psi}, \vec{\phi}_h) - (\mathrm{curl}\vec{f}, \vec{\phi}_h)] = \sum_{K \in T_h} [b_K(\vec{\psi},\vec{\phi}_h) + c_K(\vec{\psi},\vec{\phi}_h)]$$

where

$$(4.4)\quad b_K(\vec{\psi},\vec{\phi}) = \int_{\partial K} [\frac{\partial \mathrm{curl}\vec{\psi}}{\partial \nu_K} . \mathrm{curl}\vec{\phi} - \Delta \mathrm{curl}\vec{\psi} . (\vec{\phi} \wedge \nu_K)]$$

and

$$(4.5)\quad c_K(\vec{\psi},\vec{\phi}) = \sum_{i,j=1}^{3} \int_K \mathrm{grad} \frac{\partial \psi_i}{\partial x_j} . \mathrm{grad} \frac{\partial \phi_j}{\partial x_i}$$

∂K being the boundary of K and $\vec{\nu}_K$ being the unit outer normal vector with respect to ∂K.

Proof: Recalling (2.2) and (2.5) it is easy to derive

$$(4.6) \qquad a_h(\vec{\psi}, \vec{\phi}_h) = \mu[\sum_{K\in T_h} d_K(\vec{\psi}, \vec{\phi}_h) + c_K(\vec{\psi}, \vec{\phi}_h)]$$

where

$$(4.7) \qquad d_K(\vec{\psi}, \vec{\phi}_h) = \int_K \text{grad curl}\vec{\psi}.\text{grad curl}\vec{\phi}_h$$

Using Green's formulae, and by interchanging the operators Δ and curl, we get

$$(4.8) \qquad d_K(\vec{\psi}, \vec{\phi}_h) = \int_{\partial K} \frac{\partial \text{curl}\vec{\psi}}{\partial \nu_K} . \text{curl}\vec{\phi}_h - \int_{\partial K} \Delta\text{curl}\vec{\psi}.(\vec{\phi}_h \wedge \vec{\nu}_K) -$$

$$- \int_K \Delta\text{curl curl}\vec{\psi} . \vec{\phi}_h .$$

From the well-known identity

$$\text{curl curl}\vec{\psi} \equiv \text{grad div}\vec{\psi} - \Delta\vec{\psi}$$

and taking into account that $\text{div}\vec{\psi} = 0$ and $\mu \ \Delta^2\vec{\psi} = \text{curl}\vec{f}$ in Ω, the proof is completed. q.e.d.

Lemma 4.2: Under the same assumption of Lemma 4.1 we have:

$$(4.9) \qquad c_K(\vec{\psi}, \vec{\phi}_h) = \int_{\partial K} \text{grad grad } \psi_\nu.\text{grad}\vec{\phi}_h \equiv \sum_{j=1}^{3} \int_{\partial K} \text{grad } \frac{\partial \psi_\nu}{\partial x_j}. \text{ grad}\vec{\phi}_{hj}$$

where $\psi_\nu = \vec{\psi} . \vec{\nu}_K$.

Proof: Since $\text{div}\vec{\psi} = 0$ in Ω we may write

$$c_K(\vec{\psi}, \vec{\phi}_h) = \sum_{\substack{i,j=1 \\ i\neq j}}^{3} \int_K \text{grad} \frac{\partial \psi_i}{\partial x_j} \cdot \text{grad} \frac{\partial \phi_{hj}}{\partial x_i} -$$

$$- \sum_{j=1}^{3} \int_K \text{grad} \sum_{\substack{i=1 \\ i\neq j}}^{3} \frac{\partial \psi_i}{\partial x_i} \cdot \text{grad} \frac{\partial \phi_{hj}}{\partial x_j}$$

or yet

$$c_K(\vec{\psi}, \vec{\phi}_h) = \sum_{\substack{i,j=1 \\ i\neq j}}^{3} \int_K [\text{grad} \frac{\partial \psi_i}{\partial x_j} . \text{grad} \frac{\partial \phi_{hj}}{\partial x_i} - \text{grad} \frac{\partial \psi_i}{\partial x_i} . \text{grad} \frac{\partial \phi_{hj}}{\partial x_j}] .$$

Using integration by parts we get:

$$c_K(\vec{\psi}, \vec{\phi}_h) = \sum_{i\neq j} \int_{\partial K} [\text{grad} \frac{\partial \psi_i}{\partial x_j} . \text{grad}\phi_{hj} \; \nu_{ki} - \text{grad} \frac{\partial \psi_i}{\partial x_i} . \text{grad}\phi_{hj} \nu_{kj}]$$

Replacing again $-\sum_{i\neq j} \frac{\partial \psi_i}{\partial x_i}$ with $\frac{\partial \psi_j}{\partial x_j}$, we obtain:

$$c_K(\vec{\psi}, \vec{\phi}_h) = \sum_{i,j=1}^{3} \int_{\partial K} \text{grad} \frac{\partial \psi_i}{\partial x_j} . \text{grad} \; \phi_{hj} \; \nu_{ki} \;, \text{ that is (4.9). q.e.d.}$$

We are now ready to estimate the terms b_K and c_K in (4.3) in the classical way for nonconforming elements. In this respect we give below two lemmas whose proofs have been shortened, as they follow standard arguments developped in full detail in [7] and [12].

<u>Lemma 4.3</u>: Under the assumptions of Lemma 4.1 there exists C such that

$$(4.10) \quad \sum_{K\in T_h} b_K(\vec{\psi} \;\; \phi_h) \leq C \; h \; [\; \| \vec{\psi} \; \|_{3,\Omega} + \| \Delta\vec{\psi} \; \|_{2,\Omega}] \, \| \vec{\phi}_h \|_{2,h} \;\; \forall \vec{\phi}_h \in X_h$$

where $\| \, . \, \|_{\ell,\Omega}$ denotes the standard norm of $H^{\ell}(\Omega)$, $\ell \in \mathbb{N}$

Proof: Estimate (4.10) is a direct consequence of the continuity properties of X_h together with (i) and (ii). Indeed, the former states that $\mathrm{curl}\vec{\phi}_h$ is continuous at the barycenter of the inner faces of the tetrahedrons and (ii) that it vanishes at the barycenter of boundary faces. This allows us to apply the same analysis as in [7] for the Morley triangle, to the first term of b_K. Moreover, the mean values of $\vec{\phi}_h \wedge \vec{\nu}_K$ over the edges of inner faces coincide, and (i) states that they vanish if the edge lies on Γ. In this way the analysis given in [12] Lemma 3 applies to the second term of b_K too, which leads to (4.10). □

Lemma 4.4: Under the assumption of Lemma 4.1, there exists C such that

$$(4.11) \quad \sum_{K\in \mathcal{T}_h} c_K(\vec{\psi},\vec{\phi}_h) \leq C\ h\ \|\vec{\psi}\|_{3,\Omega}\ \|\vec{\phi}_h\|_{2,h} \qquad \forall\vec{\phi}_h\in X_h$$

Proof: We first recall (4.9) and letting F be an arbitrary face of ∂K, we can write:

$$c_K(\vec{\psi}\ ,\ \vec{\phi}_h) = \sum_{F\subset\partial K} \int_F \mathrm{grad}\ \mathrm{grad}\psi_\nu\cdot\mathrm{grad}\vec{\phi}_h\ , \text{ or yet}$$

$$c_K(\vec{\psi}\ ,\vec{\phi}_h) = \sum_{F\subset\partial K} \int_F [\mathrm{grad}\ \frac{\partial\psi_\nu}{\partial\sigma_K}\cdot\mathrm{grad}\phi_{h\sigma} + \mathrm{grad}\frac{\partial\psi_\nu}{\partial\tau_K}\cdot\mathrm{grad}\phi_{h\tau} +$$

$$+ \mathrm{grad}\ \frac{\partial\psi_\nu}{\partial\nu_K}\cdot\mathrm{grad}\phi_{h\nu}]$$

where, like in Theorem 3.2, $\vec{\sigma}_K$ and $\vec{\tau}_K$ are two suitable orthogonal unit vectors in the plane of F, in such a way that they form, together with $\vec{\nu}_K$, a local system of cartesian axes, for which

$\mathrm{grad}\ v = (\frac{\partial v}{\partial\sigma_K}\ ,\ \frac{\partial v}{\partial\tau_K},\ \frac{\partial v}{\partial\nu_K})$ for every differentiable function v.

The assumption that $\vec{\psi}\in[H^3(\Omega)]^3$ implies the concidence of the traces of second order derivatives of $\vec{\psi}$ a.e. on both sides of F if this is an inner face. Furthermore, the gradient of $\vec{\phi}_h$ is continous at the barycenter of such faces.

As for the boundary faces F we note the following:

$$\frac{\partial^2\psi_\nu}{\partial\sigma_K\partial\nu_K} = \frac{\partial^2\psi_\nu}{\partial\tau_K\partial\nu_K} = 0$$ a.e., since $\mathrm{div}\vec{\psi} = 0$ on Γ and

$$\frac{\partial\psi_\sigma}{\partial\sigma_K} = \frac{\partial\psi_\tau}{\partial\tau_K} = 0 \text{ on } \Gamma \;;$$

$$\frac{\partial\phi_{h\nu}}{\partial\nu_K} = 0$$ at the barycenter of F, according to property (iii) of X_h;

$$\frac{\partial\phi_{h\sigma}}{\partial\sigma_K} = \frac{\partial\phi_{h\tau}}{\partial\sigma_K} = \frac{\partial\phi_{h\sigma}}{\partial\tau_K} = \frac{\partial\phi_{h\tau}}{\partial\tau_K} = 0$$ at the barycenter of F, according to property (i) of X_h and to Remark 3.1.

Therefore we can once more apply the standard arguments that we mentioned before, more specifically those developed in [7] for the Morley element, thereby obtaining (4.11). q.e.d. □

Finally as a consequence of Lemmas 4.1, 4.3 and 4.4 we have

Theorem 4.1: Under the assumption that the solution of problem (2.4) belongs to W, the solution $\vec{\psi}$ of the approximate problem (3.2) satisfies:

$$(4.12) \qquad \|\vec{\psi} - \vec{\psi}_h\|_{2,h} \leq C\, h\, [\|\vec{\psi}\|_{3,\Omega} + \|\Delta\vec{\psi}\|_{2,\Omega}] \qquad \square$$

Notice that error estimate (4.12) is completely analogous to the one that holds for the approximation by this finite element method of scalar biharmonic problems in $\mathbb{R}^3$, as derived in [12].

5. EXTENSION TO MORE GENERAL VISCOUS FLOW PROBLEMS

The application of the finite element method and the extension of the analysis given in this paper to the case of unsteady stokesian flows, follows classical procedures, except for inequalities of the Friedrichs-Poincaré type (see below) that have to be established for space X_h. On the other hand, for the case of nonstokesian flows, even if stationary, the situation is more complex. Just to have a clear look at this questions we sketch in this Section a "would be" proof for this case. In other words, we give a series of assumptions taht need to be verified if one wishes to obtain an estimate of the type (4.12) for the case of the steady Navier-Stokes equations expressed in terms of the vector potential with zero tangencial components, namely

$$(5.1)\begin{cases}\mu\Delta^2\vec{\psi} + \text{curl}(\text{curl}\vec{\psi}\cdot\text{grad})\text{curl}\vec{\psi} = \text{curl}\ \vec{f} \quad \text{in}\ \Omega \\ \left.\begin{array}{l}\text{div}\vec{\psi} = \Delta\text{div}\vec{\psi} = 0 \\ \text{curl}\vec{\psi}\wedge\vec{\nu} = \vec{\psi}\wedge\vec{\nu} = \vec{0}\end{array}\right\} \quad \text{on}\ \Gamma\end{cases}$$

We may write equation (5.1) in variational form given in [4], namely

$$(5.2)\quad\begin{cases}\text{Find } \vec{\psi}\in X \text{ such that} \\ \mu\int_\Omega \Delta\vec{\psi}.\Delta\vec{\phi} + \int_\Omega(\text{curl}\vec{\psi}.\text{grad})\text{curl}\psi.\text{curl}\vec{\phi} = (\vec{f},\text{curl}\vec{\phi})\ \ \forall\vec{\phi}\in X\end{cases}$$

or equivalently, if Ω is a polyhedron, and using integration by parts:

$$(5.3)\quad\begin{cases}\text{Find } \vec{\psi}\in X \text{ such that} \\ a_\Omega(\vec{\psi},\vec{\phi}) + e_\Omega(\vec{\psi},\vec{\psi},\vec{\phi}) = (\text{curl}\vec{f},\vec{\phi}) \quad \forall\vec{\phi}\in X\end{cases}$$

where for $D \subseteq \Omega$, meas $(D) \neq 0$

$$(5.4)\quad e_D(\vec{\xi},\vec{\psi},\vec{\phi}) = [\ \bar{e}_D(\vec{\xi},\vec{\psi},\vec{\phi}) - \bar{e}_D(\vec{\xi},\vec{\phi},\vec{\psi})]/2$$

$$(5.5)\quad \bar{e}_D(\vec{\xi},\vec{\psi},\vec{\phi}) = \int_D (\text{curl}\vec{\xi}.\text{grad})\text{curl}\vec{\psi}.\text{curl}\vec{\phi} \quad \forall\vec{\xi},\psi,\phi\in[H^2(D)]^3.$$

The discrete problem corresponding to (5.3) would be:

$$(5.6)\quad\begin{cases}\text{Find } \vec{\psi}_h\in X_h \text{ such that} \\ a_h(\vec{\psi}_h,\vec{\phi}_h) + e_h(\vec{\psi}_h,\vec{\psi}_h,\vec{\phi}_h) = (\text{curl}\vec{f},\vec{\phi}_h) \quad \forall\vec{\phi}_h\in X_h\end{cases}$$

where

$$(5.7)\quad e_h(\vec{\xi}_h,\vec{\psi}_h,\vec{\phi}_h) = \sum_{K\in T_h} e_K(\vec{\xi}_h,\vec{\psi}_h,\vec{\phi}_h)$$

The fact that (5.6) is a well-posed problem is assumed(2).

---

(2) We refer to [14] for a well-posedness analysis of discrete problems of the same type.

Now following [14] we derive a bound similar to (4.1) for problem (5.6), that is:

$$(5.8)\qquad \|\vec{\psi}-\vec{\psi}_h\|_{2,h} \le C\left[\sup_{\vec{\phi}_h\in X_h}\frac{e_h(\vec{\psi},\vec{\psi},\vec{\phi}_h) - e_h(\vec{\psi}_h,\vec{\psi}_h,\vec{\phi}_h)|}{\|\vec{\phi}_h\|_{2,h}}\right.$$

$$\left. + \inf_{\vec{\phi}_h\in X_h}\|\vec{\psi}-\vec{\phi}_h\|_{2,h} + \sup_{\vec{\phi}_h\in X_h}\frac{|a_h(\vec{\psi},\vec{\phi}_h)+e_h(\vec{\psi},\vec{\psi},\vec{\phi}_h)-(\vec{\mathrm{curl}}\,\vec{f}\ \vec{\phi}_h)|}{\|\vec{\phi}_h\|_{2,h}}\right]$$

The second and third terms on the right hand side of (5.8) can be treated by the standard arguments already mentioned in Section 4. On the other hand, the analysis for the first term is more complicated and we have to derive several estimates.

First of all, like in [14], page 217, the boundedness of e_h in the following sense is needed.

$$(5.9)\qquad e_h(\vec{\xi}_h,\vec{\psi}_h,\vec{\phi}_h) \le E\ \|\vec{\xi}_h\|_{2,h}\|\vec{\psi}\|_{2,h}\|\vec{\phi}\|_{2,h}\qquad \forall\vec{\xi}_h,\vec{\psi}_h,\vec{\phi}_h\in X+X_h$$

E being independent of h.

Next, we need a discrete Poincaré inequality of the type:

$$(5.10)\qquad \|\vec{\phi}_h\|_{0,\Omega} \le c_1\ \|\vec{\phi}_h\|_{2,h}\qquad \forall\vec{\phi}_h\in X_h$$

Moreover, recalling [12] Lemma 1, in order to derive (5.10) it is convenient to use regularity results for the solution $\vec{w}$ of the following vector Poisson equation:

$$\begin{cases} -\Delta\vec{w} = \vec{g} & \text{in } \Omega \\ \vec{w}\wedge\vec{\nu} = \vec{0} & \\ \text{div}\vec{w} = 0 & \text{on } \Gamma \end{cases}$$

for any $\vec{g}$ given in $[L^2(\Omega)]^3$, together with the estimates

(5.11) $|\vec{w}|_{2,\Omega} \leq C_2 \|\Delta\vec{w}\|_{0,\Omega}$

(5.12) $|\vec{w}|_{1,\Omega} \leq C_3 \|\Lambda\vec{w}\|_{0,\Omega}$

(5.13) $|\vec{w}|_{0,\Omega} \leq C_4 |\vec{w}|_{1,\Omega}$

$\forall\vec{w}\in M = \{\vec{v} \,/\, \vec{v}\in[H^2(\Omega)]^3,\ \vec{v}\wedge\vec{\nu} = \vec{0}$ and $\text{div}\vec{v} = 0$ on $\Gamma\}$.

Finally we have to make the following assumption on the data $\vec{f}$ and μ:

$$\mu^2 > C \|\text{curl}\vec{f}\|_{0,\Omega}$$

where $C = C_1E$.

We conjecture that the assumptions (5.8)∿(5.13) are true but they surely deserve a careful analysis. Moreover we need some further regularity hypothesis on $\vec{\psi}$, namely $\text{curl}\vec{\psi}\in[W^{2,5/2}(\Omega)]^3$.

6. MISCELLANEOUS REMARKS

Another issue that is worth to be discussed, is the application of this method to the case of nonpolyhedral domains, even if still simply connected and convex. There are basically two questions to be adressed in this context:

- Is bilinear form a_h still applicable to this case or, as probably not, would $\bar{a}_h$ (the form defined like a_h but based on $\bar{a}_D$ instead of a_D) yield well-posed discrete problems in X_h ?

- What implications the shift of X_h-boundary conditions from the boundary of Ω to the one of $\Omega_h = \bigcup_{K \in T_h} K$ would have in the convergence analysis of Section 4 ?

Both questions cannot be answered with simple arguments and deserve a careful study. Therefore their appropriate answer is left for future work.

Nevertheless we conclude with an optimistic remark concerning both the application and the implementation of this method to the vector potential equations, at least for linear cases.

Irrespective of the kind of domain, in practice it is possible to reduce this coupled vector problem to a finite sequence of scalar inhomogeneous biharmonic problems in $\mathbb{R}^3$. This can be achieved by applying a technique suggested to the author by QUARTAPELLE [10], similar to the one proposed in [9] for second order vector problems. The resulting uncoupling of the vector boundary conditions allows us to use only the scalar version of our finite element [12] to solve this sequence of problems, whose number is $O[(\dim V_h)^{2/3}]$. Some computer tests using such a technique are now under way and they should be the object of a forthcoming paper.

REFERENCES

[1] R.A. Adams, Sobolev Spaces (1975) Academic Press, New York.

[2] P.G. Ciarlet, The Finite Element Method for Elliptic Problems, (1978) North-Holland, Amsterdam.

[3] B. Dupire, Problemas variacionais, sua aproximação e formulações mistas, Doctoral Dissertation (1985), Pontifícia Universidade Católica do Rio de Janeiro, Departamento de Informática, Rio de Janeiro.

[4] S. M. Gallic, Sistème de Stokes Stationnaire en dimension 3; Formulation en ψ et formulation en u, p dans le cas axisymétrique, Thèse de Doctorat de 3^e Cycle (1982), Université Pierre et Marie Curie, Paris.

[5] V. A. Kondratchev, Boundary value problems for elliptic equations with conical angular points, Trans, Moscow Math Soc., 16(1967), 227-313.

[6] O. A. Ladyzhenskaya & N.N. Ural'ceva, Linear and Quasilinear Elliptic Equations (1968), Academic Press, New York.

[7] P. Lascaux & P. Lesaint, Some nonconforming finite elements for the plate bending problem, RAIRO Série Rouge, Analyse Numérique R-1, (1975), 9-53

[8] L. S. D. Morley, The triangular equilibrium element in the solution of plate bending problems, Aero.Quart. 19(1968), 149-169.

[9] A. Muzzio & L. Quartapelle, Decoupled solution of vector Poisson equations with boundary condition coupling , Technical Report of the Istituto di Fisica del Politecnico di Milano, (July 1987), Milan, Italy.

[10] L. Quartapelle, Personal communication (1987)

[11] V. Ruas Santos, Finite element solution of three-dimensional viscous flow problems using nonstandard degrees of freedom, Japan Jour. Appl. Maths., 2-2 (1985), 415-431.

[12] V. Ruas, A quadratic finite element method for solving biharmonic problems in $\mathbb{R}^n$, Numerische Mathematik, in press.

[13] V. Ruas, Sur la formulation variationnelle des équations d'écoulements visqueux en fonction du potentiel vecteur , submitted to CRAS de Paris, Paris.

[14] R. Témam, Navier-Stokes Equations (1977), North-Holland , Amsterdam.