On definition-like extensions by sorts

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Abstract

The process of implementing a formal specification on another one is very important in formal software development and it can be described in terms of simple basic logical concepts. Formal specifications are presentations of theories in many-sorted logic, and an implementation of a formal specification A on another formal specification C amounts to an interpretation of A into a conservative extension B of C.

An extension consists of the addition of new symbols and axioms, a simple, but important, special case being the so-called extensions by definitions. The new symbols to be introduced correspond to sorts, functions or predicates. Extensions by the addition of function and predicate symbols are well studied in the literature. In particular, extensions by definitions of function and predicate symbols are characterized by the properties of conservativeness together with eliminability.

Our purpose here is to analyse conservative extensions that introduce new sorts. This is of importance because it occurs often in implementing formal specifications, when new sorts are "constructed" from the concrete ones.

We specify and analyse some well-known sort introduction constructs akin to those found in programming languages, namely cartesian product, discriminated union, subsorting and quotient. In each case the extension is shown to be conservative and to exhibit a generalised eliminability property, which corroborates the intuitive feeling that these amount to extension by definition of a new sort.

Key words: Formal specifications, sort constructors, extensions, implementation, abstract data types, parameterized specifications, formal program development.

Resumo

O processo de implementar uma especificação formal em outra é de grande importância no desenvolvimento formal de programas, podendo ser descrito em termos de conceitos lógicos simples que relacionam as especificações formais envolvidas. Na **proposta lógica** para especificações formais, especificações são apresentações axiomáticas em lógica poli-sortida, e uma implemantação de uma especificação A em outra especificação formal C é uma interpretação de A em uma extensão conservativa B de C.

Uma extensão consiste do acréscimo de novos símbolos e axiomas, um caso especial simples, porém importante, sendo as extensões por definições. Os novos símbolos a serem introduzidos correspondem a sortes, funções ou predicados. As extensões por acréscimo de símbolos de função e de predicado estão bem estudadas na literatura. Em particular, as extensões por definição de símbolos de função e de predicado caracterizam-se pelas propriedades de *conservatividade* e de *eliminabilidade*.

Nosso objetivo aqui é analisar extensões conservativas que introduzem novos sortes. Isto é importante pois ocorre com freqüência na implementação de especificações formais, quando novos sortes são "construídos" (ou representados) a partir dos concretos.

Especificaremos e analisaremos alguns construtores de introdução de sortes, similares aqueles encontrados nas linguagens de programação, produto cartesiano, união discriminada, subsorte e quociente. Mostraremos que estas extensões são conservativas e que possuem uma propriedade generalizada de eliminabilidade, o que concorda com o nosso sentimento intuitivo de que correspondem a extensões por definicação de um novo sorte.

Palavras chave: Especificações formais, construtores de sortes, extensões, implementação, tipos abstratos de dados, especificações parametrizadas, desenvolvimento formal de programas.

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1 Introduction

The process of implementing a formal specification on another one is very important in formal software development and it can be described in terms of simple basic logical concepts. Formal specifications are presentations of theories in many-sorted logic, and an implementation of a formal specification A on another formal specification C amounts to an interpretation of A into a conservative extension B of C.

An extension consists of the addition of new symbols and axioms, a simple, but important, special case being the so-called extensions by definitions. The new symbols to be introduced correspond to sorts, functions or predicates. Extensions by the addition of function and predicate symbols are well studied in the literature. In particular, extensions by definitions of function and predicate symbols are characterised by the property of unique expandability, which entails conservativeness and eliminability [Sho67].

Our purpose here is to analyse **conservative extensions that introduce new sorts**. This is of importance because it occurs often in implementing formal specifications, when new sorts are "constructed" from the concrete ones. We first specify and analyse some well-known sort introduction constructs, akin to those found in programming languages ([Hoa74]), namely cartesian product, discriminated union, subsort and quotient. This last construct, quotient, even though not prevalent in programming languages, corresponds to the representation of the equality on an abstract sort, which is necessary for a correct implementation. Also, in the context of implementation of formal specifications, the new sorts introduced should be somehow 'connected' to the old ones. This is what our intuition requires and is what happens in the case of the four constructs examined.

In each case the extension is shown to have unique, up to isomorphism, expandability, to be conservative and to have a generalised kind of eliminability, which corroborates the intuitive feeling that these amount to extensions by definition of a new sort. Moreover, the new sort is shown to exhibit the required behavior.

In section 2 we shall provide formal specifications for each one of these four constructs by axiomatising their usual descriptions.

In section 3 we establish the adequacy of these specifications by showing that they capture the intended behavior. This is done by characterising their models and, we also show that the extensions are conservative.

In section 4 we will present generalised eliminability properties that take into account the new variables ranging over the sort introduced. Therefore, these extensions are like extensions by definition of a new sort, even though, we have unique expandability only up to isomorphism.

In the last section we present some conclusions.

Along this paper, we shall generally employ the usual terminology and notation for logical concepts, for which the reader is referred to standard textbooks ([End72, Sho67]).

2 Constructs for introducing new sorts

A specification describes (properties of) some objects expressed in some specific language. Following the logical approach ([Vel87, TM87]), we shall formulate the specifications within the formalism of first-order logic. Thus, a specification is an axiomatic presentation of a theory in many-sorted first-order logic and the properties described by such specification are the consequences of the presentation. The semantics of the specification (i.e. the meaning which we wish to assign to it) is the theory generated by that presentation.

Actually, a specification consists of declarations and axioms. The declarations give syntactical information concerning its (extra-logical) symbols and the axioms are intended to provide the available information concerning the behavior of these symbols. So, by a specification we mean a pair $P =< L, G >$ consisting of a first-order language L and a set G of sentences of L (its axiomatisation). The equality symbols, as the equality properties, are assumed to be present, even though we may not bother to list them explicitly.

Extending specifications involves extending the languages and the axiomatisations. Given languages L and L' we say that L' extends L (or L is a sub-language of L') iff each extra-logical symbol of L is a symbol of the same kind in L' with the same declaration, which also is to occur with the variables and their declarations as well (notation $L \subseteq L'$). So, each formula of L is a formula of L', but the latter may have new formulae not in L. For theories T and T' we say that T' extends T (or T is a sub-theory of T') iff $T \subseteq T'$ as set of sentences. For specifications $P =< L, T >$ and $P' =< L', T' >$, we say that P' is an extension of P iff $L \subseteq L'$ and $Cn(T) \subseteq Cn(T')$, i.e extension of specifications amount to inclusion between theories and not between axiomatic presentations of theories.

In this section we review the formal specifications for the four usual constructs for sort introduction: cartesian product, subsort, discriminated union and quotient [MV92]. The specifications for the first three constructs are axiomatisations of their usual description [Hoa74, Vel87], while the specification for the last one axiomatises the construction of the quotient with its natural projection.

We see the four specifications as extensions of some base given specification and later on we shall establish some properties of these extensions. We shall use T to stand for the specification of the given theory, where a new sort is to be introduced, and T' for the extended theory, with the new sort and its accompanying conversion functions. Also, L stands for the language of the specification T, and L' for the language of T'. Given a structure $\mathcal{A}'$ for the language L', we use $\mathcal{A} = \mathcal{A}'/L$ to denote its reduct to the sub-language L. Finally, given a function h defined on a many-sorted structure $\mathcal{A}$, we use h_S to denote its restriction to sort $S^{\mathcal{A}}$.

2.1 Cartesian Product

Let T be a specification with two sorts S_1, S_2. Then, a specification of the cartesian product of these two sorts is:

T' = T +

sort P

operations $p_1 : P \to S_1$

$p_2 : P \to S_2$

axioms

- $\forall x_1 : S_1 \; \forall x_2 : S_2 \;\; \exists y : P \quad p_1(y) = x_1 \wedge p_2(y) = x_2$
- $\forall y, y' : P \quad p_1(y) = p_1(y') \wedge p_2(y) = p_2(y') \Longrightarrow y = y'$

These axioms make P behave as the cartesian product of S_1 and S_2. Indeed, in the next section, we will show that, in any model of T', the new sort may be regarded as consisting of the pairs $< a_1, a_2 >$ of elements of the given sorts.

2.2 Subsort

Let T be a specification with a sort S and an unary (relativisation) predicate r over S (r points which elements of S are in the subset N of S) such that $\mathrm{T} \models \exists x : S \; r(x)$. Then, a

specification of the subsort N of the sort S relativised to r is:
T' = T +

sort N
operations $j : N \to S$
axioms

- $\forall x : S \quad r(x) \iff \exists y : N \;\; x = j(y)$
- $\forall y, y' : N \;\; j(y) = j(y') \Longrightarrow y = y'$

These axioms force N to behave as the set of elements of S selected by r. In fact, as will be shown in the next section, in any model of T', the new sort can be regarded as consisting of those elements of sorts that satisfy the relativisation predicate.

Note that if we already have the sorts S and N and the function symbol j then we can introduce the predicate symbol r by definition (as the image of the function j). But we cannot define N solely from S and r.

2.3 Quotient sort

Let T be a specification with a sort S and a binary predicate q over S, which is proved in T to be an equivalence relation. Then, a specification of the quotient sort Q of the sort S by the relation q is:
T' = T +

sort Q
operations $p : S \to Q$
axioms

- $\forall x, x' : S \;\; q(x, x') \iff p(x) = p(x')$
- $\forall y : Q \;\; \exists x : S \;\; p(x) = y$

The axioms state that q is the kernel of the function p and also p is onto. We will show in the next section that the new sort in a model of T' can be regarded as consisting of the equivalence classes of the elements of the given sort under the equivalence relation.

Note that if we have sorts S and Q and a surjective $p : S \to Q$ then we can introduce q by definition via the first axiom, but we cannot define sort Q only from S and q. Also, by introducing such a quotient sort one can normalise any equivalence relation q over sort S to true identity over Q.

2.4 Discriminated Union

Let T be a specification with two sorts S_1 and S_2. Then, a specification of the (discriminated) union of these two sorts is:
T' = T +

sort U
operations $i_1 : S_1 \to U$
$i_2 : S_2 \to U$
axioms

- $\forall y : U \;\; (\exists x_1 : S_1 \quad y = i_1(x_1)) \vee (\exists x_2 : S_2 \quad y = i_2(x_2))$
- $\forall x_1 : S_1 \; \forall x_2 : S_2 \; \neg \, (i_1(x_1) = i_2(x_2))$
- $\forall x_1, x_1' : S_1 \;\; i_1(x_1) = i_1(x_1') \Rightarrow x_1 = x_1'$
- $\forall x_2, x_2' : S_2 \;\; i_2(x_2) = i_2(x_2') \Rightarrow x_2 = x_2'$

These axioms make U exhibit the behavior of the discriminated union of S_1 and S_2, as will be shown. Indeed, in the next section we will show that the new sort in a model of T' may be viewed as the discriminated union of the given sorts.

3 Adequacy of the specifications

In this section we establish some model-theoretic properties of the specifications presented in Section 2.

For each one of the four specifications, we examine their models, which leads to their characterisation, showing that the new sort indeed exhibits the desired behavior. These results indicate the adequacy of the specifications in capturing behavior that is to be expected from the intuitive view of these sort introducing constructs. Also, we obtain, up to isomorphism, unique expandability which entails conservativeness.

We shall examine each construct in turn in the following four subsection.

3.1 Cartesian product

If T and T' are the specifications involving in the presentation of the product sort, the next result characterises the models of T'. It shows that the new sort may be regarded as consisting of the desired ordered pairs and behaves as (categorical) product ([HS73]) of the given sorts.

Given a model $\mathcal{A} \models T$, consider its expansion $\mathcal{A}^\times$ to a structure of L' with $P^{\mathcal{A}^\times} = S_1^{\mathcal{A}} \times S_2^{\mathcal{A}}$ and its projections. Call $\mathcal{A}^\times$ the **canonical expansion** of $\mathcal{A}$ to L'.

Lemma 3.1 (Characterisation of Mod T') *$\mathcal{A}' \models$ T' iff, with $\mathcal{A}'/L = \mathcal{A}$,*

1. *$\mathcal{A} \models T$, and*
2. *there exists a bijection $h : P^{\mathcal{A}'} \to S_1^{\mathcal{A}} \times S_2^{\mathcal{A}}$ such that $h(c) =< p_1^{\mathcal{A}'}(c), p_2^{\mathcal{A}'}(c) >$.*

Proof:

- To prove the left-to-right direction consider the function $h : P^{\mathcal{A}'} \to S_1^{\mathcal{A}} \times S_2^{\mathcal{A}}$ defined by the given assignment. The two axioms of T' establish its surjectivity and injectivity.
- To see the right-to-left direction, it suffices to notice that $\mathcal{A}'$ is isomorphic to $\mathcal{A}^\times$.

Theorem 3.2 (Adequacy) *$\mathcal{A}' \models T'$ iff $\mathcal{A}'$ is isomorphic to the canonical expansion $\mathcal{A}^\times$ of the reduct $\mathcal{A}$ of $\mathcal{A}'$ to L.*

Corollary 3.3 (Expansiveness) *The extension $T \subseteq$ T' is expansive.*

Proposition 3.4 (Conservativeness) *The extension $T \subseteq$ T' is conservative.*

3.2 Subsort

In this subsection T and T' refer to the specifications involving in the presentation of the subsort. Notice the precondition $T \models \exists x : S\ r(x)$.

Given a model $\mathcal{A} \models T$, $\neg(r^{\mathcal{A}} = \emptyset)$. So, consider its expansion $\mathcal{A}^r$ to a structure of L' with $N^{\mathcal{A}^r} = r^{\mathcal{A}}$ and its inclusion. Call $\mathcal{A}^r$ the **canonical expansion** of $\mathcal{A}$ to L'.

Lemma 3.5 (Characterisation of Mod T') *$\mathcal{A}' \models$ T' iff, with $\mathcal{A}'/L = \mathcal{A}$,*

1. *$\mathcal{A} \models T$, and*
2. *there exists a bijection $h : N^{\mathcal{A}'} \to r^{\mathcal{A}}$, where $r^{\mathcal{A}} = \{a \in S^{\mathcal{A}} \ / \ \mathcal{A} \models r(x)[a]\}$, such that $h(c) = j^{\mathcal{A}'}(c)$.*

Proof:

- To prove the left-to-right direction consider the function $h : N^{\mathcal{A}'} \to S^{\mathcal{A}}$ defined by the given assignment. The two axioms of T' establish that, since $h = j^{\mathcal{A}'}$, h is injective and its image is the extension $r^{\mathcal{A}}$ of r in $\mathcal{A}$.
- To prove the right-to-left direction notice that $\mathcal{A}'$ is isomorphic to $\mathcal{A}^r$.

Theorem 3.6 (Adequacy) *$\mathcal{A}' \models T'$ iff $\mathcal{A}'$ is isomorphic to the canonical expansion $\mathcal{A}'$ of the reduct $\mathcal{A}$ of $\mathcal{A}'$ to L.*

This result establishes that the new sort may be thought of as consisting of those elements that satisfy the relativisation predicate. But, the elements of the new sort may very well have a different representation. This possibly different representation is one reason for the conversion function $j : N \to S$. This is illustrated by the (unsigned) naturals as a subsort of the (signed) integers.

Corollary 3.7 (Expansiveness) *The extension T ⊆ T' is expansive.*

Proposition 3.8 (Conservativeness) *The extension T ⊆ T' is conservative.*

3.3 Quotient

In this subsection T and T' are to be understood as the specifications involving in the presentation of the quotient sort. Notice the precondition on T: reflexivity, symmetry and transitivity of q must be theorems of T.

The next result characterises the models of T' by showing that the new sort can be thought to consist of the desired equivalence classes, or representatives for them. Consider a model $\mathcal{A} \models T$, so $q^{\mathcal{A}}$ is an equivalence relation on $S^{\mathcal{A}}$. Consider its expansion $\mathcal{A}^q$ to a structure of L' with $Q^{\mathcal{A}^q} = S^{\mathcal{A}}/q^{\mathcal{A}}$ together with its natural projection onto the quotient. Call $\mathcal{A}^q$ the **canonical expansion** of $\mathcal{A}$ to L'.

Lemma 3.9 (Characterisation of Mod T') *$\mathcal{A}' \models$ T' iff, with $\mathcal{A}'/L = \mathcal{A}$,*

1. *$\mathcal{A} \models T$, and*
2. *there exists a bijection $h : S^{\mathcal{A}}/q^{\mathcal{A}} \to Q^{\mathcal{A}'}$ such that $h([a]) = p^{\mathcal{A}'}(a)$, where $[a]$ is the equivalence class of $a \in S$ under q.*

Proof:

- To prove the left-to-right direction assume $\mathcal{A}' \models T'$ and $\mathcal{A} = \mathcal{A}'/L$. consider the function $p^{\mathcal{A}'} : S^{\mathcal{A}} \to Q^{\mathcal{A}'}$. The two axioms of T' show that $p^{\mathcal{A}'}$ is surjective with $\mathrm{Ker}(p^{\mathcal{A}'}) = q^{\mathcal{A}}$, where $\mathrm{Ker}(p^{\mathcal{A}'}) = \{< a_1, a_2 > \in S^{\mathcal{A}} \times S^{\mathcal{A}} \ / \ p^{\mathcal{A}'}(a_1) = p^{\mathcal{A}'}(a_2)\}$. Hence, there exists a bijection $h : S^{\mathcal{A}}/q^{\mathcal{A}} \to Q^{\mathcal{A}'}$ such that $h.n = p^{\mathcal{A}'}$ with $n : S^{\mathcal{A}} \to S^{\mathcal{A}}/q^{\mathcal{A}}$ given by $n(a) = [a]$.

- To see the right-to-left direction notice that $\mathcal{A}'$ is isomorphic to $\mathcal{A}^q$.

Theorem 3.10 (Adequacy) *$\mathcal{A}' \models T'$ iff $\mathcal{A}'$ is isomorphic to the canonical expansion $\mathcal{A}^q$ of the reduct $\mathcal{A}$ of $\mathcal{A}'$ to L.*

Corollary 3.11 (Expansiveness) *The extension $T \subseteq T'$ is expansive.*

Proposition 3.12 (Conservativeness) *The extension $T \subseteq T'$ is conservative.*

3.4 Discriminated Union

In this subsection, T and T' stand for the specifications involving in the presentation of the union sort.

The next result characterises the models of T'. It shows that the new sort may be viewed as the disjoint union of the given sorts, as desired, and exhibits the behavior of their (categorical) coproduct ([HS73]).

Given a model $\mathcal{A} \models T$, consider its expansion $\mathcal{A}^+$ to a structure of L' with $U^{\mathcal{A}^+} = S_1^{\mathcal{A}} + S_2^{\mathcal{A}}$ and its insertions. Call $\mathcal{A}^+$ the **canonical expansion** of $\mathcal{A}$ to L'.

Lemma 3.13 (Characterisation of Mod T') *$\mathcal{A}' \models T'$ iff, with $\mathcal{A}'/L = \mathcal{A}$,*

1. *$\mathcal{A} \models T$, and*
2. *there exists a bijection $h : S_1^{\mathcal{A}} + S_2^{\mathcal{A}} \to U^{\mathcal{A}'}$ such that $h(a) = i_1^{\mathcal{A}'}(a)$ for each $a \in S_1^{\mathcal{A}}$ and $h(b) = i_2^{\mathcal{A}'}(b)$ for each $b \in S_2^{\mathcal{A}}$.*

Proof:

- To prove the left-to-right direction consider the function $h : S_1^{\mathcal{A}} + S_2^{\mathcal{A}} \to U^{\mathcal{A}'}$ defined by the given assignment. The four axioms of T' establish its bijectivity.
- To prove the right-to-left direction notice that $\mathcal{A}'$ is isomorphic to $\mathcal{A}^+$.

Theorem 3.14 (Adequacy) *$\mathcal{A}' \models T'$ iff $\mathcal{A}'$ is isomorphic to the canonical expansion $\mathcal{A}^+$ of the reduct $\mathcal{A}$ of $\mathcal{A}'$ to L.*

Corollary 3.15 (Expansiveness) *The extension $T \subseteq T'$ is expansive.*

Proposition 3.16 (Conservativeness) *The extension $T \subseteq T'$ is conservative.*

4 The extensions behave like definitions

When performing an extension one expects an increase in both deductive and expressive powers. On the other hand, one often does not wish to disturb the smaller theory. When the extension only adds properties of the new symbols, without disturbing the old ones, one has a conservative extension, which does not increase the deductive power for the properties of the old symbols. Similarly, the increase in expressive power may be inessential, in which case our extension has the property of eliminability.

Extensions by definition of predicate or function symbols have some nice properties.

Consider such an extension of theory T over language L to T' over L'. We then have **Eliminability**. This property states that every formula ρ of L' is equivalent in T' to a

formula φ of L. Consequently, we have **Interpretability**: T' can be interpreted back into L.

On the model-theoretic side, we have **Invariance under elementary equivalence**. I.e., two models of T' are elementarily equivalent whenever their reducts to L are so. And also, **Definability**: a relation is definable in a model of T' iff it is definable in its reduct to L.

In the case of introduction of a sort, we have new variables over the new sort, and new universes realising the new sort. To cope with these, we use two ideas

- **connection**: a local translation of variables ranging over the new sort.
- **internalisation**: a view of relations involving the new sorts.

With these ideas we will be able to generalise the above properties of extensions by definitions of predicate or function symbols to our sort introducing constructs.

4.1 Eliminability for Product Sort

We present a generalised version of the concept of eliminability when we extend our theory with a product sort. In this case we have to face the fact that we now have variables over the product sort. This can be overcome by means of a connection, which indicates how to translate variables ranging over the sort P as pairs of variables over sorts S_1 and S_2.

Consider sets of variables $\overline{y} = \{y^1, \ldots, y^n\}$ over sort P and $\overline{v}$ of sorts in L. By a (product) connection for $\overline{y}$ over $\overline{v}$ we mean sets of variables $\overline{x_1} = \{x_1^1, \ldots, x_1^n\}$ with x_1^k of sort S_1 and $\overline{x_2} = \{x_2^1, \ldots, x_2^n\}$ with x_2^k of sort S_2 such that $\overline{v} \cap \overline{x_1} = \emptyset = \overline{v} \cap \overline{x_2}$.

Proposition 4.1 (Eliminability under connection) *given a (product) connection for $\overline{y}$ over $\overline{v}$, form the conjunction $K(\overline{y}, \overline{x_1}, \overline{x_2})$ of $p_1(y^k) = x_1^k \wedge p_2(y^k) = x_2^k$, for $k = 1, \ldots, n$.*

For every formula $\rho(\overline{v}, \overline{y})$ of L' having as its free variables those in $\overline{v} \cup \overline{y}$, there exists a formula $\varphi(\overline{v}, \overline{x_1}, \overline{x_2})$ of L with free variables in $\overline{v} \cup \overline{x_1} \cup \overline{x_2}$, such that $T' \vdash K(\overline{y}, \overline{x_1}, \overline{x_2}) \rightarrow (\rho(\overline{v}, \overline{y}) \leftrightarrow \varphi(\overline{v}, \overline{x_1}, \overline{x_2}))$. Call such a formula φ of L the **reflection of** *ρ into L (via K), and denote it by ρ^K.*

Proof: By induction on the structure of ρ. Notice that an atomic formula α' of L' is either an equality $y = y'$ between variables y and y' of P, or else, $\alpha' = \alpha'(\overline{v}, \overline{y})$ is the result $\alpha(\overline{v}, \overline{p_i(y)})$ of replacing in atomic formula $\alpha(\overline{v}, \overline{u})$ of L every occurrence of a variable u^k in $\overline{u}$ of sort S_i by $(p_i(y))^k$, and $\vdash \overline{u = p_i(y)} \rightarrow \alpha'(\overline{v}, \overline{y}) = \alpha(\overline{v}, \overline{u})$. For a formula ρ of the form $(\forall y^k : P)\theta$, we take ρ^K as $(\forall x_1^k : S_1)(\forall x_2^k : S_2)\theta^K$.

Corollary 4.2 (Eliminability of formulae without new free variables) *For each formula $\rho(\overline{v})$ of L' without free variables of sort P, there exists a formula $\varphi(\overline{v})$ of L with the same free variables, such that $T' \vdash \rho(\overline{v}) \leftrightarrow \varphi(\overline{v})$.*

Proposition 4.3 (Invariance under elementary equivalence) *Given $\mathcal{A}'_1, \mathcal{A}'_2 \models T'$ if their reducts $\mathcal{A}_1, \mathcal{A}_2$ to L are elementarily equivalent then so are $\mathcal{A}'_1$ and $\mathcal{A}'_2$.*

Proof: this follows from corollary 4.2.

We now establish that the mapping $\rho \rightarrow \rho^K$ given in proposition 4.1 has the behavior akin to that of an interpretation.

Proposition 4.4 (Generalised interpretation) *Consider a structure $\mathcal{A}$ of L with canonical expansion $\mathcal{A}^\times$, and a connection for $\overline{y}$ over $\overline{v}$.*

Each assignment δ of values in $\mathcal{A}$ to the variables in $\overline{x_1} \cup \overline{x_2}$ induces an assignment δ^K for $\overline{y}$ in $\mathcal{A}^\times$ defined by $\delta^K(y^k) =< \delta(x_1^k), \delta(x_2^k) >$.

For every formula $\rho(\overline{v}, \overline{y})$ of L' with free variables in $\overline{v} \cup \overline{y}$ and with K-reflection $\rho^K(\overline{v}, \overline{x_1}, \overline{x_2})$ and for every assignment α for $\overline{v}$ in $\mathcal{A}$:
$\mathcal{A} \models \rho^K[\alpha \cup \delta]$ iff $\mathcal{A}^\times \models \rho[\alpha \cup \delta^K]$.

Proof: Follows from proposition 4.1, since $\mathcal{A}^\times \models K(\overline{y}, \overline{x_1}, \overline{x_2})[\delta \cup \delta^K]$.

The canonical expansion of a model $\mathcal{A} \models T$ is a model $\mathcal{A}^\times \models T'$ where sort P is realised by $P^{\mathcal{A}^\times} = S_1^{\mathcal{A}} \times S_2^{\mathcal{A}}$.

For any sets M, A_1, A_2, N we have a one-to-one correspondence between $M \times (A_1 \times A_2) \times N$ and $M \times A_1 \times A_2 \times N$, which maps ternary relations $R \subseteq M \times (A_1 \times A_2) \times N$ to quaternary relations $R_\times \subseteq M \times A_1 \times A_2 \times N$. This is the basic idea behind internalisation of a relation involving a product sort.

This mapping induces natural mappings from sequences of elements of $\mathcal{A}^\times$ to sequences of elements of $\mathcal{A}$ as follows. Given $\sigma = \sigma_1 \ldots \sigma_k \in (\Sigma \cup \{P\})^+$, let $m(\sigma) = \tau_1 \ldots \tau_l \in \Sigma^+$. We then define $\sigma^{\mathcal{A}} : \sigma_1^{\mathcal{A}^\times} \times \ldots \times \sigma_k^{\mathcal{A}^\times} \to \tau_1^{\mathcal{A}} \times \ldots \times \tau_l^{\mathcal{A}}$ by $\sigma^{\mathcal{A}}(< a_1, \ldots, a_k >) = \sigma_1^{\mathcal{A}}(a_1). \ldots . \sigma_k^{\mathcal{A}}(a_k)$, where $\sigma_i^{\mathcal{A}}(a) =< a >$ if $\neg(\sigma_i = P)$ and $\sigma_i^{\mathcal{A}}(a) =< a_1, a_2 >$ if $\sigma_i = P$ and $a =< a_1, a_2 >$.

Finally, given a relation R over sorts σ in $\mathcal{A}^\times$, its **internalised version** is the relation $R_\times$ over sorts $m(\sigma)$ in $\mathcal{A}$ defined by $R_\times = \{\sigma^{\mathcal{A}}(\alpha)/\alpha \in R\}$.

Lemma 4.5 (Reflection and internalisation) *Consider a model $\mathcal{A} \models T$ with canonical expansion $\mathcal{A}^\times$ and a connection for $\overline{y}$ over $\overline{v}$.*

Given a formula $\rho(\overline{v}, \overline{y})$ with K-reflection $\rho^K(\overline{v}, \overline{x_1}, \overline{x_2})$, the relation $(\rho^K)^{\mathcal{A}}$ defined by ρ^K on $\mathcal{A}$ is the internalised version of the relation $\rho^{\mathcal{A}^\times}$ defined by ρ on $\mathcal{A}^\times$, i.e., $(\rho^{\mathcal{A}^\times})_\times = (\rho^K)^{\mathcal{A}}$.

Proof: Follows from proposition 4.4, since for every assignment η for $\overline{y}$ in $\mathcal{A}^\times$, there exists an assignment δ for $\overline{x_1} \cup \overline{x_2}$ in $\mathcal{A}$ such that $\eta = \delta^K$.

Theorem 4.6 (Definability) *Consider a model $\mathcal{A}' \models T'$ with reduct $\mathcal{A} = \mathcal{A}'/L$. A relation R is definable in $\mathcal{A}'$ iff its internalised version $R_\times$ is definable in $\mathcal{A}$.*

Proof: In view of Theorem 3.2, it suffices to consider $\mathcal{A}^\times$.

If R is defined in $\mathcal{A}^\times$ by ρ, then by lemma 4.5, ρ^K defines $R_\times$ in $\mathcal{A}$.

Notice that $R_\times$ is over sorts $m(\sigma)$ in $\mathcal{A}$. If $R_\times$ is defined in $\mathcal{A}$ by φ then φ is of the form $\varphi(\overline{v}, \overline{x_1}, \overline{x_2})$. So, we can choose $\overline{y}$ so as to have a connection for $\overline{y}$ over $\overline{v}$ to $\overline{x_1}$ and $\overline{x_2}$. Then, R is defined in $\mathcal{A}^\times$ by $(\exists \overline{x_1} : S_1)(\exists \overline{x_2} : S_2)[K(\overline{y}, \overline{x_1}, \overline{x_2}) \wedge \varphi(\overline{v}, \overline{x_1}, \overline{x_2})]$.

4.2 Eliminability for Subsort

We present a generalised version of the concept of eliminability when we extend our theory with a subsort. In this case, as in the case of the product, we have new variables over the subsort. The idea is similar, we use a connection, which indicates how to translate variables ranging over the sort N as variables over sort S.

Consider sets of variables $\overline{y} = \{y^1, \ldots, y^n\}$ of sort N and $\overline{v}$ of sorts in L. By a (subsort) connection for $\overline{y}$ over $\overline{v}$ we mean a set $\overline{x} = \{x^1, \ldots, x^n\}$ of variables of sort S such that $\overline{v} \cap \overline{x} = \emptyset$.

Proposition 4.7 (Eliminability under connection) *Given a (subsort) connection for $\overline{y}$ over $\overline{v}$, form the conjunction $K(\overline{y},\overline{x})$ of $j(y^k) = x^k$, for $k = 1,\ldots,n$.*

For every formula $\rho(\overline{v},\overline{y})$ of L' having as its free variables those in $\overline{v} \cup \overline{y}$, there exists a formula $\varphi(\overline{v},\overline{x})$ of L with free variables in $\overline{v} \cup \overline{x}$, such that $T' \vdash K(\overline{y},\overline{x}) \rightarrow (\rho(\overline{v},\overline{y}) \leftrightarrow \varphi(\overline{v},\overline{x}))$. Call such a formula φ of L the **reflection of** *ρ into L (via K), and denote it by ρ^K.*

Proof: By induction on the structure of ρ. Notice that an atomic formula α' of L' is either equality $y = y'$ between variables y and y' of N, or else, $\alpha' = \alpha'(\overline{v},\overline{y})$ is the result $\alpha(\overline{v},\overline{j(y)})$ of replacing in atomic formula $\alpha(\overline{v},\overline{u})$ of L every occurrence of a variable u^k in $\overline{u}$ of sort S by $(j(y))^k$, and
$\vdash \overline{u = j(y)} \rightarrow \alpha'(\overline{v},\overline{y}) = \alpha(\overline{v},\overline{u})$. For a formula ρ of the form $(\forall y^k : N)\theta$, we take ρ^K as $(\forall x^k : S)[r(x) \rightarrow \theta^K]$.

Corollary 4.8 (Eliminability of formulae without new free variables) *For each formula*
$\rho(\overline{v})$ of L' without free variables of sort N, there exists a formula $\varphi(\overline{v})$ of L with the same free variables, such that $T' \vdash \rho(\overline{v}) \leftrightarrow \varphi(\overline{v})$.

Proposition 4.9 (Invariance under elementary equivalence) *Given $\mathcal{A}'_1, \mathcal{A}'_2 \models T'$ if their reducts $\mathcal{A}_1, \mathcal{A}_2$ to L are elementarily equivalent then so are $\mathcal{A}'_1$ and $\mathcal{A}'_2$.*

Proof: this follows from corollary 4.8.

As in the case of product, we now establish that the mapping $\rho \rightarrow \rho^K$ given in proposition 4.6 behaves like an interpretation.

Proposition 4.10 (Generalised interpretation) *Consider a structure $\mathcal{A}$ for L with canonical expansion $\mathcal{A}^r$, and a connection for $\overline{y}$ over $\overline{v}$.*

Each assignment δ for $\overline{y}$ in $\mathcal{A}^r$ induces an assignment ${}^K\delta$ for $\overline{x}$ in $\mathcal{A}$ defined by ${}^K\delta(x^k) = \delta(y^k)$.

For every formula $\rho(\overline{v},\overline{y})$ of L' with free variables in $\overline{v} \cup \overline{y}$ and with K-reflection $\rho^K(\overline{v},\overline{x})$ and for every assignment α for $\overline{v}$ in $\mathcal{A}$: $\mathcal{A} \models \rho^K[\alpha \cup {}^K\delta]$ iff $\mathcal{A}^r \models \rho[\alpha \cup \delta]$.

Proof: Follows from proposition 4.7, since $\mathcal{A}^r \models K(\overline{y},\overline{x})[\delta \cup {}^K\delta]$.

The canonical expansion of a model $\mathcal{A} \models T$ is a models $\mathcal{A}^r \models T'$, where the realisation of sort N is $N^{\mathcal{A}^r} = r^{\mathcal{A}} \subseteq S^{\mathcal{A}}$.

We now wish to internalise relations in $\mathcal{A}^r$ involving the new subsort. Let Σ be the set of sorts of L, then the set of sorts of L' is $\Sigma \cup \{N\}$. We define $n : (\Sigma \cup \{N\})^* \rightarrow \Sigma^*$ by $n(\lambda) = \lambda$, and, given $\sigma \in (\Sigma \cup \{N\})^*$, $n(\sigma.E) = n(\sigma).E$ for $E \in \Sigma$ and $n(\sigma.N) = n(\sigma).S$.

Finally, given a relation R over sorts σ in $\mathcal{A}^r$, its **internalised version** is the relation R_r over sorts $n(\sigma)$ in $\mathcal{A}$ defined by $R_r = \{\alpha/\alpha \in R\}$. So, R_r has the same extension as R but is viewed as over sorts $n(\alpha)$ of L.

Lemma 4.11 (Reflection and internalisation) *Consider a model $\mathcal{A} \models T$ with canonical expansion $\mathcal{A}^r$ and a connection for $\overline{y}$ over $\overline{v}$. Given a formula $\rho(\overline{y},\overline{v})$ of L' with K-reflection $\rho^K(\overline{v},\overline{x})$, the internalised version of the relation $\rho^{\mathcal{A}^r}$ defined by ρ in $\mathcal{A}^r$ is the relation defined in $\mathcal{A}$ by the formula $\rho^K(\overline{v},\overline{x}) \wedge \overline{r(x)}$, where $\overline{r(x)}$ is the conjunction $r(x^1) \wedge \ldots \wedge r(x^n)$, i.e., $(\rho^{\mathcal{A}^r})_r = (\rho^K \wedge \overline{r})^{\mathcal{A}}$.*

Proof: Follows from proposition 4.10 by noticing two facts. First, for an assignment δ for $\overline{y}$ in $\mathcal{A}^r$, $\mathcal{A} \models \overline{r(x)}[\kappa\delta]$. Second, for every assignment α for $\overline{x}$ in $\mathcal{A}$ such that $\mathcal{A} \models \overline{r(x)}[\alpha]$, there exists an assignment δ for $\overline{y}$ in $\mathcal{A}^r$ such that $\alpha =_\kappa \delta$

Theorem 4.12 (Definability) *Consider a model $\mathcal{A}' \models T'$ with reduct $\mathcal{A} = \mathcal{A}'/L$.*

A relation R is definable in $\mathcal{A}'$ iff its internalised version R_r is definable in $\mathcal{A}$.

Proof: In view of Theorem 3.6, it suffices to consider $\mathcal{A}^r$.

If R is defined in $\mathcal{A}^r$ by ρ, then by lemma 4.11 $\rho^K \wedge \overline{r}$ defines R_r in $\mathcal{A}$.

If R_r is defined in $\mathcal{A}$ by $\varphi(\overline{v}, \overline{x})$ we can select $\overline{y}$ so as to have a connection for $\overline{y}$ over $\overline{v}$. Then, R is defined in $\mathcal{A}^r$ by $(\exists \overline{x} : S)[K(\overline{y}, \overline{x}) \wedge \varphi(\overline{v}, \overline{x})]$, equivalently by $\varphi(\overline{v}, \overline{x}/\overline{j(y)})$.

4.3 Eliminability for Quotient

We present a generalised version of the concept of eliminability when we extend our theory with a quotient sort. In this case we have variables over the quotient sort. This can be overcome by means of a connection, which indicates how to translate variables ranging over the sort Q as the variables over sort S.

Consider sets of variables $\overline{y} = \{y^1, \ldots, y^n\}$ of sort Q and $\overline{v}$ of sorts in L. A (quotient) connection for $\overline{y}$ over $\overline{v}$ consists of a set $\overline{x} = \{x^1, \ldots, x^n\}$ of variables of sort S such that $\overline{v} \cap \overline{x} = \emptyset$.

Proposition 4.13 (Eliminability under connection) *Given a (quotient) connection for $\overline{y}$ over $\overline{v}$, form the conjunction $K(\overline{y}, \overline{x})$ of $y^k = p(x^k)$, for $k = 1, \ldots, n$.*

For every formula $\rho(\overline{v}, \overline{y})$ of L' having as its free variables those in $\overline{v} \cup \overline{y}$, there exists a formula $\varphi(\overline{v}, \overline{x})$ of L with free variables in $\overline{v} \cup \overline{x}$, such that $T' \vdash K(\overline{y}, \overline{x}) \to (\rho(\overline{v}, \overline{y}) \leftrightarrow \varphi(\overline{v}, \overline{x}))$. Call such a formula φ of L the **reflection of** *ρ into L (via K), and denote it by ρ^K.*

Proof: By induction on the structure of ρ. Notice that each new atomic formula α of L' is an equality of one of the forms

- $y^i = y^j$ between variables y^i and y^j of Q; then take α^K as $q(x^i, x^j)$.
- $y^k = p(t)$ or $p(t) = y^k$, with variable $y^k : Q$ and term t of L with sort S; then take α^K as $q(x^k, t)$.
- $p(t) = p(t')$, for some terms t, t' of L with sort S, then take α^K as $q(t, t')$.

For a formula ρ of the form $(\forall y^k : Q)\theta$, we take ρ^K as $(\forall x^k : S)\theta^K$.

Corollary 4.14 (Eliminability of formulae without new free variables) *For each formula $\rho(\overline{v})$ of L' without free variables of sort Q, there exists a formula $\varphi(\overline{v})$ of L with the same free variables, such that $T' \vdash \rho(\overline{v}) \leftrightarrow \varphi(\overline{v})$.*

Proposition 4.15 (Invariance under elementary equivalence) *Given $\mathcal{A}'_1, \mathcal{A}'_2 \models T'$ if their reducts $\mathcal{A}_1, \mathcal{A}_2$ to L are elementarily equivalent then so are $\mathcal{A}'_1$ and $\mathcal{A}'_2$.*

Proof: this follows from corollary 4.14.

As in the previous cases, we now wish to establish that the mapping $\rho \to \rho^K$ given in proposition 4.13 behaves like an interpretation.

Proposition 4.16 (Generalised interpretation) *Consider a structure $\mathcal{A}$ for L with canonical expansion $\mathcal{A}^q$. Each assignment δ for $\overline{x}$ in $\mathcal{A}$ induces an assignment δ^K for $\overline{y}$ in $\mathcal{A}^q$ defined by $\delta^K(y^k) = [\delta(x^k)]$ (i.e. the equivalence class of $\delta(x^k)$ under q).*

For every formula $\rho(\overline{v}, \overline{y})$ of L' with free variables in $\overline{v} \cup \overline{y}$ and with K-reflection $\rho^K(\overline{x}, \overline{y})$ and for every assignment α for $\overline{v}$ in $\mathcal{A}$: $\mathcal{A} \models \rho^K[\alpha \cup \delta]$ iff $\mathcal{A}^q \models \rho[\alpha \cup \delta^K]$.

Proof: follows from proposition 4.13, because $\mathcal{A}^q \models K(\overline{y}, \overline{x})[\delta \cup \delta^K]$.

The canonical expansion of a model $\mathcal{A} \models T$ is a model $\mathcal{A}^q \models T'$ where sort Q is realised by the set of equivalence classes of the elements of $S^{\mathcal{A}}$ under equivalence relation $q^{\mathcal{A}}$.

We wish to internalise relations in $\mathcal{A}^q$ involving the new quotient sort. Let Σ be the set of sorts of L, so the set of sorts of L' is $\Sigma \cup \{Q\}$. We define a mapping $n : (\Sigma \cup \{Q\})^* \to \Sigma^*$ as in the case of subsort.

This mapping induces natural mappings $\sigma^{\mathcal{A}}$ from sequences of elements of $\mathcal{A}$ to sequences of elements of $\mathcal{A}^q$ as follows. Given $\sigma = \sigma_1 . . . . . \sigma_n$, let $n(\sigma) = \tau_1 . . . . . \tau_n$. We then define $\sigma^{\mathcal{A}}(a_1 . . . . . a_n) = b_1 . . . . . b_n$, where $b_k = [a_k]$ if $\sigma_k = Q$ (or if $a \in n(Q)^{\mathcal{A}}$) and $b_k = a_k$ otherwise (or if $a \in n(E)^{\mathcal{A}}$ with E in Σ).

Finally, given a relation R over sorts σ in $\mathcal{A}^q$ its **internalised version** is the relation R_q over sorts $n(\sigma)$ in $\mathcal{A}$ defined by $R_q = \{\overline{a} / \sigma^{\mathcal{A}}(\overline{a}) \in R\}$.

Lemma 4.17 (Reflection and internalisation) *Consider a model $\mathcal{A} \models T$ with canonical expansion $\mathcal{A}^q$ and a connection for $\overline{y}$ over $\overline{v}$.*

Consider a formula $\rho(\overline{y}, \overline{v})$ of L' with K-reflection $\rho^K(\overline{x}, \overline{v})$. Select a set $\overline{u}$ of n new variables $u^1 \dots u^n$ (disjoint from both $\overline{v}$ and $\overline{x}$) over sort S and construct the formula $(\exists \overline{x} : S)[\rho^K(\overline{x}, \overline{v}) \wedge \overline{q(x,u)}]$, where $\overline{q(x,u)}$ is the conjunction $q(x^1, u^1) \wedge \ldots \wedge q(x^k, u^k)$. Then, the internalised version of the relation $\rho^{\mathcal{A}^q}$ defined by ρ in $\mathcal{A}^q$ is the relation defined in $\mathcal{A}$ by the formula $(\exists \overline{u} : S)[\rho^K(\overline{x}, \overline{v}) \wedge \overline{q(x,u)}]$, i.e., $(\rho^{\mathcal{A}^q})_q = [(\exists \overline{u} : S)(\rho^K \wedge \overline{q})]^{\mathcal{A}}$.

Proof: Consider the variable translation $u^k \to y^k$. The above situation makes it a connection K' for $\overline{y}$ over $\overline{v}$. Thus, the assertion will follow from proposition 4.16 by noticing the following two facts:

- Given assignments α for $\overline{v}$ and ζ for $\overline{u}$ in $\mathcal{A}$, $\alpha \cup \zeta \in R_q$ iff $\alpha \cup \zeta^{K'} \in R$.
- Given assignments ξ for $\overline{x}$ and ζ for $\overline{u}$ in $\mathcal{A}$, $\mathcal{A} \models q(\overline{x}, \overline{u})[\xi \cup \zeta]$ iff $\xi^K = \zeta^{K'}$.

Theorem 4.18 (Definability) *Consider a model $\mathcal{A}' \models T'$ with reduct $\mathcal{A} = \mathcal{A}'/L$.*

A relation R is definable in $\mathcal{A}'$ iff its internalised version R_q is definable in $\mathcal{A}$.

Proof: In view of Theorem 3.10, it suffices to consider $\mathcal{A}^q$.

If R is defined in $\mathcal{A}^q$ by ρ, then by lemma 4.17 $(\exists \overline{x} : S)(\rho^K \wedge \overline{q})$ defines R_q in $\mathcal{A}$.

If R_q is defined in $\mathcal{A}$ by $\varphi(\overline{x}, \overline{v})$ we can select $\overline{y}$ so as to have a connection for $\overline{y}$ over $\overline{v}$ to $\overline{x}$. Then, in view of the definition of internalisation, R is defined in $\mathcal{A}^q$ by $(\exists \overline{x} : S)[K(\overline{x}, \overline{y}) \wedge \varphi(\overline{x}, \overline{v})]$.

4.4 Eliminability for Discriminated Union

We present a generalised version of the concept of eliminability when we extend our theory with a discriminated union sort. In this case we have variables over the union sort. This can be overcome by means of a connection, which indicates how to translate variables ranging over the sort U as, sometimes, variables over sort S_1 or, sometimes, over sort S_2.

Consider sets of variables $\overline{y} = \{y^1, \ldots, y^n\}$ of sort U and $\overline{v}$ of sorts in L. A (union) connection for $\overline{y}$ over $\overline{v}$ consists of sets $\overline{x_1} = \{x_1^1, \ldots, x_1^n\}$ of variables of sort S_1 and $\overline{x_2} = \{x_2^1, \ldots, x_2^n\}$ of variables of sort S_2 such that $\overline{v} \cap \overline{x_1} = \emptyset = \overline{v} \cap \overline{x_1}$.

Proposition 4.19 (Eliminability under connection) *Given a (union) connection for $\overline{y}$ over $\overline{v}$. For each function $\delta : \{1,\ldots,n\} \to \{1,2\}$ let $K^\delta = \wedge_{j=1}^n (y_j = i^j_{\delta(j)}(x^j_{\delta(j)}))$ and $\overline{y}^\delta = \{x^1_{\delta(1)},\ldots,x^n_{\delta(n)}\}$.*

*Given a formula $\rho(\overline{v},\overline{y})$ of L' having as its free variables those of $\overline{v}\cup\overline{y}$, there exist 2^n formulae $\rho^\delta(\overline{v},\overline{y}^\delta)$ of L with free variables of $\overline{v}\cup\overline{y}^\delta$, such that $T' \vdash K^\delta \to (\rho(\overline{v},\overline{y}) \leftrightarrow \rho^\delta(\overline{v},\overline{y}^\delta))$. Call such a formula ρ^δ of L the δ-**reflection of** ρ into L (via K).*

Proof: by induction on the structure of ρ. Each new atomic formula α of L' is an equality of one of the forms:

- $y^i = y^j$ with y^i and y^j over U; then take α^δ as $x^i_{\delta(i)} = x^j_{\delta(j)}$ if $\delta(i) = \delta(j)$ and as F otherwise.
- $y^k = i_l(t)$ or $i_l(t) = y^k$, with variable y^k over U and t a term of L with sort S_l, with $l = 1,2$; then take α^δ as $x^k = t$ if $\delta(k) = l$ and as F otherwise.
- $i_l(t) = i_l(t')$, for terms t, t' of L with sort S_l, with $l = 1,2$; then take α^δ as $t = t'$ (notice that $\overline{y} = \emptyset$).

For a formula ρ of the form $(\forall y^k : U)\theta$, we take ρ^δ as $(\forall x^k_1 : S_1)\theta^{\delta_1} \wedge (\forall x^k_2 : S_2)\theta^{\delta_2}$, where δ_1 and δ_2 extend δ by assigning 1 and 2, respectively, to k.

Corollary 4.20 (Eliminability of formulae without new free variables) *For each formula $\rho(\overline{v})$ of L' without free variables of sort U, there exists a formula $\varphi(\overline{v})$ of L with the same free variables, such that $T' \vdash \rho(\overline{v}) \leftrightarrow \varphi(\overline{v})$.*

Proposition 4.21 (Invariance under elementary equivalence) *Given $\mathcal{A}'_1, \mathcal{A}'_2 \models T'$ if their reducts $\mathcal{A}_1, \mathcal{A}_2$ to L are elementarily equivalent then so are $\mathcal{A}'_1$ and $\mathcal{A}'_2$.*

Proof: this follows from corollary 4.20.

As before we now wish to see that the mappings $\rho \to \rho^\delta$ given in proposition 4.19 behave like interpretations.

Proposition 4.22 (Generalised interpretation) *Consider a structure $\mathcal{A}$ for L with canonical expansion $\mathcal{A}^+$. Let $\delta : \{1,\ldots,n\} \to \{1,2\}$ a function associated to a connection for $\overline{y}$ over $\overline{v}$.*

Each assignment ξ for $\overline{y}^\delta$ in $\mathcal{A}$ induces an assignment ξ^δ for $\overline{y}$ in $\mathcal{A}^+$ defined by $\xi^\delta(y^k) = \xi(x^k_{\delta(k)})$.

For every formula $\rho(\overline{v},\overline{y})$ of L' with $\overline{v}\cup\overline{y}$ as its free variables and with δ-reflection $\rho^\delta(\overline{v},\overline{y}^\delta)$ and for every assignment α for $\overline{v}$ in $\mathcal{A}$: $\mathcal{A} \models \rho^\delta[\alpha\cup\xi]$ iff $\mathcal{A}^+ \models \rho[\alpha\cup\xi^\delta]$.

Proof: Follows from proposition 4.19, since $\mathcal{A}^+ \models K^\delta(\overline{y},\overline{y}^\delta)[\xi^\delta \cup \xi]$.

The canonical expansion of a model $\mathcal{A} \models T$ is a model $\mathcal{A}^+ \models T'$ where sort U is realised by $U^{\mathcal{A}^+} = S_1^{\mathcal{A}} + S_2^{\mathcal{A}}$ (disjoint union).

For any sets M, A_1, A_2, N we have $M \times (A_1 \cup A_2) \times N = (M \times A_1 \times N) \cup (M \times A_2 \times N)$. Thus a ternary relations $R \subseteq M \times (A_1 \cup A_2) \times N$ is the union of its views $R_1 \subseteq M \times A_1 \times N$ and $R_2 \subseteq M \times A_2 \times N$. This is the basic idea for internalising a relation involving a discriminated-union sort by means of its views.

We first define a mapping internalising sequences of sorts to their views. Let Σ be the set of sorts of L; so the set of sorts of L' is $\Sigma \cup \{U\}$. Given $\sigma = \sigma_1 \ldots \sigma_k \in (\Sigma \cup \{U\})^+$, let $J = \{j_1,\ldots,j_n\}$ be the set of indices i such that $\sigma_i = U$. Given a function $\delta : \{1,\ldots,n\} \to \{1,2\}$, we define $\sigma_\delta = \tau_1 \ldots \tau_n \in \Sigma^+$ where $\tau_i = \sigma_i$ if $i \notin J$ and $\tau_i = S_{\delta(k)}$ if $i = j_k \in J$.

Given a relation R over sorts σ in $\mathcal{A}^+$, its δ-**internalisation** is the relation R_δ over sorts $\sigma_\delta = \tau_1 \ldots \tau_k$ in $\mathcal{A}$ defined by $R_\delta = R \cap (\tau_1^{\mathcal{A}} \times \ldots \times \tau_k^{\mathcal{A}})$.

Lemma 4.23 (Reflection and internalisation) *Consider a model $\mathcal{A} \models T$ with canonical expansion $\mathcal{A}^+$ as well as a mapping δ associated to a connection for $\overline{y}$ over $\overline{v}$.*

For a formula $\rho(\overline{v}, \overline{y})$ of L' the relation defined in $\mathcal{A}$ by its δ-reflection $\rho^\delta(\overline{v}, \overline{y}^\delta)$ is the δ-internalisation of the relation defined in $\mathcal{A}^+$ by $\rho(\overline{v}, \overline{y})$, i.e., $(\rho^{\mathcal{A}^+})_\delta = (\rho^\delta)^{\mathcal{A}}$.

Proof: Follows from proposition 4.22. Indeed, an assignment ξ for $\overline{y}^\delta$ in $\mathcal{A}$ induces an assignment ξ^δ for $\overline{y}$ in $\mathcal{A}^+$ such that $\xi^\delta(y^k) \in S^{\mathcal{A}}_{\delta(k)}$.

Conversely, given assignment α for $\overline{v}$ and η for $\overline{y}$ in $\mathcal{A}^+$ such that $\alpha \cup \eta \in (\rho^{\mathcal{A}^+})_\delta$, $\eta = \xi^\delta$ for an assignment ξ for $\overline{y}^\delta$ in $\mathcal{A}$ (with $\xi(x^k_{\delta(k)}) = \eta(y^k)$).

Theorem 4.24 (Definability) *Consider a model $\mathcal{A}' \models T'$ with reduct $\mathcal{A} = \mathcal{A}'/L$. A relation R is definable in $\mathcal{A}'$ iff every δ-internalisation R_δ of R is definable in $\mathcal{A}$.*

Proof: In view of Theorem 3.14, it suffices to consider $\mathcal{A}^+$.

If R is defined in $\mathcal{A}^+$ by ρ, then by lemma 4.23, ρ^δ defines R_δ in $\mathcal{A}$.

Conversely, let Δ be the set of all 2^n functions $\delta : \{1, \ldots, n\} \to \{1, 2\}$. Then we have $R = \bigcup_{\delta \in \Delta} R_\delta$.

5 Conclusions

We have examined four sort introducing constructs: cartesian product, subsort, discriminated union and quotient.

Introduction of new sorts is of interest because it occurs often in implementing formal specifications. In contrast to the introduction of new function or predicate symbols [Sho67], the case of new sorts is not extensively studied.

The four constructs examined correspond to natural constructions of new sets. Three of them have found their way as data structuring constructs in programming languages [Hoa74]. The quotient construct is of importance in developing and verifying correct implementations [TM87, Vel87].

In [VM94] we have analised them as parameterised formal specifications of data types. In logical terms, each parameterised specification amounts to a presentation of a theory in the extended language which is a conservative extension of its parameter (a presentation of the theory of the old language that describes its applicability condition).

In section 2 we have presented formal specifications for these four constructs by axiomatising their usual descriptions as extensions of specifications. In each case, in addition to the new sort, we also introduce appropriate conversion functions, connecting it to the old sorts.

In section 3 we proved that, in each case, the new sort is unique, but only up to isomorphism. This contrasts with the case of introduction of new function or predicate symbols by definitions, when we have unique expandability of models. Also, these more familiar extensions by definitions exhibit eliminability: any formula of the extended language is equivalent to some formula in the original language. In our case of introduction of new sorts one should not expect full eliminability, because of the new variables ranging over the sort introduced. Nevertheless, a kind of generalised eliminability is still possible, as we have proved in the propositions called: eliminability under connection.

Actually, along this paper we have established, for each sort introduction construct, the following properties:

1. characterisation of the models, up to isomorphism;

2. the extensions are expansive, hence conservative;
3. eliminability under connection;
4. invariance under elementary equivalence;
5. generalised interpretation to formulae with new free variables;
6. reflection and internalisation;
7. definability of a relation in the extension and its internalised version.

The extension by adding a product sort and by adding a discriminated union are similar to classical extensions by definition of predicate, in that they are conservative and eliminable. The extension of the subsort and the quotient are eliminable but have requirements for conservativeness, and resemble classical extensions by definition of operation.

Another properties like the invariance under isomorphisms, the monomorphic character and the functorial character of models of the extensions were studied in [MV92].

The need for connections stems from the variables over the new sort. Unary constructs, subsort and quotient, use single connections, whereas binary constructs use double connections. These connections map variables over the new sorts to appropriate (pairs of) new variables over the parameter sorts. Our results show that connections exhibit the behavior of local eliminability interpretations.

Another tool for showing that no new relations are definable is the idea of internalising relations involving new sorts. Notice that the internalisation in the case of quotient sort can also be defined by means of choice functions, which would provide several views, much as in the case of discriminated union.

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