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FORMAL THEORIES OF PROBLEMS

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ABSTRACT

This work is based on the Algebraic Theory of Problems, developed by Veloso, Haeberer and Baum. The aim of the theory is to be a formal tool for treating diverse aspects of the software process.

This is not an independent effort; in the last years, there have been a few attempts towards this goal, for instance, [Leh83-Mai84-Sin84,85,86].

In its current version, or in slight variations thereof, this theory has been used to model the software development process, the relationships among the application concept, its specification and a program satisfying it, as well as to develop a program derivation calculus, and also to explain programming methodologies.

First of all, this work tries to complete the mathematical framework of the algebraic theory of problems, viewed as 4-tuples, called the *Theory of restricted problems* (TRP). Using TRP as a standard interpretation, an axiomatization is proposed, giving rise to the so called *Axiomatic Theory of problems* (ATP). Now, problems can also be thought of as 3-tuples -*Theory of irrestricted problems* (TIP)-. It is shown that TIP is a model of a restriction of the axiomatic theory. It is also shown that TIP cannot be extended to be a model of the complete axiomatic theory.

Both ways of seeing problems, TRP and TIP, have shown their applicability, depending on the specific aspect of the software development process one is dealing with.

Key words: Problems, relational calculus, formalisms for software development.

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RESUMO

Este trbalho basea-se na Teoria Algébrica de Problemas desenvolvida por Veloso, Haeberer e Baum. O propósito desta teoria é servir como ferramenta formal para tratar diversos aspectos do processo de desenvolvimento de software.

Este não é um esfôrço isolado; nos últimos anos tem havido alguns trabalhos neste sentido, por exemplo, [Leh83-Mai84-Sin84,85,86].

Esta versão da teoria, ou pequenas variações, tem sido utilizada para modelar o processo de desenvolvimento de software; as relações entre o conceito da aplicação, sua especificação e um programa que a satisfaça; bem como para desenvolver um cálculo de derivação de programas, e para explicar métodos de programação.

Primeiro este trabalho trata de completar a estrutura matemática da teoria algébrica de problemas considerados como 4-tuplas, que denominamos de *Teoria de problemas restritos* (TRP). Usando TRP como uma interpretação canônica, propõe-se uma axiomatização, a *Teoria axiomática de problemas* (ATP).

Problemas podem também ser considerados como 3-tuplas -*Teoria de problemas irrestritos* (TIR)-. Demostra-se que TIP è modelo de uma restrição da teoria axiomática e que não pode ser estendida para ser modelo da teoria axiomática completa.

A aplicabilidade de ambas formas de considerar problemas, TRP e TIP, depende do aspecto do processo de desenvolvimento de software que está sendo considerado.

Palabras chave: Problemas, cálculo de relações, formalismos para desenvovimento de software.

RESUMEN

Este trabajo se basa en la Teoría Algebraica de Problemas desarrollada por Veloso, Haeberer y Baum. El propósito de la misma es servir como herramienta formal para tratar diferentes aspectos del proceso de desarrollo de software.

Este no es un esfuerzo aislado; en los últimos años ha habido varios trabajos en este sentido, como por ejemplo, [Leh83-Mai84-Sin84,85,86].

Esta versión de la teoría, o pequeñas variaciones de la misma, ha sido utilizada para modelizar el proceso de desarrollo de software, las relaciones entre el concepto de la aplicación, su especificación y un programa que la satisface, así como para desarrollar un cálculo de dericación de programas y para explicar métodos de programación.

Este trabajo trata, ante todo, de completar la estructura matemática de la teoría algebraica de problemas considerados como 4-uplas, que denominamos de *Teoría de problemas restrictos* (TRP). Luego, usando TRP como uma interpretación canónica, se propone una axiomatización, la *Teoría axiomática de problemas* (ATP).

Por otra parte, los problemas pueden ser considerados como 3-uplas -*Teoría de problemas irrestrictos* (TIP)-. Se demuestra que TIP es modelo de una restricción de la teoría axiomática y que no puede ser extendida para ser modelo de la teoría axiomática completa.

La aplicabilidad de ambas formas de considerar problemas, TRP y TIP, depende del aspecto del proceso de desarrollo de software que está se esté considerando.

Palabras clave: Problemas, cálculo de relaciones, formalismos para desarrollo de software.

CONTENTS

1. THE THEORY OF RESTRICTED PROBLEMS

1.1 - BASIC DEFINITIONS

The main concepts of this theory are those of *problem* and *solution*. A problem consists of a *data domain*, a *result domain* and a binary relation between its data and result domains; this relation, called the *condition* of the problem, is its specification, in the sense that if an element *d* in the data domain is related to an element *r* of the result domain, then *r* is an acceptable result for *d* in this problem. Besides, a problem has a set of *instances of interest*, which is a subset of its data domain and models the fact that, in general, it is not desired to solve a problem for all its possible input data but just for a subset of it.

We now give a formal

<u>Definition</u>: A problem is a 4-tuple

$$P = \langle D, R, q, I \rangle,$$

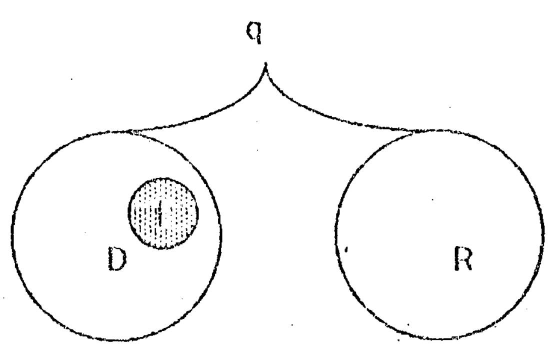
where D and R are sets over some universe $\mathcal{U}$, $q \subseteq D \times R$ and $I \subseteq D$

<u>Conventions</u>: * We will denote by D_P, R_P, q_P, I_P, the data domain, result domain, condition and instances of interest, respectively, of a problem P.

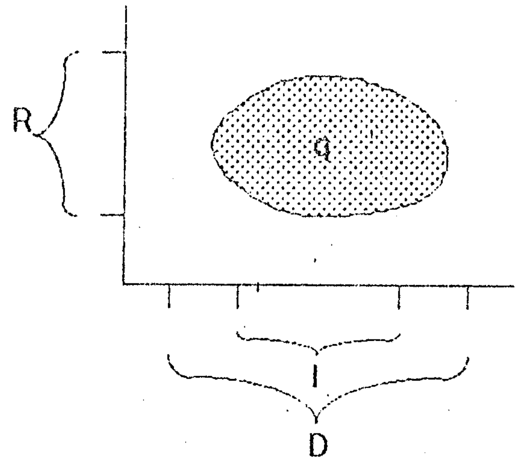
* We shall call $\mathbb{P}$ the family of all problems over $\mathcal{U}$.

<u>Definition</u>: $P = Q \quad \Leftrightarrow \quad D_P = D_Q \wedge R_P = R_Q \wedge q_P = q_Q \wedge I_P = I_Q$

Graphically, a problem could be represented as:



If we are not strict then we could also "draw" a problem in orthogonal cartesian coordinates:



As an example, consider the problem of, given a non empty set of natural numbers, finding a subset of it, containing the number 1.

This problem could be specified as follows:

$$D_p = R_p = \mathcal{P}(\mathbb{N}),$$
$$q_p = \{ (A,B) \ / \ B \subseteq A \subseteq \mathbb{N} \wedge 1 \in B \},$$
$$I_p = \mathcal{P}(\mathbb{N}) - \{ \emptyset \}$$

We chose I_p in such a way since we are interested in non empty sets. Note that if a set does not contain 1, then it is related to no set at all, and if it contains 1 then it may be related to several other sets.

A solution to a problem P is a (partial) function $\sigma: D_p \longrightarrow R_p$, such that for any $d \in I_p$, σ is defined and $\sigma(d)$ is related to d by q_p.

Instead of requiring that σ be a function from D_p to R_p we could ask that if $\sigma: A \longrightarrow B$, then $A \subseteq D_p$ and $B \subseteq R_p$. This trivial loosening of the requirement will allow a "cleaner" development of the notion of a solution of some subproblem of P being also a solution of P.

<u>Definition</u>: A function $\sigma: A \longrightarrow B$ is associated to a problem P (denoted by σ assoc P) iff $A \subseteq D_p$ and $B \subseteq R_p$.

Notation: * $\mathbb{S}$ will represent the space of functions over $\mathcal{U}$.

* Given an arbitrary binary relation $r \subseteq X \times Y$, we will call

$\mathcal{D}(r) = \{ x \in X \ / \ \exists \ y \in Y \text{ with } (x,y) \in r \}$

$\mathcal{R}(r) = \{ y \in Y \ / \ \exists \ x \in X \text{ with } (x,y) \in r \}$,

the domain and range of the relation.

Now we are able to formally define the notion of solution:

Definition: A function σ:A $\longrightarrow$ B is a **solution** of a problem P, denoted by $\sigma \leftarrow$ P, iff it is associated to P, σ is defined on every element of I_P and the restriction of σ to I_P is contained in q_P . i.e.:

$$\sigma \leftarrow P \quad \Leftrightarrow \quad \sigma \text{ assoc } P \wedge I_P \subseteq \mathcal{D}(\sigma) \wedge \forall \ d \in I_P \ , \ (\ d,\sigma(d) \ ) \in q_P$$

Why have we chosen to define the concept of solution as an Skolem function of the condition of the problem? . First of all, one should note that if we accepted the condition of the problem as a solution, then nothing would have to be done after stating the problem. So problem solving would be reduced to choosing for each input data an output data related to it by the condition. If, on the other hand, a solution is a Skolem function of the condition, then this choice is done by this function and problem solving turns out to be the activity of finding such a function.

Notation: We will denote by Ω_P the space of all solutions of a given problem P:

$$\Omega_P = \{ \sigma \in \mathbb{S} \ / \ \sigma \leftarrow P \}$$

It will be useful to consider the notion of extension of functions:

Definition: A function σ':A' $\longrightarrow$ B' is an **extension** of σ:A $\longrightarrow$ B (denoted by $\sigma \ \alpha \ \sigma'$) iff A $\subseteq$ A', B$\subseteq$ B', $\mathcal{D}(\sigma) \subseteq \mathcal{D}(\sigma')$ and, if a $\in \mathcal{D}(\sigma)$, then $\sigma'(a) = \sigma(a)$. i.e.:

$$\sigma \ \alpha \ \sigma' \quad \Leftrightarrow \quad A \subseteq A' \wedge B \subseteq B' \wedge \mathcal{D}(\sigma) \subseteq \mathcal{D}(\sigma') \wedge \sigma'\big|_{\mathcal{D}(\sigma)} = \sigma$$

In words, σ' is more defined than σ, but whenever both are defined their values coincide.

1.2 - SOLVABLE PROBLEMS

Definition: A problem P is solvable iff there exists a function σ such that $\sigma \longleftarrow P$ (equivalently, $\Omega_p \neq \emptyset$).

Proposition: P is solvable $\Leftrightarrow \forall\, d \in I_p,\ \exists\, r \in R_p \,/\, (d,r) \in q_p$

Proof: Immediate, using the axiom of choice. ∎

Corollary: P is solvable $\Leftrightarrow I_p \subseteq \mathcal{D}(q_p)$

Example: The problem stated in the previous example is not solvable since any non empty set which does not contain 1 belongs to I_p but not to $\mathcal{D}(q_p)$.

There exists a notion of α-solvability [Hae87a], which shall not be explored in this work but is nonetheless presented for the sake of completeness.

Given a property α, we say that P is α-solvable iff there exists $\sigma \in \mathbb{S}$ such that $\sigma \longleftarrow P$ and σ satisfies α.

This property α may be either extensional (e.g.: continuous, computable) or intensional (e.g.: efficient, easy)

If a solution of a problem is to be computed by a program, then this function must be Turing-computable and, moreover, its complexity should not be exponential.

In the application of the theory, it is clear that only such functions should be considered, but since we are here only interested in setting up the mathematical framework for these applications, we shall drop this notion of α-solvability and work just with the simpler concept of solvability.

1.3.2 - SUM:

The original motivation for the definition of the *sum* is to model the *decomposition* of a problem induced by a partition of the data domain. In that sense, the language construct *if... then... else... fi*, is a particular representation of this operation *sum*.

Definition: If P and Q are problems, we define the problem P+Q as follows:

$$P+Q = \langle D_P \cup D_Q,\ R_P \cup R_Q,\ q_P \cup q_Q,\ I_P \cup I_Q \rangle$$

This definition makes the sum of problems enjoy all the properties the union operation exhibits:

Proposition:
- Associativity: $\forall$ P,Q,R, (P+Q)+R = P+(Q+R)
- *Commutativity:* $\forall$ P,Q, P+Q = Q+P
- *Idempotence:* $\forall$ P, P+P = P
- *Neuter:* $\forall$ P, P+0 = P, where $0 = \langle \emptyset,\emptyset,\emptyset,\emptyset \rangle$

Proof: Immediate ∎

1.3.2.1 - ADDITIVE SUBPROBLEMS:

Definition: A problem P is an **additive subproblem** of Q (denoted by $P \leq_+ Q$) iff $\exists$ R such that P+R=Q.

Proposition: The following statements are equivalent:

i) $P \leq_+ Q$

ii) $P + Q = Q$

iii) $D_P \subseteq D_Q \wedge R_P \subseteq R_Q \wedge q_P \subseteq q_Q \wedge I_P \subseteq I_Q$.

Proof: Immediate. ∎

Proposition: $\leq_+$ is a partial order in $\mathbb{P}$

Proof: By idempotence, P+P=P and hence $P \leq_+ P\ \forall P$.

Suppose $P \leq_+ Q \wedge Q \leq_+ P \Rightarrow P + Q = Q \wedge Q + P = P$. Thus, by commutativity, $P = Q$.

Suppose $P \leq_+ Q \wedge Q \leq_+ R \Rightarrow P+R = P+(Q+R) = (P+Q)+R = Q+R = R$. $\therefore P \leq_+ R$ ∎

Remark: $\langle \mathbb{P}, \leq_+ \rangle$ has a least element, namely 0.

Proof:

$\Rightarrow$) $P \hookleftarrow Q \Rightarrow P$ solvable; let $\sigma:A \longrightarrow B$ be such that $\sigma \longleftarrow P$. Therefore,

$A \subseteq D_P \wedge B \subseteq R_P$. Consider $\tilde{\sigma}:D_P \longrightarrow R_P$, given by

$$\tilde{\sigma}(d)= \begin{cases} \sigma(d) & \text{if } d \in \mathcal{D}(\sigma) \\ \text{undefined,} & \text{otherwise} \end{cases}$$

Trivially, $\tilde{\sigma} \longleftarrow P$ and so $\tilde{\sigma} \longleftarrow Q \Rightarrow \tilde{\sigma}$ is associated to $Q \Rightarrow D_P \subseteq D_Q \wedge R_P \subseteq R_Q$.

Consider now $\sigma':I_P \longrightarrow B$ such that $\sigma'=\sigma|_{I_P}$. Then $\mathcal{D}(\sigma')=I_P$. It is clear that $\sigma' \longleftarrow P$, therefore, $\sigma' \longleftarrow Q \Rightarrow I_Q \subseteq \mathcal{D}(\sigma') \Rightarrow I_Q \subseteq I_P$.

Let $(a,b) \in q_P|_{I_Q}$, there exists σ such that $\sigma \longleftarrow P$ and $\sigma(a)=b$. As $\sigma \longleftarrow Q$ and $a \in I_Q$, we have $(a,\sigma(a)) \in q_Q$ and so, $(a,b) \in q_Q$.

$\therefore q_P|_{I_Q} \subseteq q_Q$.

$\Leftarrow$) Let $\sigma:A \longrightarrow B$ be such that $\sigma \longleftarrow P$ (such σ exists, since P is solvable)

- *σ is associated to Q:* $A \subseteq D_P \Rightarrow A \subseteq D_Q$
  $B \subseteq R_P \Rightarrow B \subseteq R_Q$
- $I_Q \subseteq \mathcal{D}(\sigma)$: $I_Q \subseteq I_P \subseteq \mathcal{D}(\sigma)$
- *Let $d \in I_Q \Rightarrow d \in I_P \Rightarrow (d,\sigma(d)) \in q_P$. So, $(d,\sigma(d)) \in q_P|_{I_Q}$ and* therefore $(d,\sigma(d)) \in q_Q$.

$\therefore \sigma \longleftarrow Q$ ∎

Definition: Let $\mathbb{P}_S = \{ P \in \mathbb{P} \;/\; P \text{ is solvable} \}$

Proposition: $\hookleftarrow$ is a preorder in $\mathbb{P}_S$.

Proof: Trivially, $\forall\, P \in \mathbb{P}_S$, $P \hookleftarrow P$.

Suppose now $P \hookleftarrow Q \wedge Q \hookleftarrow R$; $\vec{P}$ is solvable and for any σ, $\sigma \longleftarrow P \Rightarrow \sigma \longleftarrow Q \Rightarrow \sigma \longleftarrow R$. Hence, $P \hookleftarrow R$. ∎

$\hookleftarrow$ is not an order in $\mathbb{P}_S$, as is shown in the following

Example: Let $P = \langle \{1,2\}, \{3,4\}, \{ (1,3) \}, \{1\} \rangle$

and $Q = \langle \{1,2\}, \{3,4\}, \{ (1,3), (2,4) \}, \{1\} \rangle$.

We have $P \hookleftarrow Q \wedge Q \hookleftarrow P$, since $D_P = D_Q \wedge R_P = R_Q \wedge I_P = I_Q \wedge q_P|_{I_Q} = q_Q|_{I_P}$. However, $P \neq Q$.

1.3.2 - SUM:

The original motivation for the definition of the *sum* is to model the *decomposition* of a problem induced by a partition of the data domain. In that sense, the language construct *if... then... else... fi*, is a particular representation of this operation *sum*.

Definition: If P and Q are problems, we define the problem P+Q as follows:

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. *Idempotence:* $\forall$ P, P+P = P

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Proof: Immediate ∎

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Proof: Immediate. ∎

Proposition: $\le_+$ is a partial order in $\mathbb{P}$

Proof: By idempotence, P+P=P and hence $P \le_+ P\ \forall P$.

Suppose $P \le_+ Q \wedge Q \le_+ P \Rightarrow P + Q = Q \wedge Q + P = P$. Thus, by commutativity, $P = Q$.

Suppose $P \le_+ Q \wedge Q \le_+ R \Rightarrow P+R = P+(Q+R) = (P+Q)+R = Q+R = R. \therefore P \le_+ R$ ∎

Remark: $\langle \mathbb{P}, \le_+ \rangle$ has a least element, namely 0.

The notion of sum may be extended to that of summation over a set of problems A:

$$\sum_{P\in A} P = \langle \bigcup_{P\in A} D_P , \bigcup_{P\in A} R_P , \bigcup_{P\in A} q_P , \bigcup_{P\in A} I_P \rangle$$

Proposition: For any set of problems $A \subset \mathbb{P}$, there exists a least upper bound with respect to $\leq_+$:

$$\text{lub } A = \sum_{P\in A} P$$

Proof: It follows from the properties of the union operation. ∎

Theorem: $\langle \mathbb{P}, \leq_+ \rangle$ is a complete partial order.

Proof: Immediate from the previous propositions and remark.

Definition: Let $P^+ = \{ Q \in \mathbb{P} \,/\, Q \leq_+ P \}$, be the set of additive subproblems of P.

Theorem: 1. $\langle P^+, \leq_+ \rangle$ is an upper semilattice.

2. P is the lub of this semilattice.

3. $\sum_{Q\in P^+} Q = P$

Proof: 1. First note that P^+ is closed under sum. In effect:

$Q,R \in P+ \Rightarrow Q+R \in P^+$: $(Q+R)+P = Q+(R+P) = Q+P = P$. So $Q+R \in P^+$.

Since + is associative, commutative and idempotent, we have that $\langle P^+, \leq \rangle$ is an upper semilattice, where the ordering relation $\leq$ is given by $P \leq Q \Leftrightarrow P+Q=Q$, but this is just the notion of additive subproblem. We have thus shown that $\langle P^+, \leq_+ \rangle$ is an upper semilattice.

2. $\forall\, Q \in P^+$, $Q \leq_+ P$, therefore P is an upper bound of P^+. Let R be such that $Q \leq_+ R$, $\forall\, Q \in P^+$. In particular, we have that $P \in P^+$ and thus, $P \leq_+ R$. $\therefore$ P is the lub of $\langle P^+, \leq_+ \rangle$.

3. It follows from a previous proposition:

$$\sum_{Q\in P^+} Q = \text{lub } P^+ = P$$

1.3.2.2- BASIS:

There are subsets of P^+ such that, summing over their elements we obtain P and, moreover, they have the property that if one of its elements is eliminated, then the sum does not yield P. Such a subset is called a *basis* of P^+.

Definition: $B \subseteq P^+$ is a basis of P^+ iff

- $\sum_{Q \in B} Q = P$
- $\forall\ A \subset B, \sum_{Q \in A} Q \neq P$

Note that {P} is a trivial basis of P^+.

Theorem: For all P, there exists a non-trivial basis of P^+.

Proof: Let $B_1 = \{ < \{d\}, \{r\}, \{ (d,r) \}, \{d\} \cap I_p > / (d,r) \in q_p \}$,

$B_2 = \{ < \{d\}, \emptyset, \emptyset, \{d\} \cap I_p > / d \in D_p - \mathcal{D}(q_p) \}$,

$B_3 = \{ < \emptyset, \{r\}, \emptyset, \emptyset > / r \in R_p - \mathcal{R}(q_p) \}$ and

$B \ = B_1 \cup B_2 \cup B_3$

We have:

$$\sum_{Q \in B_1} Q = < \mathcal{D}(q_p),\mathcal{R}(q_p),q_p,I_p \cap \mathcal{D}(q_p) >$$

$$\sum_{Q \in B_2} Q = < D_p - \mathcal{D}(q_p),\emptyset,\emptyset,I_p - \mathcal{D}(q_p) >$$

$$\sum_{Q \in B_3} Q = < \emptyset,R_p - \mathcal{R}(q_p),\emptyset,\emptyset >$$

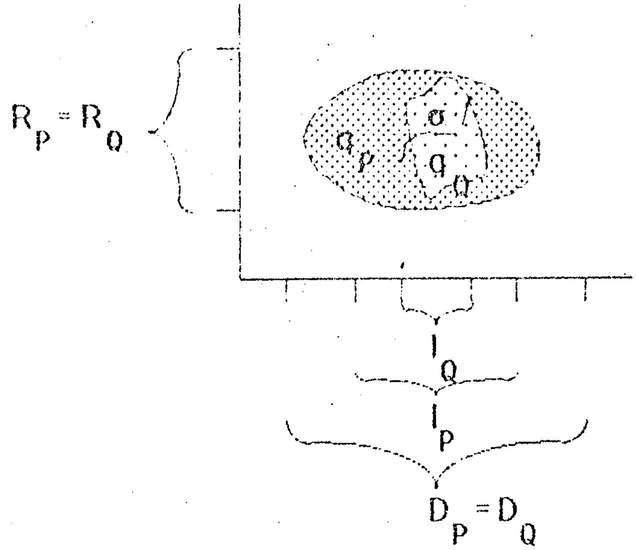
and therefore:

$$\sum_{Q \in B} Q = \sum_{Q \in B_1} Q + \sum_{Q \in B_2} Q + \sum_{Q \in B_3} Q = <D_p,R_p,q_p,I_p> = P \ .$$

It is obvious that any proper subset B' of B will fail to yield P, since, if for instance an element of B_1 is missing then the condition of $\sum_{Q \in B'} Q$ will be strictly contained in q_p. ∎

1.3.2.3 - COMPLETE ADDITIVE SUBPROBLEMS:

In general, a solution of a subproblem of P will not be a solution of P. This can be easily seen in a graph:



$\sigma \leftarrow Q$ and $Q \leq_+ P$; however σ is not a solution of P since it is not true that $I_P \subseteq \mathcal{D}(\sigma)$.

It would be interesting to study subproblems of P which do have this property; these are the so called *complete additive subproblems* of P.

Definition: R is a complete additive subproblem of P (denoted by $R \leq_{+_c} P$) iff R is both a subproblem and a relaxation of P. i.e.:

$$R \leq_{+_c} P \Leftrightarrow R \leq_+ P \wedge R \hookleftarrow P .$$

Proposition: $R \leq_{+_c} P \Leftrightarrow$ R is solvable $\wedge\ D_R \subseteq D_P \wedge R_R \subseteq R_P \wedge q_R \subseteq q_P \wedge$

$$I_R = I_P$$

Proof:

$\Rightarrow$) Since $R \leq_+ P$ we have $D_R \subseteq D_P \wedge R_R \subseteq R_P \wedge q_R \subseteq q_P \wedge I_R \subseteq I_P$. But $R \hookleftarrow P \Rightarrow I_P \subseteq I_R \wedge$ R is solvable.

$\Leftarrow$) We only have to show that $R \hookleftarrow P$; now, $D_R \subseteq D_P \wedge R_R \subseteq R_P \wedge I_P \subseteq I_R \wedge q_R|_{I_P} \subseteq q_R \subseteq q_P$. ∎

1.3.2.4 - DIFFERENCE OF PROBLEMS:

This operation is derived from the sum and is intuitively based on the difference of sets; however, we will consider P-Q to be defined only in the interesting cases, namely when $Q \subseteq_+ P$. This is just like natural numbers, where x-y is defined only when $y \leq x$.

Definition: Let $Q \subseteq_+ P$, then

$$R = P-Q \Leftrightarrow R+Q = P \wedge \forall S,\ S+Q = P \Rightarrow R \subseteq_+ S.$$

In words, P-Q is the smallest problem such that summed to Q yields P.

At first glance, one may think that $P-Q = \langle D_P-D_Q, R_P-R_Q, q_P-q_Q, I_P-I_Q \rangle$ but this is not always a problem! I_P-I_Q may not be necessarily contained in D_P-D_Q. The question is straightened out in the following

Theorem: Let $Q \subseteq_+ P$, then $R = P-Q \Leftrightarrow$

$$R = \langle (D_P-D_Q)\cup\mathcal{D}(q_P-q_Q)\cup(I_P-I_Q), (R_P-R_Q)\cup\mathcal{R}(q_P-q_Q), q_P-q_Q, I_P-I_Q \rangle \ (*)$$

Proof:

$\Rightarrow$) Let Z be defined as in (*).

It is obvious that $q_R \geq q_P-q_Q$, $I_R \geq I_P-I_Q$, $D_R \geq D_P-D_Q$, $R_R \geq R_P-R_Q$ must hold. And, since R is a problem, $\mathcal{D}(q_R)\cup I_R \subseteq D_R$ and $\mathcal{R}(R_R) \subseteq R_R$ must also hold. So we must have $Z \subseteq_+ R$. But it is easily seen that Z+Q=P and hence $R \subseteq_+ Z$. $\therefore$ R=Z

$\Leftarrow$) Clearly, R+Q=P. Let S be such that S+Q=P.

$$q_R = q_P-q_Q = (q_S \cup q_Q)-q_Q = q_S-q_Q \subseteq q_S$$

Hence,

$$\mathcal{D}(q_R) \subseteq \mathcal{D}(q_S) \subseteq D_S \qquad (1)$$

$$\mathcal{R}(q_R) \subseteq \mathcal{R}(q_S) \subseteq R_S \qquad (2)$$

We also have:

$$I_R = I_P-I_Q = (I_S \cup I_Q)-I_Q = I_S-I_Q \subseteq I_S \subseteq D_S \qquad (3)$$

$$D_P-D_Q = (D_S \cup D_Q)-D_Q = D_S-D_Q \subseteq D_S \qquad (4)$$

$$R_P-R_Q \subseteq R_S \qquad (5)$$

From (1),(3) and (4): $D_R \subseteq D_S$ and from (2) and (5): $R_R \subseteq R_S$

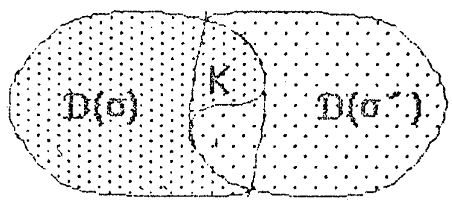
$\therefore R \subseteq_+ S$. Therefore R = P-Q ▮

1.3.2.5 - SOLUTIONS OF THE SUM OF PROBLEMS:

Consider P+Q. We are interested in finding when solutions of P and Q can be "summed" to obtain a solution of P+Q. We also have to define what does it mean to sum solutions. At first glance, we may try to do the union of these solutions but the result is not necessarily a function.

Definition: Let $\sigma:A \longrightarrow B$ and $\sigma':A' \longrightarrow B'$, be functions and $K \subseteq \mathcal{D}(\sigma) \cap \mathcal{D}(\sigma')$. We define $\sigma \oplus_K \sigma':A \cup A' \longrightarrow B \cup B'$ by

$$\sigma \oplus_K \sigma'(d) := \begin{cases} \sigma(d) & \text{if } d \in (\mathcal{D}(\sigma) - \mathcal{D}(\sigma')) \cup K \\ \sigma'(d) & \text{if } d \in \mathcal{D}(\sigma') - K \\ \text{undefined} & \text{otherwise} \end{cases}$$



Remark: $\forall K \subseteq \mathcal{D}(\sigma) \cap \mathcal{D}(\sigma')$, $\sigma \oplus_K \sigma'$ is a function whose domain is

$$\mathcal{D}(\sigma \oplus_K \sigma') = \mathcal{D}(\sigma) \cup \mathcal{D}(\sigma')$$

Proposition: Let $\sigma:A \longrightarrow B$ and $\sigma':A' \longrightarrow B'$ be such that $\sigma \longleftarrow P$ and $\sigma' \longleftarrow Q$ and let $K \subseteq \mathcal{D}(\sigma) \cap \mathcal{D}(\sigma')$, then

$$I_P - I_Q \subseteq (\mathcal{D}(\sigma) - \mathcal{D}(\sigma')) \cup K \wedge I_Q - I_P \subseteq \mathcal{D}(\sigma') - K \Rightarrow \sigma \oplus_K \sigma' \longleftarrow P+Q$$

Proof: . $\sigma \oplus_K \sigma'$ is associated to P + Q: $A \cup A' \subseteq D_P \cup D_Q \wedge B \cup B' \subseteq R_P \cup R_Q$, since σ and σ' are associated to P and Q, respectively.

. $I_{P+Q} = I_P \cup I_Q \subseteq \mathcal{D}(\sigma) \cup \mathcal{D}(\sigma') = \mathcal{D}(\sigma \oplus_K \sigma')$

. Let $d \in I_{P+Q}$, *there are 3 possible cases:*

i) $d \in I_P - I_Q \Rightarrow d \in (\mathcal{D}(\sigma) - \mathcal{D}(\sigma')) \cup K \Rightarrow \sigma \oplus_K \sigma'(d) = \sigma(d)$. So $(d,\sigma(d)) \in q_P \Rightarrow (d,\sigma \oplus_K \sigma'(d)) \in q_P \Rightarrow (d,\sigma \oplus_K \sigma'(d)) \in q_{P+Q}$

ii) $d \in I_Q - I_P \Rightarrow d \in \mathcal{D}(\sigma') - K \Rightarrow \sigma \oplus_K \sigma'(d) = \sigma'(d)$. So $(d,\sigma'(d)) \in q_Q \Rightarrow (d,\sigma \oplus_K \sigma'(d)) \in q_Q \Rightarrow (d,\sigma \oplus_K \sigma'(d)) \in q_{P+Q}$.

iii) $d \in I_P \cap I_Q$ .If $\sigma \oplus_K \sigma'(d) = \sigma(d)$, then since $d \in I_P$ $(d,\sigma(d)) \in q_P \Rightarrow (d,\sigma \oplus_K \sigma'(d)) \in q_P \Rightarrow (d,\sigma \oplus_K \sigma'(d)) \in q_{P+Q}$

Analogously, if $\sigma \oplus_K \sigma'(d) = \sigma'(d)$.

$\therefore \forall d \in I_{P+Q}$, $(d, \sigma \oplus_K \sigma'(d)) \in q_{P+Q}$

We have thus shown that $\sigma \oplus_K \sigma' \leftarrow P+Q$ ∎

In defining the *sum* of σ and σ' we may choose any K, since all are in principle equally suitable.

Definition: **(Sum of solutions)** We define

$$\oplus: \mathbb{S} \times \mathbb{S} \longrightarrow \mathbb{S}$$

by $\quad \sigma \oplus \sigma' = \delta(\{ \sigma \oplus_K \sigma' \ / \ K \subseteq \mathcal{D}(\sigma) \cap \mathcal{D}(\sigma') \})$,

where δ is a function that for any non empty set yields an element belonging to it.

Proposition: 1. $\oplus$ is commutative

2. $\oplus$ is idempotent

3. $\oplus$ has a neutral element: $\emptyset$, the empty function.

Proof: 1. Note that, for any $K \subseteq \mathcal{D}(\sigma) \cap \mathcal{D}(\sigma')$,

$\sigma \oplus_K \sigma' = \sigma' \oplus_{(\mathcal{D}(\sigma)\cap\mathcal{D}(\sigma'))-K} \sigma$, and so,

$\{ \sigma \oplus_K \sigma' \ / \ K \subseteq \mathcal{D}(\sigma) \cap \mathcal{D}(\sigma') \} = \{ \sigma' \oplus_K \sigma \ / \ K \subseteq \mathcal{D}(\sigma) \cap \mathcal{D}(\sigma') \}$

2. For any $K \subseteq \mathcal{D}(\sigma) \cap \mathcal{D}(\sigma)$, $\sigma \oplus_K \sigma = \sigma$, and therefore,

$\{ \sigma \oplus_K \sigma \ / \ K \subseteq \mathcal{D}(\sigma) \} = \{ \sigma \}$

3. Trivial ∎

Remark: If $\mathcal{D}(\sigma) \cap \mathcal{D}(\sigma') = \emptyset$ then $\sigma \oplus \sigma'$ is expressible as:

$$\sigma \oplus \sigma'(d) = \begin{cases} \sigma(d) & \text{if } d \in \mathcal{D}(\sigma) \\ \sigma'(d) & \text{if } d \in \mathcal{D}(\sigma') \\ \text{undefined} & \text{otherwise} \end{cases}$$

since $\sigma \oplus \sigma' = \mathfrak{F}(\{\sigma \oplus_{\emptyset} \sigma'\})$

<u>Definition</u>: Two problems P and Q are **disjoint** (denoted by P **disj** Q) iff

$$D_P \cap D_Q = \emptyset$$

<u>Theorem</u>: If P **disj** Q then

$$\forall\, \sigma,\sigma',\quad \sigma \leftarrow P \wedge \sigma \leftarrow Q \Rightarrow \sigma \oplus \sigma' \leftarrow P{+}Q$$

Proof: $\mathcal{D}(\sigma) \cap \mathcal{D}(\sigma') \subseteq D_P \cap D_Q \Rightarrow \mathcal{D}(\sigma) \cap \mathcal{D}(\sigma') = \emptyset$ and so

$$\sigma \oplus \sigma' = \mathfrak{F}(\{\sigma \oplus_{\emptyset} \sigma'\}) = \sigma \oplus_{\emptyset} \sigma'$$

Now, since $D_P \cap D_Q = \emptyset$, we must also have, $I_P \cap I_Q = \emptyset$. So, $I_P - I_Q = I_P \subseteq \mathcal{D}(\sigma) = (\mathcal{D}(\sigma)-\mathcal{D}(\sigma')) \cup \emptyset$ and $I_Q - I_P = I_Q \subseteq \mathcal{D}(\sigma') = \mathcal{D}(\sigma') - \emptyset$. Applying a previous proposition, we conclude that $\sigma \oplus_{\emptyset} \sigma' \leftarrow P{+}Q$ and, therefore, $\sigma \oplus \sigma' \leftarrow P{+}Q$ ∎

The converse also holds:

<u>Theorem</u>: If P **disj** Q then

$$\sigma \leftarrow P{+}Q \Rightarrow \exists\, \rho,\tau \;/\; \rho \leftarrow P \wedge \tau \leftarrow Q \wedge \sigma = \rho \oplus \tau$$

Proof: Let $\rho{:}D_P \longrightarrow R_P$ and $\tau{:}D_Q \longrightarrow R_Q$ be given by

$$\rho(d) = \begin{cases} \sigma(d) & \text{if } d \in D_P \cap \mathcal{D}(\sigma) \\ \text{undefined} & \text{otherwise} \end{cases} \qquad \tau(d) = \begin{cases} \sigma(d) & \text{if } d \in D_Q \cap \mathcal{D}(\sigma) \\ \text{undefined} & \text{otherwise} \end{cases}$$

So, $\mathcal{D}(\sigma) = \mathcal{D}(\rho) \cup \mathcal{D}(\tau)$. It is immediate that $\rho \leftarrow P \;\wedge \tau \leftarrow Q$.

Now, $\mathcal{D}(\rho) \cap \mathcal{D}(\tau) \subseteq D_P \cap D_Q \Rightarrow \mathcal{D}(\rho) \cap \mathcal{D}(\tau) = \emptyset$ and hence

$$\rho \oplus \tau(d) = \begin{cases} \rho(d) & \text{if } d \in \mathcal{D}(\rho) \\ \tau(d) & \text{if } d \in \mathcal{D}(\tau) \\ \text{undefined} & \text{otherwise} \end{cases}$$

Suppose $d \in \mathcal{D}(\rho)$, then $\rho \oplus \tau(d) = \rho(d) = \sigma(d)$.

Suppose $d \in \mathcal{D}(\tau)$, then $\rho \oplus \tau(d) = \tau(d) = \sigma(d)$.

Moreover, if $d \notin \mathcal{D}(\rho) \cup \mathcal{D}(\tau)$ then $d \notin \mathcal{D}(\sigma)$ either, and so $\rho\oplus\tau(d)$ and $\sigma(d)$ are both undefined.

$\therefore\ \rho \oplus \tau = \sigma.$ ▮

Corollary: If P disj Q then $\Omega_{P+Q} = \Omega_P \oplus \Omega_Q$,
where $\Omega_P \oplus \Omega_Q = \{ \sigma \oplus \sigma' \ / \ \sigma \in \Omega_P \wedge \sigma' \in \Omega_Q \}$

If P and Q are solvable, albeit not necessarily disjoint, then P+Q is also solvable as states the following

Proposition: If P and Q are solvable then P+Q is also solvable
Proof: Let $d \in I_{P+Q}$. If $d \in I_P$, then since P is solvable, there exists $r \in R_P \subseteq R_{P+Q} \ / \ (d,r) \in q_P$, but then $(d,r) \in q_{P+Q}$.
Analogously if $d \in I_Q$. ▮

The converse does not hold, as shows the following

Example: Let $P = \langle \{1,2\} , \{3\} , \{ (1,3) \} , \{1,2\} \rangle$
and $Q = \langle \{1,2\} , \{3\} , \{ (2,3) \} , \{1,2\} \rangle$.

Neither P nor Q are solvable but

$$P+Q = \langle \{1,2\} , \{3\} , \{ (1,3),(2,3) \} , \{1,2\} \rangle$$

is a solvable problem.

1.3.2.6 - RESTRICTING A PROBLEM TO ITS INSTANCES OF INTEREST:

One may be led to think that if one wants to solve a problem just for the elements contained in its instances of interest, then why dealing with a bigger data domain and, thus, with a bigger condition? The answer is: That is just what happens in real world computing. Programs (which compute solutions to problems) accept as input a wider set than that for which we are sure the program does what is intended to (Of course, there is no guarantee about what the program will do with this input!).

Anyway, if one is too bothered because of this fact, one can restrict oneself to the instances of interest of the problem.

Definition: Let Φ: $\mathbb{P} \longrightarrow \mathbb{P}$ be given by:

$$\Phi(P) = \langle I_P, R_P, q_P|_{I_P}, I_P \rangle$$

As we will see in section 3, this transformation Φ will give us the link between the theory of restricted problems (4-tuples) and that of unrestricted problems (3-tuples).

Note that $\Phi(P)$ is a *minimal* problem, in the sense that it has all the necessary information and only that.

Definition: A problem P is minimal iff $D_P = I_P$

This concept will lead us to another interesting characterization of $\Phi(P)$. First, let us state some properties.

Theorem: 1. $\Phi(P) \subseteq_+ P$

2. P is solvable $\Leftrightarrow$ $\Phi(P)$ is solvable

3. If P is solvable then $\Phi(P) \subseteq_{+_c} P$

4. $\sigma \longleftarrow P \Rightarrow \sigma|_{I_P} \longleftarrow \Phi(P)$

5. $\sigma \longleftarrow \Phi(P) \Rightarrow \sigma \longleftarrow P$

Proof: 1. Trivial.

2. P is solvable $\Leftrightarrow \forall\ d \in I_P,\ \exists\ r \in R_P\ /\ (d,r) \in q_P \Leftrightarrow \forall\ d \in I_{\Phi(P)},$ $\exists\ r \in R_{\Phi(P)}\ /\ (d,r) \in q_{\Phi(P)} \Leftrightarrow \Phi(P)$ is solvable.

3. P solvable $\Rightarrow \Phi(P)$ solvable. Besides $\Phi(P) \subseteq_+ P$ and $I_{\Phi(P)} = I_P$.

4. Let σ:A $\longrightarrow$ B be such that $\sigma \longleftarrow$ P. Define $\sigma|_{I_P}:I_P \longrightarrow B$ by $\sigma|_{I_P}(d) = \sigma(d)\ \forall\ d \in I_P$. (Remark that $\sigma|_{I_P} \propto \sigma$)

- $\sigma|_{I_P}$ is associated to $\Phi(P)$: We have $I_P \subseteq I_P = D_{\Phi(P)}$ and, since σ is associated to P, $B \subseteq R_P = R_{\Phi(P)}$.
- $I_P = \mathcal{D}(\sigma|_{I_P})$: Since $\sigma \longleftarrow P$, we have $I_P \subseteq \mathcal{D}(\sigma)$. Thus $\forall\ d \in I_P$ $\sigma(d)$ is defined and hence so is $\sigma|_{I_P}(d)$. So, $I_{\Phi(P)} \subseteq \mathcal{D}(\sigma|_{I_P})$ Moreover, $I_{\Phi(P)} = \mathcal{D}(\sigma|_{I_P})$.
- Let $d \in I_{\Phi(P)} = I_P \Rightarrow (d,\sigma(d)) \in q_P$. Now, $d \in I_P \Rightarrow \sigma|_{I_P}(d) = \sigma(d)$ and we have $(d,\sigma|_{I_P}(d)) \in q_P \Rightarrow (d,\sigma|_{I_P}(d)) \in q_{\Phi(P)}$.

5. Let σ:A $\longrightarrow$ B be such that $\sigma \longleftarrow \Phi(P)$. $A \subseteq I_P \subseteq D_P \wedge B \subseteq R_P \Rightarrow \sigma$ is associated to P. The other two conditions are immediately satisfied.

$\therefore\ \sigma \longleftarrow P$ ∎

<u>Definition</u>: Let $MIN_P = \{ Q \ /\ Q \leq_+ P \wedge Q$ is minimal $\}$

<u>Theorem</u>: $\Phi(P) = \sum_{Q \in MIN_P} Q$

Proof: Let $Q \in MIN_P$. Q minimal $\Rightarrow D_Q = I_Q$, but $I_Q \subseteq I_P \Rightarrow D_Q \subseteq I_P = D_{\Phi(P)}$. We have also $R_Q \subseteq R_P = R_{\Phi(P)}$ and $I_Q \subseteq I_P = I_{\Phi(P)}$. Now, $q_Q \subseteq q_P$, but since $\mathcal{D}(q_Q) \subseteq D_Q \subseteq I_P$ we have $q_Q \subseteq q_P|_{I_P} = q_{\Phi(P)}$.

$\therefore\ Q \leq_+ \Phi(P)$ and so $\Phi(P)$ is an upper bound of MIN_P . But $\Phi(P) \in MIN_P$ and hence $\Phi(P)$ is the least upper bound of MIN_P

$$\therefore\ \Phi(P) = \text{lub } MIN_P = \sum_{Q \in MIN_P} Q$$ ∎

1.3.3 - PRODUCT:

The original motivation for the definition of the *product* is to model the *decomposition* of a problem induced by the interpolation with an arbitrary data domain.

<u>Notation</u>: Given two binary relations r and s, we denote by $r|s$ the relative product of these relations. i.e.:

$$r|s = \{ (a,c) \ / \ \exists \ b \text{ with } (a,b) \in r \wedge (b,c) \in s \}$$

and by r^{-1} the inverse of r. i.e.:

$$r^{-1} = \{ (b,a) \ / \ (a,b) \in r \}$$

This product of relations has some properties that will be used latter on.

<u>Lemma</u>: Let r,s,t be binary relations:

1. $r|(s|t) = (r|s)|t$
2. $r|(s \cup t) = (r|s) \cup (r|t)$
3. $(s \cup t)|r = (s|r) \cup (t|r)$
4. $r|\emptyset = \emptyset|r = \emptyset$

Proof: The proof of any of these statements is straightforward. As an example we will show that 1. holds:
$(a,d) \in r|(s|t) \Leftrightarrow \exists \ b \ / \ (a,b) \in r \wedge (b,d) \in s|t \Leftrightarrow \exists \ b,c \ / \ (a,b) \in r \wedge (b,c) \in s \wedge (c,d) \in t \Leftrightarrow \exists \ c \ / \ (a,c) \in r|s \wedge (c,d) \in t \Leftrightarrow (a,d) \in (r|s)|t$ ∎

<u>Definition</u>: If P and Q are problems we define the problem P.Q by

$$P.Q = \langle D_P \ , \ R_Q \ , \ q_P|q_Q \ , I_P \rangle$$

Let us now analyze the properties of the product of problems:

<u>Theorem</u>: The product is associative and left and right distributive with respect to the sum. i.e.: $\forall$ P,Q,R,

1. $P.(Q.R) = (P.Q).R$
2. $P.(Q+R) = P.Q + P.R$
3. $(Q+R).P = Q.P + R.P$

Proof:

1. $D_{P.(Q.R)} = D_P = D_{P.Q} = D_{(P.Q).R}$
$R_{P.(Q.R)} = R_{Q.R} = R_R = R_{(P.Q).R}$
$I_{P.(Q.R)} = I_P = I_{P.Q} = I_{(P.Q).R}$
$q_{P.(Q.R)} = q_P | q_{Q.R} = q_P | (q_Q | q_R) = (q_P | q_Q) | q_R = q_{P.Q} | q_R = q_{(P.Q).R}$

2. $D_{P.(Q+R)} = D_P = D_P \cup D_P = D_{P.Q} \cup D_{P.R} = D_{P.Q+P.R}$
$R_{P.(Q+R)} = R_{Q+R} = R_Q \cup R_R = R_{P.Q} \cup R_{P.R} = R_{P.Q+P.R}$
$q_{P.(Q+R)} = q_P | q_{Q+R} = q_P | (q_Q \cup q_R) = (q_P | q_Q) \cup (q_P | q_R) = q_{P.Q} \cup q_{P.R}$
$= q_{P.Q+P.R}$
$I_{P.(Q+R)} = I_P = I_P \cup I_P = I_{P.Q} \cup I_{P.R} = I_{P.Q+P.R}$

3. Analogous to 2. ▪

Note that P.0 = 0 does not hold, in fact

$$P.0 = \langle D_P , \emptyset , q_P | \emptyset , I_P \rangle = \langle D_P , \emptyset , \emptyset , I_P \rangle$$

This suggests the following

Definition: Given a problem P, we define the problem

$$0_P = \langle D_P , \emptyset , \emptyset , I_P \rangle$$

Proposition: $P.0_P = 0_P \wedge \forall R, P.R = R \Rightarrow 0_P \leq_+ R$

Proof:

$P.0_P = \langle D_P , R_P , q_P , I_P \rangle . \langle D_P , \emptyset , \emptyset , I_P \rangle =$
$= \langle D_P , \emptyset , \emptyset , I_P \rangle = 0_P$

Let R be such that $P.R = R \Rightarrow$ $D_R = D_{P.R} = D_P = D_{0_P}$
$I_R = I_{P.R} = I_P = I_{0_P}$

Now, $R_{0_P} = \emptyset \subseteq R_R \wedge q_{0_P} = \emptyset \subseteq q_R$

$\therefore 0_P \leq_+ R$ ▪

Analogously,

Definition: Given a problem P, we define the problem

$$0^P = \langle \emptyset , R_P , \emptyset , \emptyset \rangle$$

Proposition: $0^P.P = 0^P \wedge \forall R, R.P = R \Rightarrow 0^P \leq_+ R$

Proof: Similar to the one above ▮

A brief digression:

We are now able to characterize in another way when two problems are disjoint, without explicitly referring to the intersection of their data domains.

Theorem: $P \text{ disj } Q \Leftrightarrow \forall S, S \leq_+ P \wedge S \leq_+ Q \Rightarrow S \leq_+ 0^P \wedge S \leq_+ 0^Q$

Proof:

$\Rightarrow$) Let S be such that $S \leq_+ P \wedge S \leq_+ Q \Rightarrow D_S \subseteq D_P \wedge D_S \subseteq D_Q \Rightarrow D_S \subseteq D_P \cap D_Q$ $\Rightarrow D_S = \emptyset$ .Hence we must also have $I_S = q_S = \emptyset$. Now, $R_S \subseteq R_P \Rightarrow S \leq_+ 0^P$ and $R_S \subseteq R_Q \Rightarrow S \leq_+ 0^Q$

$\Leftarrow$) Let $S = \langle D_P \cap D_Q, R_P \cap R_Q , \emptyset , \emptyset \rangle$. We have $S \leq_+ P \wedge S \leq_+ Q$ and therefore, $S \leq_+ 0^P$, thus $D_S \subseteq D_{0^P} = \emptyset \Rightarrow D_P \cap D_Q = \emptyset \Rightarrow P \text{ disj } Q$ ▮

In general, we are not able to find a problem 1 such that $1.P = P.1 = P$ for any P. However, for every problem P there exist a left and a right identity.

Definition: Given P, let

$$1_P = \langle D_P , D_P , id_{D_P} , I_P \rangle$$

and

$$1^P = \langle R_P , R_P , id_{R_P} , R_P \rangle$$

Proposition: **1.** $1_P.P = P$

2. $P.1^P = P$

Proof:

1. $D_{1_P.P} = D_{1_P} = D_P$; $R_{1_P.P} = R_P$; $I_{1_P.P} = I_{1_P} = I_P$.
Now, $id_{D_P}|q_P \subseteq q_P$: $(a,b) \in id_{D_P}|q_P \Rightarrow \exists\ c \ /\ (a,c) \in id_{D_P} \wedge (c,b) \in q_P$. $(a,c) \in id_{D_P} \Rightarrow a=c \Rightarrow (a,b) \in q_P$. The converse is also true: $q_P \subseteq id_{D_P}|q_P$: let $(a,b) \in q_P$. Since $a \in D_P$, $(a,a) \in id_{D_P}$. Hence, $(a,b) \in id_{D_P}|q_P$.

$\therefore q_{1_P.P} = q_{1_P}|q_P = id_{D_P}|q_P = q_P$

2. Analogously to 1. ∎

Given P, these left and right identities may be called canonical, however they are not in general unique. Moreover, they are not the smallest problems such that multiplied with P yield P. In effect, suppose R.P = P, all we can say is:

$$D_R = D_P = D_{1_P} \qquad \text{and} \qquad I_R = I_P = I_{1_P}$$

but we can tell nothing about R's result domain or condition. Perhaps they are contained in R_{1_P} and q_{1_P} or vice versa; they may as well not be related to R_{1_P} and q_{1_P} !!

Definition: Given P,

- $\ast$ P is functional iff $(a,b) \in q_P \wedge (a,c) \in q_P \Rightarrow b=c$
- $\ast$ P is injective iff $(a,b) \in q_P \wedge (c,b) \in q_P \Rightarrow a=c$
- $\ast$ P is surjective iff $\mathcal{R}(q_P) = R_P$.
- $\ast$ P covers its domain iff $\mathcal{D}(q_P) = D_P$.

Definition: Given a problem P, we define the problem P^{-1} by:

$$P^{-1} = \langle R_P , D_P , q_P^{-1}, R_P \rangle$$

If some special conditions hold then P^{-1} is a sort of "inverse" of P, in the sense that, left multiplied by P it yields 1_P and, right multiplied by P it yields 1^P.

Theorem: $P.P^{-1} = 1_P \Leftrightarrow$ P is injective and covers its domain.

Proof:

$\Rightarrow$) Let (a,b), $(c,b) \in q_P$. Thus, $(a,b) \in q_P \wedge (b,c) \in q_P^{-1} \Rightarrow (a,c) \in q_P | q_P^{-1} = q_{P.P^{-1}} \Rightarrow (a,c) \in q_{1_P} = id_{D_P} \Rightarrow a=c$. $\therefore$ P is injective.

Let $a \in D_P$, $(a,a) \in id_{D_P} \Rightarrow (a,a) \in q_{1_P} = q_{P.P^{-1}} \Rightarrow \exists\ b \ / \ (a,b) \in q_P \wedge (b,a) \in q_P^{-1}$. Thus $a \in \mathcal{D}(q_P)$.

$\therefore D_P \subseteq \mathcal{D}(q_P)$. Now P is a problem and so $\mathcal{D}(q_P) \subseteq D_P$. Hence P covers its domain.

$\Leftarrow$) $D_{P.P^{-1}} = D_P = D_{1_P}$

$R_{P.P^{-1}} = R_{P^{-1}} = D_P = R_{1_P}$

$I_{P.P^{-1}} = I_P = I_{1_P}$

Let $(a,b) \in q_P | q_P^{-1} \Rightarrow \exists\ c \ / \ (a,c) \in q_P \wedge (c,b) \in q_P^{-1} \Rightarrow (a,c) \in q_P \wedge (b,c) \in q_P$. But P is injective and so we must have a=b. Thus we can conclude that $(a,b) \in id_{D_P}$. $\therefore q_P | q_P^{-1} \subseteq id_{D_P}$.

Let $(a,a) \in id_{D_P}$, since P covers its domain, there exists $b \in R_P$ $(a,b) \in q_P$. Now, $(b,a) \in q_P^{-1} \Rightarrow (a,a) \in q_P | q_P^{-1}$ $\therefore id_{D_P} \subseteq q_P | q_P^{-1}$

Therefore,

$$q_{P.P^{-1}} = q_P | q_P^{-1} = id_{D_P} = q_{1_P}$$ ∎

<u>Theorem</u>: $P^{-1}.P = 1^P \Leftrightarrow$ P is functional and surjective

Proof:

$\Rightarrow$) Let (a,b) , $(a,c) \in q_P \Rightarrow (b,a) \in q_P^{-1} \wedge (a,c) \in q_P \Rightarrow (b,c) \in q_P^{-1} | q_P = id_{R_P} \Rightarrow b=c$. $\therefore$ P is functional.

Let $b \in R_P \Rightarrow (b,b) \in id_{R_P} \Rightarrow (b,b) \in q_{P^{-1}.P} = q_P^{-1} | q_P \Rightarrow \exists\ a \ / \ (b,a) \in q_P^{-1} \wedge (a,b) \in q_P$. Thus $b \in \mathcal{R}(q_P)$ and so $R_P \subseteq \mathcal{R}(q_P)$. But P is a problem and hence $\mathcal{R}(q_P) \subseteq R_P$. $\therefore$ P is surjective.

$\Leftarrow$) $D_{P^{-1}.P} = D_{P^{-1}} = R_P = D_{1^P}$

$R_{P^{-1}.P} = R_P = R_{1^P}$

$I_{P^{-1}.P} = I_{P^{-1}} = R_P = I_{1^P}$

$q_P^{-1} | q_P \subseteq id_{R_P}$: $(a,b) \in q_P^{-1} | q_P \Rightarrow \exists\ c\ /\ (a,c) \in q_P^{-1} \wedge (c,b) \in q_P \Rightarrow$ $(c,a) \in q_P \wedge (c,b) \in q_P$. Since P is functional, we must have a=b and so $(a,b) \in id_{R_P}$.

$id_{R_P} \subseteq q_P^{-1} | q_P$: Let $(b,b) \in id_{R_P}$. Since P is surjective and $b \in R_P$, there exists $a \in D_P\ /\ (a,b) \in q_P \Rightarrow (b,a) \in q_P^{-1}$. Thus $(b,b) \in q_P^{-1} | q_P$

$\therefore q_{P^{-1}.P} = q_P^{-1} | q_P = id_{R_P} = q_1{}^P$ ∎

Proposition: $P.R + Q.S \subseteq_+ (P+Q) . (R+S)$

Proof: Since the product is left and right distributive with respect to the sum and the latter is commutative and idempotent, we have:

$$
\begin{aligned}
P.R + Q.S + \big((P+Q) . (R+S) \big) &= P.R + Q.S + (P+Q).R + (P+Q).S \\
&= P.R + Q.S + P.R + Q.R + P.S + Q.S \\
&= (P.R + P.R) + Q.R + P.S + (Q.S + Q.S) \\
&= P.R + Q.R + P.S + Q.S \\
&= (P+Q).R + (P+Q).S \\
&= (P+Q) . (R+S)
\end{aligned}
$$

$\therefore P.R + Q.S \subseteq_+ (P+Q) . (R+S)$ ∎

Corollary: If $\mathcal{R}(q_P) \cap \mathcal{D}(q_S) = \mathcal{D}(q_Q) \cap \mathcal{R}(q_R) = \emptyset$ then the equality holds

Proof: $D_{(P+Q).(R+S)} = D_{P+Q} = D_P \cup D_Q = D_{P.R} \cup D_{Q.S} = D_{P.R+Q.S}$

$R_{(P+Q).(R+S)} = R_{R+S} = R_R \cup R_S = R_{P.R} \cup R_{Q.S} = R_{P.R+Q.S}$

$I_{(P+Q).(R+S)} = I_{P+Q} = I_P \cup I_Q = I_{P.R} \cup I_{Q.S} = I_{P.R+Q.S}$

We thus only have to prove that $q_{(P+Q).(R+S)} \subseteq q_{P.R+Q.S}$:
Let $(a,b) \in (q_P \cup q_Q) \mid (q_R \cup q_S) \Rightarrow \exists\ c\ /\ (a,c) \in q_P \cup q_Q \wedge (c,b) \in q_R \cup q_S$
Suppose $(a,c) \in q_P$. Hence $c \in \mathcal{R}(q_P)$ and so $c \notin \mathcal{D}(q_S)$. Therefore (c,b) must belong to q_R. $\therefore (a,b) \in q_P | q_R$
Analogously, if $(a,c) \in q_Q$, then (c,b) must belong to q_S and so $(a,b) \in q_Q | q_S$.
In any case, $(a,b) \in (q_P | q_R) \cup (q_Q | q_S)$. ∎

1.3.3.1 - SOLUTIONS OF THE PRODUCT OF PROBLEMS:

As we did with the sum, we shall investigate when solutions of two problems can be "multiplied" so as to obtain a solution of the product problem. But first, we need some preliminary concepts.

Often, the result of the product of two problems is not interesting at all, as it may have, for instance, an empty condition. Actually, multiplying two problems makes full sense only when the condition of the first relates an element of its set of instances of interest to elements which are in the set of instances of interest of the second. Problems satisfying this requirement are worth studying, since in the applications of the theory only such problems are multiplied. They thus deserve a special name.

Definition: P may be coupled to Q (and is denoted by P coup Q) iff

$$\mathcal{R}(q_P|_{I_P}) \subseteq I_Q$$

Definition: (Product of solutions) We define

$$\odot : \mathbb{S} \times \mathbb{S} \longrightarrow \mathbb{S}$$

by

$$\sigma \odot \sigma' = \sigma' \circ \sigma$$

where $\circ$ is the usual function composition.

Remember we are dealing with partial functions, so the above definition is perfectly legal and $\sigma \odot \sigma'$ is, of course, a partial function too. (It may even be the empty function!)

We are now in position to state the theorem we were aiming at.

Theorem: If P may be coupled to Q then

$$\forall\, \sigma,\sigma' \,,\ \sigma \leftarrow P \wedge \sigma' \leftarrow Q \Rightarrow \sigma \odot \sigma' \leftarrow P.Q$$

Proof:

. *$\sigma \odot \sigma'$ is associated to P.Q : If σ:A $\longrightarrow$ B and σ':A' $\longrightarrow$ B' then*
$\sigma \odot \sigma'$:A $\longrightarrow$ B' , since σ is associated to P and σ' to Q, we have:
$A \subseteq D_P = D_{P.Q}$ and $B' \subseteq R_Q = R_{P.Q}$

. $I_{P.Q} \subseteq \mathcal{D}(\sigma \odot \sigma')$: We have that

$$\mathcal{D}(\sigma \odot \sigma') = \{ a \in \mathcal{D}(\sigma) \ / \ \sigma(a) \in \mathcal{D}(\sigma') \}$$

Let $d \in I_{P.Q} = I_P$, since $\sigma \leftarrow P$, $d \in \mathcal{D}(\sigma)$ and $(d,\sigma(d)) \in q_P$. Thus $\sigma(d) \in \mathcal{R}(q_P|_{I_P}) \Rightarrow \sigma(d) \in I_Q \Rightarrow \sigma(d) \in \mathcal{D}(\sigma)$. $\therefore$ $d \in \mathcal{D}(\sigma \odot \sigma')$

. Let $d \in I_P \Rightarrow (d,\sigma(d)) \in q_P$; now, $\sigma(d) \in I_Q \Rightarrow (\ \sigma(d),\sigma'(\sigma(d))\ ) \in q_Q \Rightarrow$ $(\ d,\sigma'(\sigma(d))\ ) \in q_P|q_Q \Rightarrow (\ d,\sigma \odot \sigma'(d)\ ) \in q_{P.Q}$

$\therefore \sigma \odot \sigma' \leftarrow P.Q$ ▪

Corollary: If P coup Q then $\Omega_P \odot \Omega_Q \subseteq \Omega_{P.Q}$, where

$$\Omega_P \odot \Omega_Q = \{ \sigma \odot \sigma' \ / \ \sigma \in \Omega_P \wedge \sigma' \in \Omega_Q \}$$

Corollary: Let P coup Q, then

P and Q solvable $\Rightarrow$ P.Q solvable

In contrast to the sum, the solvability of P and Q does not mean the solvability of P.Q . To see this, take for instance the extreme case where the condition of the product turns out to be empty.

Remark: In order to be closed with respect to the property of solvability, we could choose as an alternative definition of the product of P and Q the greatest viable additive subproblem of P.Q . i.e.:

$$P{*}Q = \langle D_P , R_Q , q_P|q_Q , I_P \cap \mathcal{D}(q_P|q_Q) \rangle$$

but then some properties are lost, the most important one being right distributivity with respect to the sum.

The requirement that P may be coupled to Q is a sufficient condition for P.Q to be solvable, but it is by no means necessary:

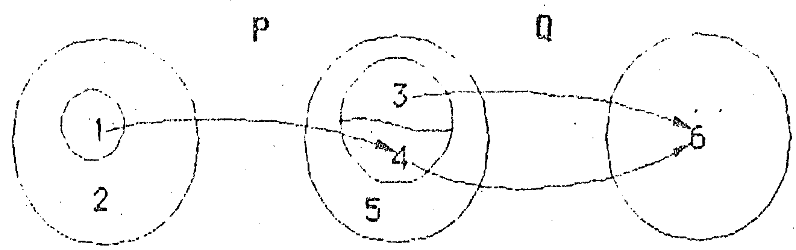
Example: Let $P = \langle \{1,2\} , \{3,4,5\} , \{ (1,4) \} , \{1\} \rangle$

$Q = \langle \{3,4\} , \{6\}, \{ (3,6),(4,6) \} , \{3\} \rangle$

Observe that $\mathcal{R}(q_P|_{I_P}) \cap I_Q = \{4\} \cap \{3\} = \emptyset$ and so P may not be coupled to Q. However,

$$P.Q = \langle \{1,2\} , \{6\}, \{ (1,6) \} , \{1\} \rangle$$

is a solvable problem.



Unfortunately, there does not seem to be a statement of a necessary condition other than the trivial:

$\forall\ d \in I_P\ \exists\ r \in \mathcal{R}(q_P) \cap \mathcal{D}(q_Q),\ s \in \mathcal{R}(q_Q)\ \ /\ (d,r) \in q_P \wedge (r,s) \in q_Q$.

1.3.4 - DIRECT PRODUCT:

This operation tries to model the *decomposition* of a problem by decomposing the input data. The sum is also motivated by decomposition, but of the data domain not of its elements, and it does not rely on the inner structure of the input data.

Definition: Let P and Q be problems,
we define the direct product of P and Q by

$$P \times Q = \langle D_P \times D_Q, R_P \times R_Q , q_{P\times Q} , I_P \times I_Q \rangle$$

where

$$q_{P\times Q} = \{ ((d,d'),(r,r')) \ / \ (d,r) \in q_P \wedge (d',r') \in q_Q \}$$

Let us examine the properties of this operation:

Theorem: The direct product is left and right distributive with respect to the sum and 0 is an absorbing element. i.e.: ∀ P,Q,R,

1. $P \times (Q+R) = P\times Q + P\times R$
2. $(Q+R) \times P = Q\times P + R\times P$
3. $P \times 0 = 0 \times P = 0$

Proof:

1. The only non trivial check is that $q_{P\times(Q+R)} = q_{P\times Q+P\times R}$. Now
   $((a,b),(c,d)) \in q_{P\times(Q+R)} \Leftrightarrow (a,c) \in q_P \wedge (b,d) \in q_{(Q+R)} = q_Q \cup q_R \Leftrightarrow$
   $((a,b) , (c,d)) \in q_{P\times Q} \vee ((a,b) , (c,d)) \in q_{P\times R}$
   $\Leftrightarrow ((a,b) , (c,d)) \in q_{P\times Q+P\times R}$
2. Analogous to 1.
3. Note that the cartesian product of any set with the empty set gives as result ∅, therefore $D_{P\times O} = R_{P\times O} = I_{P\times O} = \emptyset$. Now $q_{P\times O} \subseteq D_{P\times O} \times R_{P\times O} = \emptyset \times \emptyset = \emptyset$, and so $q_{P\times O} = \emptyset$. ∴ P×0 = P
   In a similar way we show that 0xP = P ∎

Although the direct product is neither associative nor commutative, we have that Px(QxR) is isomorphic to (PxQ)xR and also PxQ ≅ QxP. Therefore, in the application of the theory we could consider them as equivalent, since these assumptions do not actually invalidate any other result.

Proposition: $P.Q \times R.S = (P\times R) . (Q\times S)$

Proof: $D_{(P.Q)x(R.S)} = D_{P.Q} \times D_{R.S} = D_P \times D_R = D_{PxR} = D_{(PxR).(QxS)}$

$R_{(P.Q)x(R.S)} = R_{P.Q} \times R_{R.S} = R_Q \times R_S = R_{QxS} = R_{(PxR).(QxS)}$

$I_{(P.Q)x(R.S)} = I_{P.Q} \times I_{R.S} = I_P \times I_R = I_{PxR} = I_{(PxR).(QxS)}$

Moreover,

$((a,b) , (c,d)) \in q_{(P.Q)x(R.S)} \Leftrightarrow (a,c) \in q_{P.Q} \wedge (b,d) \in q_{R.S} \Leftrightarrow \exists\ x,y\ /$
$(a,x) \in q_P \wedge (x,c) \in q_Q \wedge (b,y) \in q_R \wedge (y,d) \in q_S \Leftrightarrow \exists\ x,y\ /$
$((a,b) , (x,y)) \in q_{PxR} \wedge ((x,y) , (c,d)) \in q_{QxS} \Leftrightarrow$
$((a,b) , (c,d)) \in q_{(PxR).(QxS)}$

$\therefore q_{(P.Q)x(R.S)} = q_{(PxR).(QxS)}$ ∎

Proposition: $(PxQ) + (RxS) \leq_+ (P+R) \times (Q+S)$

Proof: Exactly analogous to the proof given for the "." product, replacing "." by "x" ∎

1.3.4.1 - SOLUTIONS OF THE DIRECT PRODUCT OF PROBLEMS:

Definition: **(Direct product of solutions)** We define

$$\otimes : \mathbb{S} x \mathbb{S} \longrightarrow \mathbb{S}$$

by

let $\sigma : A \longrightarrow B \wedge \sigma' : A' \longrightarrow B'$ then $\sigma \otimes \sigma' : AxA' \longrightarrow BxB'$ is defined as:

$$\sigma \otimes \sigma' (a,a') = (\sigma(a) , \sigma'(a'))$$

Fortunately, for the direct product, the transference of solutions is immediate, we do not need any precondition.

Theorem: $\forall$ P,Q, σ, σ', $\quad \sigma \longleftarrow P \wedge \sigma' \longleftarrow Q \Rightarrow \sigma \otimes \sigma' \longleftarrow PxQ$

Proof:

- $\sigma \otimes \sigma'$ is associated to PxQ: If $\sigma : A \longrightarrow B$ and $\sigma' : A' \longrightarrow B'$ then $\sigma \otimes \sigma' : AxA' \longrightarrow BxB'$; now σ is associated to P and σ' is associated to Q, thus $AxA' \subseteq D_P x D_Q = D_{PxQ}$ and $BxB' \subseteq R_P x R_Q = R_{PxQ}$.
- $I_P \subseteq \mathcal{D}(\sigma \otimes \sigma')$: $I_P \subseteq \mathcal{D}(\sigma) \wedge I_Q \subseteq \mathcal{D}(\sigma') \Rightarrow I_{PxQ} = I_P x I_Q \subseteq \mathcal{D}(\sigma) x \mathcal{D}(\sigma') = \mathcal{D}(\sigma \otimes \sigma')$

. Let $(d,d') \in I_{PxQ} \Rightarrow d \in I_P \wedge d' \in I_Q \Rightarrow (d,\sigma(d)) \in q_P \wedge (d,\sigma'(d)) \in q_Q$
$\Rightarrow ((d,d') , (\sigma(d),\sigma'(d'))) \in q_{PxQ} \Rightarrow ((d,d') , \sigma\otimes\sigma'(d,d')) \in q_{PxQ}$
$\therefore \sigma \otimes \sigma' \leftarrow PxQ$ ▯

The converse is also true:

Theorem: Let $\sigma \leftarrow PxQ$, then there exist ρ,τ / $\rho \leftarrow P \wedge \tau \leftarrow Q \wedge \sigma = \rho \otimes \tau$

Proof: Take $\rho = \Pi_1 \circ \sigma$ and $\tau = \Pi_2 \circ \sigma$, where Π_1 and Π_2 are the projections along the first and second coordinate, respectively. It is immediate that they satisfy $\rho \leftarrow P \wedge \tau \leftarrow Q \wedge \sigma = \rho \otimes \tau$ ▯

These two theorems allow us to state two very nice properties of the direct product.

Corollary: P and Q are solvable $\Leftrightarrow$ PxQ is solvable

Corollary: $\Omega_{PxQ} = \Omega_P \otimes \Omega_Q$,

where $\Omega_P \otimes \Omega_Q = \{ \sigma \otimes \sigma' \ / \ \sigma \in \Omega_P \wedge \sigma' \in \Omega_Q \}$.

1.3.5 - ⊖ SUM:

This operation characterizes the *enrichment* of the condition of a problem. This is achieved by intersecting the instances of interest, obtaining so a smaller one, but all the while making the union of the conditions (Of course, in order to obtain a problem, we must also make the union of the data and result domain).

Definition: If P and Q are problems, we define the problem $P \ominus Q$ as the problem whose components are the union of the components of P and Q, except the set of instances of interest, which is the intersection of I_P and I_Q i.e.:

$$P \ominus Q = \langle D_P \cup D_Q ,\ R_P \cup R_Q ,\ q_P \cup q_Q ,\ I_P \cap I_Q \rangle$$

Let us analyze the properties of the ⊖ sum:

Theorem: ⊖ is associative, commutative and idempotent.

Proof: Immediate, from the properties of set union and intersection. ▒

Unfortunately, this operation does not have a neutral element. Suppose there exists X such that $\forall$ P, $P \ominus X = P$; thus, $\forall$ P, $D_P \cup D_X = D_P$ This says that D_X must be empty. But then $I_X = \emptyset$ must also hold. So, if P is such that $I_P \neq \emptyset$ then $I_P \cap I_X \neq I_P$, contradicting $P+X = P$

Theorem: . and x are left and right distributive with respect to ⊖. i.e.: $\forall$ P,Q,R,

1. $P . (Q \ominus R) = P.Q \ominus P.R$
2. $(Q \ominus R) . P = Q.P \ominus R.P$
3. $P \times (Q \ominus R) = P{\times}Q \ominus P{\times}R$
4. $(Q \ominus R) \times P = Q{\times}P \ominus R{\times}P$

Proof: The proof that the data and result domains and the conditions coincide is identical to the one given for the left and right distributivity of . and x with respect to the sum. We only have to show that the sets of instances of interest are equal:

1. $I_{P.(Q\ominus R)} = I_P = I_P \cap I_P = I_{P.Q} \cap I_{P.R} = I_{P.Q\ominus P.R}$

2. $I_{(Q\ominus R).P} = I_{Q\ominus R} = I_Q \cap I_R = I_{Q.P} \cap I_{R.P} = I_{Q.P\ominus R.P}$

3. $I_{P\times(Q\ominus R)} = I_P \times I_{Q\ominus R} = I_P \times (I_Q \cap I_R) = (I_P \times I_Q) \cap (I_P \times I_R) =$
$= I_{P\times Q} \cap I_{P\times R} = I_{P\times Q\ominus P\times R}$

4. $I_{(Q\ominus R)\times P} = I_{Q\ominus R} \times I_P = (I_Q \cap I_R) \times I_P = (I_Q \times I_P) \cap (I_R \times I_P) =$
$= I_{Q\times P} \cap I_{R\times P} = I_{Q\times P\ominus R\times P}$ ∎

Theorem: $\ominus$ is distributive with respect to the sum. i.e.: $\forall$ P,Q,R

$$P \ominus (Q+R) = (P \ominus Q) + (P \ominus R)$$

Proof: $D_{P\ominus(Q+R)} = D_P \cup D_{Q+R} = D_P \cup (D_Q \cup D_R) = (D_P \cup D_Q) \cup (D_P \cup D_R) =$
$= D_{P\ominus Q} \cup D_{P\ominus R} = D_{(P\ominus Q)+(P\ominus R)}$

Analogously, $R_{P\ominus(Q+R)} = R_{(P\ominus Q)+(P\ominus R)}$ and $q_{P\ominus(Q+R)} = q_{(P\ominus Q)+(P\ominus R)}$

Now, $I_{P\ominus(Q+R)} = I_P \cap I_{Q+R} = I_P \cap (I_Q \cup I_R) = (I_P \cap I_Q) \cup (I_P \cap I_R) =$
$= I_{P\ominus Q} \cup I_{P\ominus R} = I_{(P\ominus Q)+(P\ominus R)}$

$\therefore P \ominus (Q+R) = (P \ominus Q) + (P \ominus R)$ ∎

Just as we did with the usual sum, we can define the notion of subproblem and we obtain similar results:

Definition: $Q \leq_\ominus P \quad \Leftrightarrow \quad \exists R \,/\, Q \ominus R = P$

Proposition: The following statements are equivalent:

i) $Q \leq_\ominus P$

ii) $Q \ominus P = P$

iii) $D_Q \subseteq D_P \wedge R_Q \subseteq R_P \wedge q_Q \leq q_P \wedge I_P \subseteq I_Q$

Proposition: $\langle P^\ominus, \ominus \rangle$ is an upper semilattice,

where $P^\ominus = \{ Q \,/\, Q \leq_\ominus P \}$

Proposition: $\leq_\ominus$ is a partial order in $\mathbb{P}$

Proposition: For any $A \subset \mathbb{P}$, lub $A = \ominus_{Q\in A} Q$

We cannot say that $< \mathbb{P}, \leq_{\ominus} >$ is a complete partial order since it has no least element. In effect, suppose there exists X such that $\forall$ P, $X \leq_{\ominus} P$. Thus $\forall$ P, $X \ominus P = P$, and so $\ominus$ would have a neutral element.

1.3.5.1. - SOLUTIONS OF THE $\ominus$ SUM OF PROBLEMS:

Proposition: $\forall\ P,Q,\sigma,\ \sigma \leftarrow P \vee \sigma \leftarrow Q \Rightarrow \sigma \leftarrow P \ominus Q$

Proof: Let σ:A $\longrightarrow$ B be such that $\sigma \leftarrow P \vee \sigma \leftarrow Q$. Suppose $\sigma \leftarrow P$

. *σ is associated to* $P \ominus Q : A \subseteq D_P \subseteq D_{P\ominus Q} \wedge B \subseteq R_P \subseteq R_{P\ominus Q}$

. $I_{P\ominus Q} \subseteq \mathcal{D}(\sigma) : I_{P\ominus Q} = I_P \cap I_Q \subseteq I_P \subseteq \mathcal{D}(\sigma)$

. Let $d \in I_{P\ominus Q} \Rightarrow d \in I_P \Rightarrow (d,\sigma(d)) \in q_P \Rightarrow (d,\sigma(d)) \in q_{P\ominus Q}$

$\therefore\ \sigma \leftarrow P \ominus Q$

Analogously if $\sigma \leftarrow Q$. ▨

Corollary: $\Omega_P \cup \Omega_Q \subseteq \Omega_{P\ominus Q}$

The converse is in general not true, if $\sigma \leftarrow P \ominus Q$, then it may happen that $\mathcal{D}(\sigma) = I_{P\ominus Q} = I_P \cap I_Q$, and if $I_P \cap I_Q$ is *strictly* contained in I_P and in I_Q then neither $I_P \subseteq \mathcal{D}(\sigma)$ nor $I_Q \subseteq \mathcal{D}(\sigma)$ will hold. And so σ will not be a solution of either P or Q.

Up till now, for every operation defined on problems we have introduced a dual operation on solutions, so that under certain special conditions, the following holds:

$$\sigma \leftarrow P \wedge \sigma' \leftarrow Q \Rightarrow \sigma \boxdot \sigma' \leftarrow P \boxtimes Q$$

For the $\ominus$ operation there is no real need to define the corresponding operation on solution, since, by the previous proposition, solutions are transferred directly, but we will do so for symmetry.

Definition: We define $\circledast : \mathbb{S} \times \mathbb{S} \longrightarrow \mathbb{S}$
by

$$\sigma \circledast \sigma' = \aleph(\{\sigma,\sigma'\}),$$

where, as before $\aleph$ is a choice function.

Proposition: $\forall\ P,Q,\sigma,\sigma'$, $\sigma \leftarrow P \wedge \sigma' \leftarrow Q \Rightarrow \sigma \circledast \sigma' \leftarrow P \ominus Q$

1.4 - THE RELATIONSHIP BETWEEN RELAXATION AND THE OPERATIONS ON PROBLEMS

As relaxation is the main relation defined between problems, and seems to play an important role on problem abstraction, it is worth to take a closer look at the interaction between relaxation and operations on problems.

<u>Theorem</u>: Let $P \hookleftarrow Q \wedge R \hookleftarrow S$. If $(q_R - q_P)|_{I_Q - I_S} = (q_P - q_R)|_{I_S - I_Q} = \emptyset$ then

$$P{+}R \hookleftarrow Q{+}S$$

Proof: P and R solvable $\Rightarrow$ P+R solvable. Since $P \hookleftarrow Q \wedge R \hookleftarrow S$, we have

- $D_{P+R} = D_P \cup D_R \subseteq D_Q \cup D_S = D_{Q+S}$
- $R_{P+R} = R_P \cup R_R \subseteq R_Q \cup R_S = R_{Q+S}$
- $I_{Q+S} = I_Q \cup I_S \subseteq I_P \cup I_R = I_{P+R}$
- $q_{P+R}|_{I_{Q+S}} \subseteq q_{Q+S}$: Let $(a,b) \in q_P \cup q_R$ such that $a \in I_Q \cup I_S$. There are three possible cases:
  - a) $a \in I_Q - I_S$: Since $(q_R - q_P)|_{I_Q - I_S} = \emptyset$, (a,b) cannot belong to $q_R - q_P$ and so $(a,b) \in q_P - q_R \vee (a,b) \in q_P \cap q_R$; in any case, $(a,b) \in q_P$; but $a \in I_Q \Rightarrow (a,b) \in q_P|_{I_Q} \Rightarrow (a,b) \in q_Q \Rightarrow (a,b) \in q_{Q+S}$
  - b) $a \in I_S - I_Q$: Analogous to a)
  - c) $a \in I_S \cap I_Q$: c_1) $(a,b) \in q_P$: $a \in I_Q \Rightarrow (a,b) \in q_P|_{I_Q} \Rightarrow (a,b) \in q_Q \Rightarrow (a,b) \in q_{Q+S}$

    c_2) $(a,b) \in q_Q$: Analogous to c_1)

Since P+R is solvable $\wedge\ D_{P+R} \subseteq D_{Q+S} \wedge R_{P+R} \subseteq R_{Q+S} \wedge I_{Q+S} \subseteq I_{P+R} \wedge$ $q_{P+R}|_{I_{Q+S}} \subseteq q_{Q+S}$ then $P{+}R \hookleftarrow Q{+}S$ ∎

<u>Corollary</u>: Let $P \hookleftarrow Q \wedge R \hookleftarrow S$. If Q and S are disjoint then $P{+}R \hookleftarrow Q{+}S$

Proof: $D_Q \cap D_S = \emptyset \Rightarrow D_P \cap D_R = \emptyset \wedge I_Q \cap I_S = \emptyset$. Therefore, $q_P \cap q_R = \emptyset$ must also hold.

So, $(q_R - q_P)|_{I_Q - I_S} = q_R|_{I_Q}$ Suppose $\exists\ (a,b) \in q_R|_{I_Q}$, then $a \in \mathcal{D}(q_R) \cap I_Q \Rightarrow$ $a \in D_R \cap D_Q$. Absurd. Thus, $q_R|_{I_Q} = \emptyset$.

Analogously, we show $(q_P - q_R)|_{I_S - I_Q} = q_P|_{I_S} = 0$.

Hence, the previous theorem may be applied to yield $P{+}R \hookleftarrow Q{+}S$ ∎

Theorem: Let Q coup S and P coup R. Then $P \hookleftarrow Q \wedge R \hookleftarrow S \Rightarrow P.R \hookleftarrow Q.S$

Proof:

. $D_{P.R} = D_P \subseteq D_Q = D_{Q.S}$

. $R_{P.R} = R_R \subseteq R_S = R_{Q.S}$

. $I_{Q.S} = I_Q \subseteq I_P = I_{P.R}$

. $q_{P.R}\,|_{I_{Q.S}} \subseteq q_{Q.S}$: Let $(a,b) \in q_{P.R}\,|_{I_{Q.S}} \Rightarrow (a,b) \in q_{P.R} \wedge a \in I_{Q.S} = I_Q \Rightarrow \exists\ c\ /\ (a,c) \in q_P \wedge (c,b) \in q_R$, since $(a,c) \in q_P|_{I_Q}$, we have $(a,c) \in q_Q$. But $a \in I_Q \Rightarrow c \in \mathcal{R}(q_Q|_{I_Q}) \Rightarrow c \in I_S$ (Remember that Q coup S). So, $(c,b) \in q_R\,|_{I_S} \Rightarrow (c,b) \in q_S$.

So $(a,c) \in q_Q \wedge (c,b) \in q_S \Rightarrow (a,b) \in q_{Q.S}$

. P.R is solvable: Since P coup Q and P and R are solvable, we have that P.R is solvable.

$\therefore$ $P.R \hookleftarrow Q.S$ ▨

Notice that we cannot drop the requirement that P coup R, since otherwise, P.R may not be solvable, as shows the following

Example: Let $P = \langle \{1,2\} , \{3,4,5\} , \{ (1,3),(1,4),(2,5) \} , \{1,2\} \rangle$,

$Q = \langle \{1,2\} , \{3,4,5\} , \{ (1,3),(1,4) \} , \{1\} \rangle$,

$R = \langle \{3,4\} , \{6,7\} , \{ (3,6) \} , \{3,4\} \rangle$ and

$S = \langle \{3,4\} , \{6,7\} , \{ (3,6) \} , \{3,4\} \rangle$.

We have $P \hookleftarrow Q$, $R \hookleftarrow S$; Q coup S, but it is not true that P coup R , since $5 \in \mathcal{R}(q_P|_{I_P})$ and $5 \notin I_R$.

Now

$$P.R = \langle \{1,2\} , \{6,7\} , \{ (1,6) \} , \{1,2\} \rangle$$

is not solvable, and thus we cannot say that $P.R \hookleftarrow Q.S$

Theorem: $P \hookleftarrow R \wedge Q \hookleftarrow S \Rightarrow PxR \hookleftarrow QxS$

Proof: P and R solvable $\Rightarrow$ PxR solvable.

Let $\sigma \longleftarrow PxR$, then $\sigma = \sigma' \otimes \sigma''$, with $\sigma' \longleftarrow P \wedge \sigma'' \longleftarrow R$.

$P \hookleftarrow R \wedge Q \hookleftarrow S \Rightarrow \sigma' \longleftarrow Q \wedge \sigma'' \longleftarrow S \Rightarrow \sigma' \otimes \sigma'' \longleftarrow QxS \Rightarrow \sigma \longleftarrow QxS$

$\therefore$ $PxR \hookleftarrow QxS$ ▨

Theorem: Let $P \hookleftarrow Q \wedge R \hookleftarrow S$. If $(q_R - q_P)|_{I_Q - I_S} = (q_P - q_R)|_{I_S - I_Q} = \emptyset$ then

$$P \ominus R \hookleftarrow Q \ominus S$$

Proof: The proof is similar to the one given for the sum, except for the instances of interest. We only have to check that $I_{Q\ominus S} \subseteq I_{P\ominus R}$: $I_{Q\ominus S} = I_Q \cap I_S \subseteq I_P \cap I_R = I_{P\ominus R}$ ▪

Corollary: Let $P \hookleftarrow Q \wedge R \hookleftarrow S$. If Q and S are disjoint then $P\ominus R \hookleftarrow Q\ominus S$

Theorem: Let P_1 and P_2 be solvable problems. Then

$$P_i \hookleftarrow P_1 \ominus P_2 \qquad (i = 1,2)$$

and

$$\forall Q, P_i \hookleftarrow Q \; (i = 1,2) \Rightarrow P_1 \ominus P_2 \hookleftarrow Q$$

Proof: Let $i \in \{1,2\}$

- $D_{P_i} \subseteq D_{P_1} \cup D_{P_2} = D_{P_1 \ominus P_2}$
- $R_{P_i} \subseteq R_{P_1} \cup R_{P_2} = R_{P_1 \ominus P_2}$
- $I_{P_1 \ominus P_2} = I_{P_1} \cap I_{P_2} \subseteq I_{P_i}$
- $q_{P_i}|_{I_{P_1 \ominus P_2}} \subseteq q_{P_i} \subseteq q_{P_1} \cup q_{P_2} = q_{P_1 \ominus P_2}$

$\therefore P_i \hookleftarrow P_1 \ominus P_2$

Let Q be such that $P_i \hookleftarrow Q \; (i = 1,2)$

- $D_{P_i} \subseteq D_Q \Rightarrow D_{P_1 \ominus P_2} = D_{P_1} \cup D_{P_2} \subseteq D_Q$
- $R_{P_i} \subseteq R_Q \Rightarrow R_{P_1 \ominus P_2} = R_{P_1} \cup R_{P_2} \subseteq R_Q$
- $I_Q \subseteq I_{P_i} \Rightarrow I_Q \subseteq I_{P_1} \cap I_{P_2} = I_{P_1 \ominus P_2}$
- $q_{P_i}|_{I_Q} \subseteq q_Q \Rightarrow q_{P_1}|_{I_Q} \cup q_{P_2}|_{I_Q} \subseteq q_Q \Rightarrow (q_{P_1} \cup q_{P_2})|_{I_Q} \subseteq q_Q \Rightarrow$

  $q_{P_1 \ominus P_2}|_{I_Q} \subseteq q_Q$

$\therefore P_1 \ominus P_2 \hookleftarrow Q$ ▪

In general, for fixed P_1 and P_2 there does not exist P such that $P \hookleftarrow P_1$ and $P \hookleftarrow P_2$:

Example: Let P_1 and P_2 be such that $D_{P_i} = I_{P_i}$ and $D_{P_1} \neq D_{P_2}$ and suppose there exists P such that $P \hookleftarrow P_i$ ($i = 1,2$). $P \hookleftarrow P_i \Rightarrow D_P \subseteq D_{P_i} \wedge I_{P_i} \subseteq I_P \Rightarrow D_{P_1} \cup D_{P_2} = I_{P_1} \cup I_{P_2} \subseteq I_P \subseteq D_P \subseteq D_{P_1} \cap D_{P_2}$. We get a contradiction.

1.5 - THE RELATIONSHIP BETWEEN Φ AND THE OPERATIONS ON PROBLEMS

Theorem: If P and Q are disjoint then $\Phi(P+Q) = \Phi(P) + \Phi(Q)$

Proof:
- $D_{\Phi(P+Q)} = I_{P+Q} = I_P \cup I_Q = D_{\Phi(P)} \cup D_{\Phi(Q)} = D_{\Phi(P)+\Phi(Q)}$
- $\mathcal{R}_{\Phi(P+Q)} = \mathcal{R}_{P+Q} = \mathcal{R}_P \cup \mathcal{R}_Q = \mathcal{R}_{\Phi(P)} \cup \mathcal{R}_{\Phi(Q)} = \mathcal{R}_{\Phi(P)+\Phi(Q)}$
- $I_{\Phi(P+Q)} = I_{P+Q} = I_P \cup I_Q = I_{\Phi(P)} \cup I_{\Phi(Q)} = I_{\Phi(P)+\Phi(Q)}$
- $q_{\Phi(P+Q)} = q_{P+Q}|_{I_{\Phi(P+Q)}} = (q_P \cup q_Q)|_{I_P \cup I_Q}$

  $q_{\Phi(P)+\Phi(Q)} = q_{\Phi(P)} \cup q_{\Phi(Q)} = q_P|_{I_P} \cup q_Q|_{I_Q}$

  Trivially, $q_P|_{I_P} \cup q_Q|_{I_Q} \subseteq (q_P \cup q_Q)|_{I_P \cup I_Q}$. Now let $(a,b) \in (q_P \cup q_Q)|_{I_P \cup I_Q}$. Suppose $(a,b) \in q_P$. Then $a \in D_P \Rightarrow a \notin D_Q \Rightarrow a \notin I_Q \Rightarrow a \in I_P$. Therefore, $(a,b) \in q_P|_{I_P}$.

  Analogously, $(a,b) \in q_Q \Rightarrow (a,b) \in q_Q|_{I_Q}$.

  Thus, $(q_P \cup q_Q)|_{I_P \cup I_Q} \subseteq q_P|_{I_P} \cup q_Q|_{I_Q}$.

  $\therefore q_{\Phi(P+Q)} = q_{\Phi(P)+\Phi(Q)}$ ∎

Theorem: If P may be coupled to Q then $\Phi(P.Q) = \Phi(P) . \Phi(Q)$

Proof:
- $D_{\Phi(P.Q)} = I_{P.Q} = I_P = D_{\Phi(P)} = D_{\Phi(P).\Phi(Q)}$
- $\mathcal{R}_{\Phi(P.Q)} = \mathcal{R}_{P.Q} = \mathcal{R}_Q = \mathcal{R}_{\Phi(Q)} = \mathcal{R}_{\Phi(P).\Phi(Q)}$
- $I_{\Phi(P.Q)} = I_{P.Q} = I_P = I_{\Phi(P)} = I_{\Phi(P).\Phi(Q)}$
- $q_{\Phi(P.Q)} = q_{P.Q}|_{I_{\Phi(P.Q)}} = (q_P \mid q_Q)|_{I_P}$

  $q_{\Phi(P).\Phi(Q)} = q_{\Phi(P)} \mid q_{\Phi(Q)} = (q_P|_{I_P}) \mid (q_Q|_{I_Q})$

  We obviously have $(q_P|_{I_P}) \mid (q_Q|_{I_Q}) \subseteq (q_P \mid q_Q)|_{I_P}$. Now let $(a,b) \in (q_P \mid q_Q)|_{I_P} \Rightarrow a \in I_P \wedge \exists\ c \ /\ (a,c) \in q_P \wedge (c,b) \in q_Q$. But $c \in \mathcal{R}(q_P|_{I_P}) \Rightarrow c \in I_Q$ and so $(a,c) \in q_P|_{I_P} \wedge$

$(c,b) \in q_Q|_{I_Q} \Rightarrow (a,b) \in (q_P|_{I_P}) \mid (q_Q|_{I_Q})$. Thus

$(q_P \mid q_Q)|_{I_P} \subseteq (q_P|_{I_P}) \mid (q_Q|_{I_Q})$ and therefore

$q_{\Phi(P.Q)} = q_{\Phi(P).\Phi(Q)}$ ∎

<u>Theorem</u>: $\Phi(P \times Q) = \Phi(P) \times \Phi(Q)$

Proof: . $D_{\Phi(P\times Q)} = I_{P\times Q} = I_P \times I_Q = D_{\Phi(P)} \times D_{\Phi(Q)} = D_{\Phi(P)\times\Phi(Q)}$

. $R_{\Phi(P\times Q)} = R_{P\times Q} = R_P \times R_Q = R_{\Phi(P)} \times R_{\Phi(Q)} = R_{\Phi(P)\times\Phi(Q)}$

. $I_{\Phi(P\times Q)} = I_{P\times Q} = I_P \times I_Q = I_{\Phi(P)} \times I_{\Phi(Q)} = I_{\Phi(P)\times\Phi(Q)}$

. $(\ (a,b)\ ,\ (c,d)\ ) \in q_{\Phi(P\times Q)} \Leftrightarrow (\ (a,b)\ ,\ (c,d)\ ) \in q_{P\times Q}$
$\wedge\ (a,b) \in I_{P\times Q} \Leftrightarrow (a,c) \in q_P \wedge a \in I_P \wedge (b,d) \in q_Q \wedge b \in I_Q \Leftrightarrow$
$(a,c) \in q_P|_{I_P} \wedge (b,d) \in q_Q|_{I_Q} \Leftrightarrow (a,c) \in q_{\Phi(P)} \wedge (b,d) \in q_{\Phi(Q)} \Leftrightarrow$
$(\ (a,b)\ ,\ (c,d)\ ) \in q_{\Phi(P)\times\Phi(Q)}$.

$\therefore q_{\Phi(P\times Q)} = q_{\Phi(P)\times\Phi(Q)}$ ∎

16 - MONOTONICITY AND CONTINUITY OF THE OPERATIONS

The importance of analyzing these properties on the operations lies in the fact that continuity guarantees the existence of a least fixed point solution of recursive equations involving sum, product , et cetera on problems. These recursive equations are extensively used throughout the process of program derivation, as can be seen in [Vaz89-Elu88,89].

We have already seen that $\langle \mathbb{P}, \subseteq_+ \rangle$ is a complete partial order (cpo). In a cpo there exist the concepts of monotonic and continuous functions. Let us recall these notions:

<u>Definition</u>: Let $\langle D, \leq \rangle$ and $\langle D', \leq_o \rangle$ be cpo's and $f: D \longrightarrow D'$

- A subset $K \subseteq D$ is a chain $\Leftrightarrow \forall\, k,k' \in K$, $k \leq k' \vee k' \leq k$. Any (denumerable) chain may be thus "ordered": $K = \{ k_1, k_2, \ldots \}$ where $i \leq j \Rightarrow k_i \leq k_j$. We denote

$$f(K) = \{ f(k_i) \,/\, k_i \in K \}$$

- f is monotonic $\Leftrightarrow \forall\, k,k',\ k \leq k' \Rightarrow f(k) \leq_o f(k')$
- f is continuous in $\langle D', \leq_o \rangle \Leftrightarrow \forall$ chain $K \subseteq D$, $f(\text{lub } K) = \text{lub } f(K)$

<u>Remark</u>: If $\langle D, \leq \rangle$ is a cpo then $\langle D\times D, \leq \rangle$ is a cpo, where the ordering relation $\leq$ is given by $(d_1,d_2) \leq (d_3,d_4) \Leftrightarrow d_1 \leq d_3 \wedge d_2 \leq d_4$.

$S = (S_1 , S_2)$ is a chain in DxD iff S_1 and S_2 are chains in D and

$$\text{lub } S = (\text{lub } S_1 , \text{lub } S_2)$$

<u>Lemma</u>: + , . , x and $\ominus$ are monotonic in $\langle \mathbb{P}, \subseteq_+ \rangle$

Proof: Suppose $P \subseteq_+ Q$ and $R \subseteq_+ S$, we have to show that

$$P * R \subseteq_+ Q * S , \qquad \text{where } * \in \{ + , . , x , \ominus \}$$

$-\ (P+R)+(Q+S) = (P+Q) + (R+S)$ (commutativity and associativity of +)

$= Q + S$ (since $P \subseteq_+ Q \wedge R \subseteq_+ S$)

$\therefore\ P+R \subseteq_+ Q+S$

$- (P.R)+(Q.S) = P.R + (P+Q) . (R+S)$ (since $P \subseteq_+ Q \wedge R \subseteq_+ S$)

$= P.R + (P+Q).R + (P+Q).S$ (left distributivity of . wrt +)

$= P.R + P.R + Q.R + P.S + Q.S$ (right distributivity)

$= P.R + Q.R + P.S + Q.S$ (idempotence of +)

$= (P+Q).R + (P+Q).S$

$= (P+Q) . (R+S)$

$= Q . S$

$\therefore P.R \subseteq_+ Q.S$

In this proof we have only used the properties of right and left distributivity of . with respect to + . Since x and $\ominus$ also enjoy these properties we can prove in a completely analogous way that x and $\ominus$ are monotonic.

<u>Theorem</u>: + , . , x and $\ominus$ are continuous in $\langle \mathbb{P}, \subseteq_+ \rangle$

Proof: Let $S = (S_1 , S_2)$ be a chain in $\mathbb{P} \times \mathbb{P}$

$\ast$ $lub\ (+S) = lub\ \{ S_1^i + S_2^i \ / \ i \in \mathbb{N} \} = \sum_i (S_1^i + S_2^i) =$

$= \sum_i S_1^i + \sum_i S_2^i$ (by associativity and commutativity of +)

$= lub\ S_1 + lub\ S_2$

$= +\ (lub\ S_1 , lub\ S_2)$

$= +\ (lub\ S)$

$\ast$ $lub\ (.S) = lub\ \{ S_1^i . S_2^i \ / \ i \in \mathbb{N} \} = \sum_i (S_1^i . S_2^i)$

$. (lub\ S) = . (lub\ S_1 , lub\ S_2) = lub\ S_1 . lub\ S_2 =$

$= \left[\sum_i S_1^i \right] . \left[\sum_j S_2^j \right]$

$= \sum_i \left[S_1^i . \sum_j S_2^j \right]$ (right distributivity of . wrt +)

$= \sum_i \sum_j S_1^i . S_2^j$ (left distributivity of . wrt +)

Trivially, $\sum_i (S_1^i . S_2^i) \subseteq_+ \sum_i \sum_j S_1^i . S_2^j$ Now let $i,j \in \mathbb{N}$ and take

$u = \max\{i,j\}$. Since S and S are chains in $\mathbb{P}$, we have that $S_1^i \subseteq_+ S_1^u \wedge S_2^j \subseteq_+ S_2^u$. Thus, $S_1^i \cdot S_2^j \subseteq_+ S_1^u \cdot S_2^u \subseteq_+ \sum_i (S_1^i \cdot S_2^i)$ (by monotonicity of .)

Being i,j arbitrarily chosen, $\sum_i \sum_j S_1^i \cdot S_2^j \subseteq_+ \sum_i (S_1^i \cdot S_2^i)$

$$\therefore \sum_i \sum_j S_1^i \cdot S_2^j = \sum_i (S_1^i \cdot S_2^i) \Rightarrow . (\text{lub } S) = \text{lub } (.S) \Rightarrow . \text{ is continuous}$$

Just as what happened in the previous lemma, we have used in this proof properties of . (namely, left and right distributivity and monotonicity) which $\times$ and $\ominus$ also enjoy. Thus, we can show analogously that $\times$ and $\ominus$ are continuous in $\langle \mathbb{P}, \subseteq_+ \rangle$ ∎

1.7 - CONCLUSIONS

Many of the concepts and results here exposed were gathered from previous works on the theory like [Vel84,89-Hae87,89a-Elu88]. We have tried to take a deeper look on these concepts, completing them if necessary or clarifying some obscure points. In addition, quite a few notions were thoroughly studied here for the first time, for instance, relaxation, direct product, $\ominus$ sum, the transformation Φ, etc.

As we have already said, there have been attempts to apply the theory to different aspects of the software development process. But the field which gave the fundamental ideas to the theory and in fact motivated the development of it is program derivation: starting from a formal specification, to obtain an executable program which satisfies it. From the problem theoretical point of view this means: starting with a problem to obtain a program which computes a solution of it. To achieve this goal, one may need to replace a problem by another (this step is theoretically interpreted by relaxation) or decompose it into the sum, product or direct product of other simpler problems. Thus, it is crucial in this step to dispose of theoretical background allowing to compose solutions of the latter so as to obtain a solution of the original one. We have seen that, theoretically, such theorems may not always be applied: for the sum P+Q, we require that P and Q be disjoint; for the product P.Q, that P may be coupled to Q. Fortunately in practice, and this is just an empirical observation, almost every problem decomposition into subproblems satisfies these requirements. In fact, it is almost a tacit convention that if we write a problem R as, for instance, the product of P and Q, then P may be coupled to Q. Thus, in practice, solutions may always be transferred and moreover, some additional nice properties may also be exploited, like for example, that Φ is a homomorphism (i.e. : $\Phi(P*Q) = \Phi(P) * \Phi(Q)$, with $* \in \{ +, . ,\times \}$)

Finally, it should be remarked that although introduced in the theory, the $\ominus$ operation has not been so far used in practice, but we believe that this operation will be fundamental throughout the process of abstraction and formalization of an application concept, yielding a formal specification

2. AN AXIOMATIZATION OF THE ALGEBRAIC THEORY OF PROBLEMS

2.1 – WHY AXIOMATIZING?

By axiomatizing the theory we will be able to achieve a certain degree of freedom as to what are problems and solutions, while keeping all the relevant properties the theory enjoys.

One may then wonder what do we want this freedom for.

When using the theory of problems to derive programs, the objects of the data or result domains of a problem are usually considered as embedded into a given universe of discourse. That is, they are supposed to possess some attributes which are exploited throughout the derivation. For instance, if the objects are lists, these attributes may be the operations of constructing a list or returning the first element or the rest of a given list, etc. However, these attributes are not expressible by means of the theory, as presented in the previous chapter. Now if problems were, for instance, abstract data types, then all attributes of abstract data types could be used for inducing a particular decomposition of a problem, this is what happens in the so called *data driven* programming methodologies.

Hence, providing a set of axioms for the theory is a good starting point to investigate the possibility of defining problems on abstract data types. But this is by no means the only possible way of viewing problems. The axioms will test whether a new view is acceptable by checking whether the interpretation built from this view is a model of the axioms.

Finally, the set of axioms is a quick reference guide to all the properties of the theory of restricted problems, already presented.

Naturally, the axioms are stated with the aim that the interpretation of problems as 4-tuples , i.e.: the theory of restricted problems (c.f. section 1) , be a model of these axioms. This interpretation will be called hereafter the canonical interpretation.

Thus, most of the axioms are statements of theorems proven for this interpretation. Others are given so as to restrict (and hopefully wipe out) the appearance of "demon models", that is, interpretations which are identical to the canonical one, except for a slight change in the meaning given to some predicate and are nonetheless models.

To axiomatize the algebraic theory we employ a two-sorted first order logic, which is just like an ordinary first order logic except that it has two universes of discourse or sorts (one for problems and one for solutions). Of course, it has two distinct sorts of variables and its relations and functions (including 0-ary functions: constants) are defined on the union of these two universes.

Although employing two-sorted instead of one-sorted first order logic is much more natural for this particular case, it could be treated as well in one-sorted first order logic, since any two-sorted structure $(\mathfrak{M}, \mathfrak{N}, \ldots)$ can be turned into an ordinary structure $(\mathfrak{M} \cup \mathfrak{N}, M, N, \ldots)$, with unary predicates M and N to sort out the different kinds of elements.

Let $\mathfrak{P} = \langle A, C, F, R \rangle$,

where

A is the set of sorts of $\mathfrak{P}$,

C is the set of constant (0-ary functions) symbols of $\mathfrak{P}$,

F is the set of function symbols of $\mathfrak{P}$.

R is the set of predicate symbols of $\mathfrak{P}$ and

Specifically,

$$A = \{\mathbb{P}, \mathbb{S}\} ,$$

$$C = \{ 0: \longrightarrow \mathbb{P} \}$$

$$\begin{aligned} F = \{ & +: \mathbb{P} \times \mathbb{P} \longrightarrow \mathbb{P} , \\ & .: \mathbb{P} \times \mathbb{P} \longrightarrow \mathbb{P} , \\ & \times: \mathbb{P} \times \mathbb{P} \longrightarrow \mathbb{P} , \\ & \ominus: \mathbb{P} \times \mathbb{P} \longrightarrow \mathbb{P} , \\ & \oplus: \mathbb{S} \times \mathbb{S} \longrightarrow \mathbb{S} , \\ & \circ: \mathbb{S} \times \mathbb{S} \longrightarrow \mathbb{S} , \\ & \otimes: \mathbb{S} \times \mathbb{S} \longrightarrow \mathbb{S} \ \} \end{aligned}$$

$$\begin{aligned} R = \{ & \mathbf{funct} \subseteq \mathbb{P} , \\ & \mathbf{inject} \subseteq \mathbb{P} , \\ & \mathbf{surject} \subseteq \mathbb{P} , \\ & \mathbf{cov} \subseteq \mathbb{P} , \\ & \mathbf{min} \subseteq \mathbb{P} , \\ & \mathbf{coup} \subseteq \mathbb{P} \times \mathbb{P} , \\ & \propto \subseteq \mathbb{S} \times \mathbb{S} , \\ & \longleftarrow \subseteq \mathbb{S} \times \mathbb{P} , \\ & \mathbf{assoc} \subseteq \mathbb{S} \times \mathbb{P} \ \} \end{aligned}$$

<u>Notation:</u>

∗ Elements of sort $\mathbb{P}$ will be called in the sequel "problems" and those of sort $\mathbb{S}$, "solutions".

∗ We shall employ $P,Q,R,S,\ldots$ as variables of sort $\mathbb{P}$, and $\sigma,\sigma',\ldots$ as variables of sort $\mathbb{S}$.

* Predicate	*should be read as*
funct(P)	P is functional
inject(P)	P is injective
surject(P)	P is surjective
cov(P)	P covers its domain
min(P)	P is minimal
P coup Q	P may be coupled to Q
$\sigma \propto \sigma'$	σ' is an extension of σ
$\sigma \leftarrow P$	σ solves P or σ is a solution of P
σ assoc P	σ is associated to P

AXIOMS:

1. SUM:

1.1 - $(\forall P,Q,R) \quad (P + Q) + R = P + (Q + R)$

1.2 - $(\forall P,Q) \quad P + Q = Q + P$

1.3 - $(\forall P) \quad P + 0 = P$

1.4 - $(\forall P) \quad P + P = P$

Definition: P is a subproblem of Q ($P \le_+ Q$) iff $(\exists R)\ P + R = Q$

1.5 - $(\forall P,Q) \quad Q \le_+ P \Rightarrow$

$(\exists R)\ (R + Q = P \wedge (\forall S)\ (S + Q = P \Rightarrow R \le_+ S))$

Notation: Such a problem R is called the **difference** between P and Q and is denoted by P−Q

2. PRODUCT:

2.1 - $(\forall P,Q,R) \quad (P \,.\, Q) \,.\, R = P \,.\, (Q \,.\, R)$

2.2 - $(\forall P)\ (\exists 1_P) \quad 1_P \,.\, P = P \quad \wedge$

$((\exists P^{-1}) \quad P \,.\, P^{-1} = 1_P \quad \Leftrightarrow \quad \text{inject}(P) \wedge \text{cov}(P))$

2.3 - $(\forall P)\ (\exists 1^P) \quad P \,.\, 1^P = P \quad \wedge$

$((\exists P^{-1}) \quad P^{-1} \,.\, P = 1^P \Leftrightarrow \text{surject}(P) \wedge \text{funct}(P))$

2.4 - $(\forall P)\ (\exists Q)\ (P \,.\, Q = Q \wedge (\forall X)\ (P \,.\, X = X \Rightarrow Q \le_+ X))$

Definition: Given P, we define 0_P as the *unique* Q satisfying 2.4

$2.5 - (\forall P)(\exists Q)(Q . P = Q \wedge (\forall X)(X . P = X \Rightarrow Q \leq_+ X))$

Definition: Given P, we define 0^P as the *unique* Q satisfying 2.5

$2.6 - (\forall P,Q) \quad P \leq_+ Q \Rightarrow 0_P \leq_+ 0_Q \wedge 0^P \leq_+ 0^Q$

$2.7 - (\forall P,Q,R) \quad P . (Q + R) = P . Q + P . R$

$2.8 - (\forall P,Q,R) \quad (Q + R) . P = Q . P + R . P$

3. DIRECT PRODUCT:

$3.1 - (\forall P,Q,R) \quad P \times (Q + R) = P \times Q + P \times R$

$3.2 - (\forall P,Q,R) \quad (Q + R) \times P = Q \times P + R \times P$

$3.3 - (\forall P) \quad P \times 0 = 0 \times P = 0$

$3.4 - (\forall P,Q,R,S) \quad (P . Q) \times (R . S) = (P \times R) . (Q \times S)$

4. $\ominus$ SUM:

$4.1 - (\forall P,Q,R) \quad (P \ominus Q) \ominus R = P \ominus (Q \ominus R)$

$4.2 - (\forall P,Q) \quad P \ominus Q = Q \ominus P$

$4.3 - (\forall P) \quad P \ominus P = P$

$4.4 - (\forall P)(\exists Q) \quad P \ominus Q \neq Q$ (No neutral element)

$4.5 - (\forall P,Q,R) \quad P \ominus (Q + R) = P \ominus Q + P \ominus R$

$4.6 - (\forall P,Q,R) \quad P . (Q \ominus R) = P . Q \ominus P . R$

$4.7 - (\forall P,Q,R) \quad (Q \ominus R) . P = Q . P \ominus R . P$

$4.8 - (\forall P,Q,R) \quad P \times (Q \ominus R) = P \times Q \ominus P \times R$

$4.9 - (\forall P,Q,R) \quad (Q \ominus R) \times P = Q \times P \ominus R \times P$

5. EXTENSION:

$\propto$ is a partial order in $\mathbb{S}$

6. RELATIONSHIP BETWEEN A PROBLEM AND ITS SOLUTIONS:

$(\forall P,\sigma) \quad \sigma \leftarrow P \Rightarrow \sigma$ assoc P

Definition: P is solvable (solv(P)) iff $(\exists \sigma) \quad \sigma \leftarrow P$

7. MINIMALITY:

$7.1 - (\forall P)(\exists Q) \; Q \leq_+ P \wedge \min(Q)$

$7.2 - (\forall P) \min(P) \Rightarrow (\forall \sigma)(\sigma \leftarrow P \Rightarrow \neg(\exists \sigma')(\sigma \neq \sigma' \wedge \sigma \propto \sigma' \wedge \sigma' \text{ assoc } P))$

7.3 - $(\forall P)\ \min(P) \wedge solv(P) \Rightarrow cov(P)$

Definition: $R = \hat{P} \Leftrightarrow R \subseteq_+ P \wedge (\forall Q)\ (Q \subseteq_+ P \wedge \min(Q) \Rightarrow Q \subseteq_+ R)$

As we will see below, $\hat{P}$ could also be characterized in the following way:

Let $MIN_P = \{ Q \ / \ Q \subseteq_+ P \wedge \min(Q) \}$; note that, by axiom 7.1, MIN_P is always non empty. Then

$$\hat{P} = \sum_{Q \in MIN_P} Q$$

8. RELATIONSHIP BETWEEN P AND $\hat{P}$:

8.1 - $(\forall P)\ (\forall \sigma)\ (\ \sigma \leftarrow P \Rightarrow (\exists \sigma')\ (\ \sigma' \propto \sigma \wedge \sigma' \leftarrow \hat{P}\ )$

8.2 - $(\forall P)\ (\forall \sigma)\ (\ \sigma \leftarrow \hat{P} \Rightarrow \sigma \leftarrow P\ )$

Definition: P and Q are **disjoint** (P **disj** Q) **iff**

$$(\forall S)\ S \subseteq_+ P \wedge S \subseteq_+ Q \Rightarrow S \subseteq_+ 0^P \wedge S \subseteq_+ 0^Q$$

9. DISJOINTEDNESS:

9.1 - $(\forall P,Q,R,S)\ \ P \hookleftarrow Q \wedge R \hookleftarrow S \Rightarrow (\ Q$ **disj** $S \Rightarrow P$ **disj** $R\ )$

10. OPERATIONS ON PROBLEMS AND ON SOLUTIONS: $(\forall P,Q)\ (\forall \sigma,\sigma')$

10.1 - P solvable $\wedge$ Q solvable $\Rightarrow$ P + Q solvable

10.2 - P **disj** Q $\Rightarrow (\ \sigma \leftarrow P \wedge \sigma' \leftarrow Q \Rightarrow \sigma \oplus \sigma' \leftarrow P + Q\ )$

10.3 - P **disj** Q $\Rightarrow (\ \sigma \leftarrow P + Q \Rightarrow$

$(\exists \rho,\tau)\ \rho \leftarrow P \wedge \tau \leftarrow Q \wedge \sigma = \rho \oplus \tau$

10.4 - P **coup** Q $\Rightarrow (\ \sigma \leftarrow P \wedge \sigma' \leftarrow Q \Rightarrow \sigma \odot \sigma' \leftarrow P\ .\ Q\ )$

10.5 - $\sigma \leftarrow P \wedge \sigma' \leftarrow Q \Rightarrow \sigma \otimes \sigma' \leftarrow P \times Q\ )$

10.6 - $\sigma \leftarrow P \times Q \Rightarrow (\exists \rho,\tau)\ \rho \leftarrow P \wedge \tau \leftarrow Q \wedge \sigma = \rho \otimes \tau$

10.7 - $\sigma \leftarrow P \vee \sigma \leftarrow Q \Rightarrow \sigma \leftarrow P \ominus Q$

We believe that this axiomatization really characterizes the theory of restricted problems since, so far, any important result proven in the algebraic theory is also deducible from the axioms, as we will see below. This strongly suggests that the axiomatization in fact is a good characterization of the theory of restricted problems, but it is by no means a proof of this statement.

In some sense, the axiomatic theory is also minimal, since if we eliminate any axiom then either we can introduce some demon model or some facts that are true in the canonical interpretation (i.e.: the theory of restricted problems) are no longer deducible from the resulting set of axioms.

2.3 - CONSEQUENCES OF THE AXIOMS

We present here some results deducible from the axioms alone. Most of them were already proven in the first section for the canonical interpretation. There we tried not to use the particular definition of a problem as a 4-tuple, but instead its properties (which would be axiomatized later). Hence, many proofs are directly applicable here and will not be repeated.

Others, although not being proven in the first section, are easy consequences and shall not be included for the sake of readability.

1. - $(\forall P,Q)\ P \subseteq_+ Q \Leftrightarrow P + Q = Q$

2. - $\subseteq_+$ is a partial order in $\mathbb{P}$

3. - For any $A \subset \mathbb{P}$, $\mathrm{lub}\ A = \sum_{Q \in A} Q$

4. - 0 is the least element of $\langle \mathbb{P}, \subseteq_+ \rangle$

5. - $\langle \mathbb{P}, \subseteq_+ \rangle$ is a complete partial order

6. - $+$, $.$, $\times$ and $\ominus$ are monotonic in $\langle \mathbb{P}, \subseteq_+ \rangle$. i.e.:
   $P \subseteq_+ Q \wedge R \subseteq_+ S \Rightarrow P * R \subseteq_+ Q * S$, where $* \in \{ + , . , \times , \ominus \}$

7. - $+$, $.$, $\times$ and $\ominus$ are continuous in $\langle \mathbb{P}, \subseteq_+ \rangle$. i.e.: for any chain $S \subset \mathbb{P} \times \mathbb{P}$, $\mathrm{lub}(* S) = * (\mathrm{lub}\ S)$, where $* \in \{ + , . , \times , \ominus \}$

Thus, any recursive equation in $\mathfrak{P}$ involving sum, product and direct product of problems has a minimal fix-point, which can be easily characterized:

Given $P = f(P)$

then

$$\mathrm{fix}\ f = \mathrm{lub}\ \{ f^i(0) \ /\ i \in \mathbb{N} \} = \sum_{i \in \mathbb{N}} f^i(0)$$

<u>Definition</u>: Let $P^+ = \{ Q \ /\ Q \subseteq_+ P \}$ be the set of additive subproblems of P.

8. - i) $\langle \mathbb{P}^+, \leq_+ \rangle$ is an upper semilattice

ii) P is the lub of this semilattice

iii) $\sum_{Q \in P^+} Q = P$

Definition: P is a relaxation of Q ($P \hookleftarrow Q$) iff P is solvable and any solution of P is also a solution of Q

$$P \hookleftarrow Q \quad \Leftrightarrow \quad P \text{ is solvable} \wedge (\forall \sigma)\ \sigma \leftarrow P \Rightarrow \sigma \leftarrow Q$$

Definition: $\mathbb{P}_S = \{ P \ / \ P \text{ is solvable} \}$

9. - $\hookleftarrow$ is a preorder in $\mathbb{P}_S$

10. - $P \hookleftarrow Q \wedge R \hookleftarrow S \wedge Q$ **disj** $S \Rightarrow P + R \hookleftarrow Q + S$

Proof: As $P \hookleftarrow Q \wedge R \hookleftarrow S$ then P and R are solvable and so is P + R (ax. 10.1).

Q **disj** S $\Rightarrow$ P **disj** R (ax. 9)

Let σ be such that $\sigma \leftarrow P + R$. P **disj** R $\Rightarrow \exists\ \rho,\tau \ / \ \rho \leftarrow P \wedge \tau \leftarrow R \wedge \sigma = \rho \oplus \tau$ (ax. 10.3). $P \hookleftarrow Q \wedge R \hookleftarrow S \Rightarrow \rho \leftarrow Q \wedge \tau \leftarrow S$. But Q **disj** S and so, $\rho \oplus \tau \leftarrow Q + S$ (ax. 10.2) $\Rightarrow \sigma \leftarrow Q + S$

$\therefore P + R \hookleftarrow Q + S$ ∎

11. - $P \hookleftarrow Q \wedge R \hookleftarrow S \Rightarrow P \times R \hookleftarrow Q \times S$

12. - P solvable $\wedge$ Q solvable $\Rightarrow P \hookleftarrow P \ominus Q \wedge Q \hookleftarrow P \ominus Q$

13. - P solvable $\wedge$ Q solvable $\wedge$ P **coup** Q $\Rightarrow$ P . Q solvable

14. - P solvable $\wedge$ Q solvable $\Leftrightarrow P \times Q$ solvable

15. - P solvable $\vee$ Q solvable $\Rightarrow P \ominus Q$ solvable

16. - $P \leq_{+_c} Q \wedge R \leq_{+_c} S \wedge Q$ **disj** $S \Rightarrow P + R \leq_{+_c} Q + S$

17. - $P \leq_{+_c} Q \wedge R \leq_{+_c} S \Rightarrow P \times R \leq_{+_c} Q \times S$

2.4 - THE CANONICAL INTERPRETATION IS INDEED A MODEL OF THE AXIOMATIC THEORY

The canonical interpretation is the one presented in section 1. Let us recall it:

A problem is a 4-tuple $P = \langle D_P, R_P, q_P, I_P \rangle$, where D_P and R_P are sets over some universe $\mathfrak{U}$, $q_P \subseteq D_P \times R_P$ and $I_P \subseteq D_P$.

Thus, we take as $\mathbb{P}$ the family of all problems over $\mathfrak{U}$.

Operations between problems are interpreted as follows:

$$P + Q = \langle D_P \cup D_Q, R_P \cup R_Q, q_P \cup q_Q, I_P \cup I_Q \rangle$$
$$P \,.\, Q = \langle D_P , R_Q, q_P \mid q_Q, I_P \rangle$$
$$P \times Q = \langle D_P \times D_Q, R_P \times R_Q, q_{P\times Q}, I_P \times I_Q \rangle ,$$

where

$$q_{P\times Q} = \{ ((d_1,d_2) , (r_1,r_2)) / (d_1,r_1) \in q_P \wedge (d_2,r_2) \in q_Q \}$$

$$P \ominus Q = \langle D_P \cup D_Q, R_P \cup R_Q, q_P \cup q_Q, I_P \cap I_Q \rangle$$

The constant **0** is interpreted as

$$\mathbf{0} = \langle \emptyset,\emptyset,\emptyset,\emptyset \rangle$$

and the predicates between problems as

P is injective $\Leftrightarrow (a,c) , (b,c) \in q_P \Rightarrow a = b$

P is surjective $\Leftrightarrow \mathcal{R}(q_P) = R_P$

P is functional $\Leftrightarrow (a,b) , (a,c) \in q_P \Rightarrow b = c$

P is covers its domain $\Leftrightarrow \mathcal{D}(q_P) = D_P$

P is minimal $\Leftrightarrow D_P = I_P$

P may be coupled to Q $\Leftrightarrow \mathcal{R}(q_P|_{I_P}) \subseteq I_Q$

Let $\mathbb{S}$ be the space of all partial functions over $\mathfrak{U}$, with operations:

Suppose $\sigma:A \longrightarrow B \wedge \sigma':A' \longrightarrow B'$

$$\sigma \oplus \sigma' = \mathfrak{S}(\{ \sigma \oplus_K \sigma' / K \in \mathcal{D}(\sigma) \cap \mathcal{D}(\sigma') \}) \quad ,$$

where

$\sigma \oplus_K \sigma':A\cup A' \longrightarrow B\cup B'$ is given by

$$\sigma \oplus_K \sigma'(d) = \begin{cases} \sigma(d) & \text{if } d \in (\mathcal{D}(\sigma) - \mathcal{D}(\sigma')) \cup K \\ \sigma'(d) & \text{if } d \in \mathcal{D}(\sigma') - K \\ \text{undefined} & \text{otherwise} \end{cases}$$

$\sigma \odot \sigma' = \sigma' \circ \sigma$, where $\circ$ is the usual functional composition

$\sigma \otimes \sigma' : A \times A' \longrightarrow B \times B'$, given by

$$\sigma \otimes \sigma' (a,a') = (\sigma(a) , \sigma'(a'))$$

The relation $\propto$ between solutions is interpreted as extension of functions. i.e.:

Suppose $\sigma : A \longrightarrow B \wedge \sigma' : A' \longrightarrow B'$ then

$$\sigma \propto \sigma' \Leftrightarrow A \subseteq A' \wedge B \subseteq B' \wedge \mathcal{D}(\sigma) \subseteq \mathcal{D}(\sigma') \wedge \sigma'|_{\mathcal{D}(\sigma)} = \sigma$$

The relations assoc, $\longleftarrow \subseteq \mathbb{P} \times \mathbb{S}$ are interpreted as follows:

* A function $\sigma : A \longrightarrow B$ is associated to a problem P iff $A \subseteq D_P \wedge B \subseteq R_P$

* $\sigma \longleftarrow P \Leftrightarrow \sigma$ assoc $P \wedge I_P \subseteq \mathcal{D}(\sigma) \wedge \forall d \in I_P , (d,\sigma(d)) \in q_P$

Notice that in the first section we defined an operation on solutions: $\otimes$, the "dual" of $\ominus$, which is not included in the axiomatic theory. As we have already explained in that chapter, there is no real need to have such an operation, since solutions of the components of a $\ominus$ sum are immediately solutions of the $\ominus$ sum. Thus, it would have been redundant to include such an operation in the axiomatic theory.

Returning to our point, we have given the interpretation, but we still have to prove that it satisfies the axioms. Fortunately this has been mostly shown in section 1, and only a few axioms remain to be checked:

2.6 - $(\forall P,Q)\ P \leq_+ Q \Rightarrow 0^P \leq 0^Q \wedge 0_P \leq 0_Q$

Proof: $P \leq_+ Q \Rightarrow D_P \subseteq D_Q \wedge R_P \subseteq R_Q \wedge I_P \subseteq I_Q$. Hence,

$0_P = \langle D_P , \emptyset , \emptyset , I_P \rangle \leq_+ \langle D_Q , \emptyset , \emptyset , I_Q \rangle = 0_Q$

$0^P = \langle \emptyset , R_P , \emptyset , \emptyset \rangle \leq_+ \langle \emptyset , R_Q , \emptyset , \emptyset \rangle = 0^Q$ ∎

7.2 - $\min(P) \Rightarrow$

$$(\forall \sigma)\ \sigma \longleftarrow P \Rightarrow \neg(\exists \sigma')\ (\sigma \neq \sigma' \wedge \sigma \propto \sigma' \wedge \sigma' \text{ assoc } P)$$

Proof: Let $\sigma:A \longrightarrow B$ be such that $\sigma \longleftarrow P \Rightarrow \sigma$ is associated to $P \Rightarrow \mathcal{D}(\sigma) \subseteq A \subseteq D_P$. Now $\sigma \longleftarrow P \Rightarrow I_P \subseteq \mathcal{D}(\sigma) \Rightarrow D_P \subseteq \mathcal{D}(\sigma)$, since $D_P = I_P$.

$\therefore \mathcal{D}(\sigma) = A = D_P$.

Suppose there exists $\sigma':A' \longrightarrow B'$ such that $\sigma \propto \sigma' \wedge \sigma$ is associated to P

σ assoc $P \Rightarrow A' \subseteq D_P$. $\sigma \propto \sigma' \Rightarrow A \subseteq A' \Rightarrow D_P \subseteq A'$.

$\therefore A' = D_P$

Now, $\sigma \propto \sigma' \Rightarrow \sigma'|_{\mathcal{D}(\sigma)} = \sigma \Rightarrow \sigma'|_{D_P} = \sigma \Rightarrow \sigma'|_{A'} = \sigma \Rightarrow \sigma' = \sigma$. ∎

7.3 - $\min(P) \wedge \text{solv}(P) \Rightarrow \text{cov}(P)$

Proof: P solvable $\Rightarrow I_P \subseteq \mathcal{D}(q_P)$

P minimal $\Rightarrow D_P = I_P \Rightarrow D_P \subseteq \mathcal{D}(q_P)$

Now, P is a problem and so, we have $\mathcal{D}(q_P) \subseteq D_P$

$\therefore \mathcal{D}(q_P) = D_P$ and so P covers its domain. ∎

8.1 - $(\forall \sigma)\ \sigma \longleftarrow P \Rightarrow (\exists \sigma')\ \sigma' \propto \sigma \wedge \sigma' \longleftarrow \hat{P}$

Proof: Remark that in the canonical interpretation,

$$\hat{P} = \Phi(P) = \langle I_P\ ,\ R_P\ ,\ q_P|_{I_P}\ ,\ I_P \rangle$$

Let $\sigma:A \longrightarrow B$ be such that $\sigma \longleftarrow P$ and take $\sigma':I_P \longrightarrow B$ given by $\sigma' = \sigma|_{I_P}$.

* $\sigma' \propto \sigma$: $I_P \subseteq \mathcal{D}(\sigma) \subseteq A$, $B \subseteq B$. Since $\sigma \longleftarrow P$, we have $\mathcal{D}(\sigma') = I_P \subseteq \mathcal{D}(\sigma)$ and $\sigma' = \sigma|_{I_P} = \sigma|_{\mathcal{D}(\sigma')}$
* $\sigma' \longleftarrow \hat{P}$: . σ' is associated to $\hat{P}$: $I_P \subseteq D_{\Phi(P)}$. σ associated to $P \Rightarrow$

  $B \subseteq R_P \Rightarrow B \subseteq R_{\Phi(P)}$

  . $\mathcal{D}(\sigma') = I_P = I_{\Phi(P)}$

  . Let $d \in I_{\Phi(P)} \Rightarrow d \in I_P \Rightarrow \sigma'(d) = \sigma(d)$

  $\sigma \longleftarrow P \Rightarrow (d,\sigma(d)) \in q_P \Rightarrow (d,\sigma'(d)) \in q_P$. Since $d \in I_P$, we have $(d,\sigma'(d)) \in q_P|_{I_P} \Rightarrow (d,\sigma'(d)) \in q_{\Phi(P)}$ ∎

9. - $(\forall P,Q,R,S)\ P \hookleftarrow Q \wedge R \hookleftarrow S \Rightarrow (\ Q$ **disj** $S \Rightarrow P$ **disj** $R)$

Proof: Let $P \hookleftarrow Q \wedge R \hookleftarrow S$. In this interpretation

T **disj** $U \Leftrightarrow D_T \cap D_U = \emptyset$

Suppose Q **disj** S.

$P \hookleftarrow Q \wedge R \hookleftarrow S \Rightarrow D_P \subseteq D_Q \wedge D_R \subseteq D_S$

So $D_P \cap D_R \subseteq D_Q \cap D_S = \emptyset \Rightarrow D_P \cap D_R = \emptyset \Rightarrow P$ **disj** R ∎

3. THE THEORY OF UNRESTRICTED PROBLEMS

3.1- $\hat{\mathbb{P}}$: PROBLEMS AS 3-TUPLES

Let us reexamine the definition of a problem as a 4-tuple

$$P = \langle D, R, q, I \rangle \quad , \text{ where } I \subseteq D \wedge q \subseteq D \times R$$

If all that is really important for the problem is in the set of instances of interest then one may wonder why carry along the data domain too.

Here is another definition of problem which considers this objection.

<u>Definition</u>: A problem is a 3-tuple

$$P = \langle D, R, q \rangle ,$$

where

D and R are sets over some universe $\mathfrak{U}$ and $q \subseteq D \times R$.

Although in this definition there is no set of instances of interest, what we have really eliminated is the data domain, keeping in D what was kept before in I.

Anyway, we will call D the *data domain,* just to follow the tradition. As before, R is the *result domain* and q, the *condition* of the problem.

Viewing problems as 4-tuples is equivalent to viewing them as 3-tuples, in the sense that any problem expressed as a 4-tuple may also be expressed as a 3-tuple and conversely, any problem as a 3-tuple may be expressed as a 4-tuple, by means of the following transformations:

<u>Definition</u>: $\gamma : \mathbb{P} \longrightarrow \hat{\mathbb{P}}$

$\delta : \hat{\mathbb{P}} \longrightarrow \mathbb{P}$

$$\gamma \left(\langle D,R,q,I \rangle \right) = \langle I,R,q|_I \rangle$$
$$\delta \left(\langle D,R,q \rangle \right) = \langle D,R,q,D \rangle$$

If we have a 3-tuple, transform it into a 4-tuple and then back again to a 3-tuple, we obtain the same 3-tuple.

However, starting from a 4-tuple, converting it to a 3-tuple and back to a 4-tuple, what we obtain is the restriction of the original problem to its instances of interest, that is, the same as applying the transformation Φ. This is shown in the following

Proposition: i) $\delta \circ \gamma = \Phi$

ii) $\gamma \circ \delta = id_{\hat{\mathbb{P}}}$

Proof:

i) $\delta \circ \gamma (\langle D,R,q,I \rangle) = \delta (\langle I,R,q|_I \rangle) = \langle I,R,q|_I,I \rangle = \Phi (\langle D,R,q,I \rangle)$

ii) $\gamma \circ \delta (\langle D,R,q \rangle) = \gamma (\langle D,R,q,D \rangle) = \langle D,R,q|_D,D \rangle = \langle D,R,q \rangle$ ∎

This equivalence suggests us that $\hat{\mathbb{P}}$ should also be a model of the axiomatic theory. This is true except for a detail: there does not seem to be a plausible interpretation of the operation $\ominus$ which satisfies the axiom

$$(\forall P,Q) (\forall \sigma) (\sigma \leftarrow P \vee \sigma \leftarrow Q \Rightarrow \sigma \leftarrow P \ominus Q)$$

So far we have not given a whole interpretation of $\hat{\mathbb{P}}$; we have only interpreted a part of the domain: problems. To interpret solutions, the constant 0 and the predicate and function symbols we will adapt the meaning given to them by the canonical interpretation. Whenever a reference is made there to the instances of interest of a problem, we will replace it by its data domain.

3.2.- THE $\hat{\mathbb{P}}$ INTERPRETATION

A problem is a 3-tuple $P = \langle D, R, q \rangle$, where D and R are sets over some universe $\mathcal{U}$ and $q \subseteq D \times R$.

Operations between problems are defined as:

$$P + Q = \langle D_P \cup D_Q , R_P \cup R_Q , q_P \cup q_Q \rangle$$
$$P . Q = \langle D_P , R_Q , q_P \mid q_Q \rangle$$
$$P \times Q = \langle D_P \times D_Q , R_P \times R_Q , q_{P\times Q} \rangle ,$$

where

$$q_{P\times Q} = \{ ((d_1,d_2) , (r_1,r_2)) / (d_1,r_1) \in q_P \wedge (d_2,r_2) \in q_Q \}$$

The constant 0 is

$$0 = \langle\emptyset,\emptyset,\emptyset\rangle$$

and we have the following predicates:

P is injective iff $\langle a,c\rangle$, $\langle b,c\rangle \in q_P \Rightarrow a = b$

P is surjective iff $\mathcal{R}(q_P) = R_P$

P is functional iff $\langle a,b\rangle$, $\langle a,c\rangle \in q_P \Rightarrow b = c$

P is covers its domain iff $\mathcal{D}(q_P) = D_P$

$(\forall P)$ P is minimal

P may be coupled to Q iff $\mathcal{R}(q_P) \subseteq D_Q$

$\mathbb{S}$, the space of solutions, is exactly the same as in the canonical interpretation, and the definitions of operations on solutions and the relation $\propto$ is also identical.

The relations assoc, $\leftarrow \subseteq \mathbb{P} \times \mathbb{S}$ are defined as

_ A function $\sigma: A \longrightarrow B$ is associated to a problem P iff $A \subseteq D_P \wedge B \subseteq R_P$

_ $\sigma \leftarrow P \;\Leftrightarrow\; \sigma$ assoc $P \wedge \mathcal{D}(\sigma) = D_P \wedge \forall\, d \in D_P$, $\langle d,\sigma(d)\rangle \in q_P$

As an immediate consequence of these definitions we have that

* If $\sigma: A \longrightarrow B$ is such that $\sigma \leftarrow P$ then $A = D_P$

* P is solvable $\Leftrightarrow$ P covers its domain

3.3.- $\hat{\mathbb{P}}$ AS A MODEL OF $\hat{\mathfrak{P}}$

<u>Definition</u>: Let $\hat{\mathfrak{P}}$ be the theory which results after eliminating from $\mathfrak{P}$ the function symbol $\ominus$ and all the axioms in which it appears.

<u>Theorem</u>: $\hat{\mathbb{P}}$ is a model of $\hat{\mathfrak{P}}$

Proof: Since almost every predicate or function symbol has been interpreted in $\hat{\mathbb{P}}$ as in the canonical interpretation, the proofs that the latter satisfies the axioms are also proofs that $\hat{\mathbb{P}}$ satisfies the axioms as well (Of course, changing all references to instances of interest for references to data domains). ∎

In some cases we can give simpler proofs. Consider for instance axiom **8.1** . Since every problem P is minimal, we have $P \in MIN_P$, where $MIN_P = \{ Q \,/\, Q \subseteq_{+} P \wedge Q \text{ is minimal} \}$ and hence

$$\hat{P} = \sum_{Q \in MIN_P} Q = P$$

Let $\sigma \leftarrow P$, since α is a reflexive relation we have $\sigma \; \alpha \; \sigma$ and also $\sigma \leftarrow \hat{P}$. Hence the axiom $(\forall P)\ (\forall \sigma)\ \sigma \leftarrow P \Rightarrow (\exists \sigma')\ (\sigma' \; \alpha \; \sigma \wedge \sigma' \leftarrow \hat{P})$ is satisfied.

The following proposition tells us what is a relaxation and a complete additive subproblem in $\hat{\mathbb{P}}$:

<u>Proposition</u>: i) $P \hookleftarrow Q \quad \Leftrightarrow \quad \dot{P}$ is solvable $\wedge\ D_Q \subseteq D_P \wedge q_P|_{D_Q} \subseteq q_Q$

ii) $P \leq_{+_C} Q \quad \Leftrightarrow \quad D_P = D_Q \wedge R_P \subseteq R_Q \wedge q_P \subseteq q_Q$

Proof: As before we refer the reader to the corresponding proof in section 1. ∎

We will now see why $\hat{\mathbb{P}}$ cannot be extended to be a model of $\mathfrak{P}$.

Following the canonical interpretation, we may think that $\ominus$ can be interpreted as

$$P \ominus Q = \langle D_P \cap D_Q ,\ R_P \cup R_Q ,\ q_P \cup q_Q \rangle$$

but then although knowing that P and Q are problems it does not follow that $P \ominus Q$ is also a problem, since from $\mathcal{D}(q_P) \subseteq D_P \wedge \mathcal{D}(q_Q) \subseteq D_Q$ we cannot conclude that $\mathcal{D}(q_P \cup q_Q) \subseteq D_P \cap D_Q$.

In fact, the next theorem will show that there is no possible definition of $\ominus$ such that it satisfies the axiom

$$(\forall P,Q)\ (\forall \sigma)\ (\ \sigma \leftarrow P \vee \sigma \leftarrow Q \Rightarrow \sigma \leftarrow P \ominus Q\ )$$

<u>Theorem</u>: $\hat{\mathbb{P}}$ cannot be extended to be a model of $\mathfrak{P}$

Proof: Suppose there exists an interpretation of $\ominus$ such that it satisfies the above axiom and let P,Q be solvable problems such that $D_P \neq D_Q$. Let $\sigma \leftarrow P \Rightarrow \mathcal{D}(\sigma) = D_P$. But $\sigma \leftarrow P \ominus Q$ and so $\mathcal{D}(\sigma) = D_{P\ominus Q}$.
$\therefore D_P = D_{P\ominus Q}$
Let $\sigma' \leftarrow Q \Rightarrow \mathcal{D}(\sigma') = D_Q$. But $\sigma' \leftarrow P \ominus Q$ and so $\mathcal{D}(\sigma') = D_{P\ominus Q}$.
$\therefore D_Q = D_{P\ominus Q}$
Thus $D_P = D_Q$. We get a contradiction and so, there cannot exist such an interpretation. ∎

We do not have yet enough feedback from the applications of the axiomatic theory but the importance of the previous result appears to be as follow: As long as we are not dealing with problem decomposition by abstraction the theory of unrestricted problems captures (and in a simpler way) every aspect the restricted version does.

On the other hand, problem abstraction is characterized by relaxation, and there seems to be a close relationship between relaxation and the $\ominus$ sum. In such a case, we are thus forced to use restricted problems (i.e.: with instances of interest) instead of unrestricted ones.

3.4.- SOLUTIONS OF THE PRODUCT OF PROBLEMS

To close this chapter we will investigate how we can express the space of solutions of the product of two problems (as 3-tuples) in terms of the spaces of solutions of these problems.

We shall obtain some stronger results than those stated for problems as 4-tuples.

<u>Theorem</u>: Let P and Q be solvable problems, then

$$\Omega_P \odot \Omega_Q \subseteq \Omega_{P.Q} \iff \text{P } \mathbf{coup} \text{ Q}$$

Proof:

$\Rightarrow$) We have to show that $\mathcal{R}(q_P) \subseteq D_Q$. Let $b \in \mathcal{R}(q_P) \Rightarrow \exists\ a \ / \ (a,b) \in q_P$. There exists σ such that $\sigma \leftarrow P \wedge \sigma(a) = b$. Let σ' be such that $\sigma' \leftarrow Q$ By hypothesis, $\sigma' \circ \sigma = \sigma \odot \sigma'$ solves P.Q and so $\langle\ a, \sigma'(\sigma(a))\ \rangle \in q_P | q_Q$ $\Rightarrow \langle\ a, \sigma'(b)\ \rangle \in q_P | q_Q$. In particular $\sigma'(b)$ must be defined. Thus $b \in \mathcal{D}(\sigma') \Rightarrow b \in D_Q$, since $\sigma' \leftarrow Q$.

$\Leftarrow$) Let $\sigma : D_P \longrightarrow B \wedge \sigma' : D_Q \longrightarrow B'$ be such that $\sigma \leftarrow P \wedge \sigma' \leftarrow Q$.

* $\sigma' \circ \sigma$ is associated to P.Q : $\sigma' \circ \sigma : D_P \longrightarrow B'$. $D_P = D_{P.Q} \wedge$

$$B' \subseteq R_Q = R_{P.Q}$$

* $\mathcal{D}(\sigma' \circ \sigma) = D_{P.Q}$: We only have to show that $D_P \subseteq \mathcal{D}(\sigma' \circ \sigma) =$
$= \{\ a \in \mathcal{D}(\sigma) \ / \ \sigma(a) \in \mathcal{D}(\sigma')\ \}$. Let $a \in D_P \Rightarrow$
$a \in \mathcal{D}(\sigma) \wedge (a,\sigma(a)) \in q_P \Rightarrow \sigma(a) \in \mathcal{R}(q_P) \Rightarrow$
$\sigma(a) \in D_Q \Rightarrow \sigma(a) \in \mathcal{D}(\sigma') \therefore a \in \mathcal{D}(\sigma' \circ \sigma)$

* Let $a \in D_P \Rightarrow (a,\sigma(a)) \in q_P \Rightarrow \sigma(a) \in \mathcal{R}(q_P) \Rightarrow \sigma(a) \in D_Q \Rightarrow$ $\langle\ \sigma(a), \sigma'(\sigma(a))\ \rangle \in q_Q \Rightarrow \langle\ a, \sigma'(\sigma(a))\ \rangle \in q_P | q_Q \Rightarrow$ $\langle\ a, \sigma \odot \sigma'(a)\ \rangle \in q_{P.Q}$

Thus $\sigma \odot \sigma' \in \Omega_{P.Q}$

$\therefore \Omega_P \odot \Omega_Q \subseteq \Omega_{P.Q}$ ∎

The following example shows that P **coup** Q does not imply $\Omega_{P.Q} \subseteq \Omega_P \odot \Omega_Q$:

<u>Example</u>: Let $P = \langle\ \{1,2\}\ ,\ \{3,4\}\ ,\ \{\ (1,4),(2,4)\ \}\ \rangle$

and $Q = \langle\ \{3,4\}\ ,\ \{5,6\}\ ,\ \{\ (3,5),(4,5),(4,6)\ \}\ \rangle$

Obviously, P may be coupled to Q, since $R_P = D_Q$
We have

$$P \, . \, Q = \langle \langle 1,2 \rangle , \langle 5,6 \rangle , \{ (1,5),(1,6),(2,5),(2,6) \} \rangle$$

Let $\sigma = \{ (1,4),(2,4) \}$, $\sigma' = \{ (3,5),(4,5) \}$, $\sigma'' = \{ (3,5),(4,6) \}$
then

$$\Omega_P = \{ \sigma \} \qquad \text{and} \qquad \Omega_Q = \{ \sigma',\sigma'' \}$$

Consider $\tau = \{ (1,5),(2,6) \}$ then $\tau \in \Omega_{P.Q}$ but

$$\tau \neq \sigma \odot \sigma' \quad \wedge \quad \tau \neq \sigma \odot \sigma'' \quad \Rightarrow \quad \tau \notin \Omega_P \odot \Omega_Q$$

Theorem: Let P and Q be solvable. If P coup Q and Q is functional then

$$\Omega_{P.Q} = \Omega_P \odot \Omega_Q$$

Proof: P coup Q $\Rightarrow \Omega_P \odot \Omega_Q \subseteq \Omega_{P.Q}$
Since P and Q are solvable and P may be coupled to Q we have that P.Q is solvable.
Hence, let σ be such that $\sigma \leftarrow P.Q \Rightarrow \forall\ a \in D_P$, $(a,\sigma(a)) \in q_P | q_Q \Rightarrow$ $\forall\ a \in D_P\ \exists\ t_a \in \mathcal{R}(q_P) \cap \mathcal{D}(q_Q) \ / \ (a,t_a) \in q_P \wedge (t_a,\sigma(a)) \in q_Q$
Define $\sigma_P : D_P \longrightarrow R_P \ \wedge \ \sigma_Q : D_Q \longrightarrow R_Q$ by

$$\sigma_P(a) = t_a \qquad \sigma_Q(b) = \begin{cases} \sigma(a) & \text{if } b \in \mathcal{R}(\sigma_P) \wedge b = \sigma_P(a) \\ t_b & \text{otherwise, where } (b,t_b) \in q_Q \end{cases}$$

We will show that σ_Q is well defined: Let $b \in D_Q$
If $b \notin \mathcal{R}(\sigma_P)$, as Q is solvable, we can find $t_b \in R_Q \ / \ (b,t_b) \in q_Q$
If $b \in \mathcal{R}(\sigma_P)$, suppose there exist $a,a' \in D_P \ / \ \sigma_P(a) = b = \sigma_P(a')$. By the construction of σ_P we have
$(\sigma_P(a), \sigma(a)) \in q_Q \wedge (\sigma_P(a'), \sigma(a')) \in q_Q \Rightarrow (b, \sigma(a)) \in q_Q \wedge$ $(b, \sigma(a')) \in q_Q \Rightarrow \sigma(a) = \sigma(a')$, as Q is functional.
Thus σ_Q is well defined.
It is immediate that $\sigma_P \leftarrow P \wedge \sigma_Q \leftarrow Q \wedge \sigma = \sigma_P \odot \sigma_Q$
$\therefore \ \sigma \in \Omega_P \odot \Omega_Q$ ▩

Using analogous arguments we can show that the same result holds when P is injective.

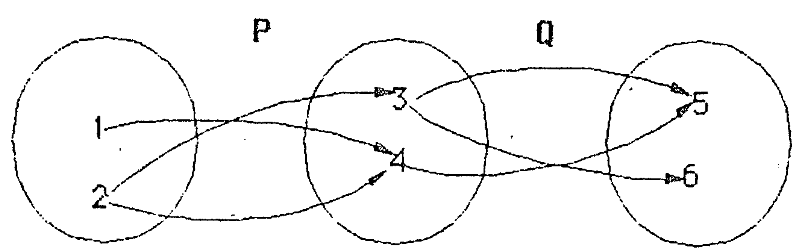
Theorem: Let P and Q be solvable. If P coup Q and Q is injective then

$$\Omega_{P.Q} = \Omega_P \odot \Omega_Q$$

Proof: The only change with respect to the above proof is in showing that $\sigma_Q(b)$ is well defined when $b \in \mathcal{R}(\sigma_P)$.
Remember that, in this case, $\sigma_Q(b) = \sigma(a)$, where a is such that $b = \sigma_P(a)$.
Suppose $\exists\ a,a' \ /\ \sigma_P(a) = b = \sigma_P(a')$. By the construction of σ_P,
$(\ a,\ \sigma_P(a)\ ) \in q_P \wedge (\ a',\ \sigma_P(a')\ ) \in q_P \Rightarrow (a,b) \in q_P \wedge (a',b) \in q_P \Rightarrow$
$a = a'$, since P is injective.
Thus $\sigma(a) = \sigma(a')$ and so σ_Q is well defined. ∎

There are cases in which neither P is injective nor Q is functional but however $\Omega_{P.Q} \subseteq \Omega_P \odot \Omega_Q$:

Example: Let $P = \langle\ \{1,2\}\ ,\ \{3,4\}\ ,\ \{\ (1,4),(2,3),(2,4)\ \}\ \rangle$
and $Q = \langle\ \{3,4\}\ ,\ \{5,6\}\ ,\ \{\ (3,5),(3,6),(4,5)\ \}\ \rangle$.



Note that neither P is injective nor Q is functional.
Now,

$$P.Q = \langle\ \{1,2\}\ ,\ \{5,6\}\ ,\ \{\ (1,5),(2,5),(2,6)\ \}$$

Let $\sigma = \{\ (1,5),(2,5)\ \}$ $\qquad$ $\sigma' = \{\ (1,5),(2,6)\ \}$
$\rho = \{\ (1,4),(2,3)\ \}$ $\qquad$ $\rho' = \{\ (1,4),(2,4)\ \}$
$\tau = \{\ (3,5),(4,5)\ \}$ $\qquad$ $\sigma' = \{\ (3,6),(4,5)\ \}$

We have

$\Omega_{P.Q} = \{\ \sigma,\sigma'\}$ $\qquad$ $\Omega_P = \{\ \rho,\rho'\}$ $\qquad$ $\Omega_Q = \{\ \tau,\tau'\}$

now, $\sigma = \rho \odot \tau = \rho' \odot \tau = \rho' \odot \tau'$ and $\sigma' = \rho \odot \tau'$

$\therefore\ \Omega_{P.Q} = \Omega_P \odot \Omega_Q$

4 CONCLUSIONS

This work had two goals:

* To investigate as deep as possible the "classical" view of problems, that is, 4-tuples. The reason why we have put special emphasis in proving every result and giving counter examples is that we intend this work to be useful as a reference of what is presently known on the theory of problems.

* To develop an axiomatic theory of problems, which as we have said at the beginning of section 2 attempts to definitively lift the formal aspects of the algebraic theory from its interpretations. In doing so we have achieved the necessary freedom to view problems in other promising ways, for instance, with domains on abstract data types.
  As a byproduct of this axiomatization we were able to prove that the theory of unrestricted problems [Vel81] -which actually is an ancestor of the 4-tuple version- is almost equivalent to the latter, in the sense that this one is a model of the axiomatic theory as a whole while the former is a model of only a restriction of it.

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