ON DEFINABILITY PROPERTIES FOR SORT INTRODUCTION CONSTRUCTS

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Abstract

The process of implementing a formal specification on another one is very important in formal software development and can be described in terms of simple logical concepts as an interpretation into a conservative extension. An extension consists of the addition of new symbols and axioms, a simple, but important, special case being the so-called extensions by definitions. The new symbols to be introduced correspond to sorts, functions and predicates. Extensions by the addition of function and predicate symbols are well studied in the literature.

Our purpose here is to analyse definability properties in conservative extensions that introduce new sorts. We regard some usual sort introduction constructs, akin those found in programming languages (namely cartesian product, discriminated union, subsort and quotient), as extensions of a parameter specification with the given sorts and study some important properties to these extensions.

Key words: Formal specifications, sort constructors, extensions, definability, eliminability, internalisation, implementation, abstract data types, parameterized specifications, formal program development.

Resumo

O processo de implementar uma especificação formal em outra é de grande importância no desenvolvimento formal de programas, podendo ser descrito em termos de conceitos lógicos simples como uma interpretação em uma extensão conservativa. Uma extensão consiste do acréscimo de novos símbolos e axiomas, um caso especial simples, porém importante, sendo as extensões por definições. Os novos símbolos a serem introduzidos correspondem a sortes, funções ou predicados. Extensões por acréscimo de símbolos de função e de predicado estão bem estudadas na literatura.

Nosso objetivo aqui é analisar propriedades de definibilidade em extensões conservativas que introduzem novos sortes. Encaramos alguns construtores de introdução de sortes, similares aos encontrados em linguagens de programação (produto cartesiano, união discriminada, subsorte e quociente), como extensões de uma especificação base com os sortes dados e estudamos algumas propriedades importante destas extensões.

Palavras chave: Especificações formais, construtores de sortes, extensões, definibilidade, eliminabilidade, internalização, implementação, tipos abstratos de dados, especificações parametrizadas, desenvolvimento formal de programas.

Contents

1 Introduction

The process of implementing a formal specification on another one is very important in formal software development and it can be described in terms of simple basic logical concepts.

Formal specifications are presentations of theories in many-sorted logic, and an *implementation* of a formal specification A on another formal specification C amounts to an interpretation of A into a conservative extension B of C. The idea here is that the 'concrete' specification C is extended so as to incorporate versions to which the abstract symbols of A can then be mapped [TM87, Vel87].

An *extension* consists of the addition of new symbols and axioms; a simple, but important, special case being the so-called *extensions by definitions*. The new symbols to be introduced correspond to sorts, functions or predicates. Extensions by the addition of function and predicate symbols are well studied in the literature. In particular, extensions by definitions of function and predicate symbols are characterized by the property of *unique expandability*, which entails *conservativeness* and *eliminability* [Sho67].

Following our previous work [MV92, MV94], our purpose is to analyse **conservative extensions that introduce new sorts**. This is of importance because it occurs often in implementing formal specifications, when new sorts are "constructed" from the concrete ones.

In [MV92] we centred our efforts in axiomatising the specifications and establishing their adequacy in the sense of exhibiting the intented behaviour and in [VM94] we viewed these specifications as parameterised specifications, whose parameter describe the concretes sorts. In this work we will focus our attention on establishing that these extensions are like extensions by definition of a new sort, even though in [MV92] we proved that we have unique expandability only up to isomorphism.

When performing an extension one expects an increase in both deductive and expressive powers. On the other hand, one often does not wish to disturb the smaller theory. When the extension only adds properties of the new symbols, without disturbing the old ones, one has a conservative extension, which does not increase the deductive power for the properties of the old symbols. Similarly, the increase in expressive power may be inessential, in which case our extension has the property of eliminability. This property was examined in more detail in [MV94].

Extensions by definition of predicate or function symbols have some nice properties. Consider such an extension of theory T over language L to T' over L'. We then have

- **Eliminability**. Every formula ρ of L' is equivalent in T' to a formula φ of L.
- **Conservativeness**. Every formula φ of L provable in T' is already provable in T.

Consequently, we have the following properties:

- **Interpretability**. T' can be interpreted back into T.
- **Maximal conservativeness** No proper extension of T' with language L' can be a conservative extension of T.

On the model-theoretic side, we have

- **Invariance under elementary equivalence**. Two models of T' are elementarily equivalent whenever their reducts to L are so.
- **Definability**. A relation is definable in a model of T' iff it is definable in its reduct to L.

In the case of introduction of a sort, we have new variables over the new sort, and new universes, with their equalities, realising the new sort. To cope with these, we use two ideas

- **connection**: a local translation of variables ranging over the new sort.
- **internalisation**: a view of relations involving the new sorts.

With these ideas we will be able to generalise the above properties of extensions by definitions of predicate or function symbols to our sort introducing constructs.

In each one of the next four sections we examine the sort introduction constructs product, subsort, quotient and discriminated union. They have the same structure: for each sort introduction construct we prove some properties, that corroborate the intuitive feeling that these amount to extensions by definition of a new sort. In particular, we will prove that a relation is definable in the extension iff its internalisation is definable in the old language. Thus, the mathematical concept of internalisation corresponds to ref l ection of a formula into the old language.

In section 6 we present some conclusions and some sugestions of future work.

We shall be dealing with extensions by introduction of a new sort. It is convenient to establish some notational conventions. Consider an extension of theory T over language L to theory T' over language L'. We shall generally use letters like u, v, w and x (perhaps with subscripts or superscripts) for variables of the smaller language L. Similarly, we shall employ letters like y and z (perhaps with subscripts or superscripts) for variables ranging over the new sort: new variables of the larger language L'. We also use the notational device of an overbar to indicate a set of variables, e. g. $\overline{w}$ for a set of variables of the smaller language L and $\overline{z}$ for a set of variables ranging over the new sort. Generally when we display a formula φ as $\varphi(\overline{v}, \overline{y})$ we mean that φ has no free occurrences of variables other than those in $\overline{v} \cup \overline{y}$.

A many-sorted language L has an underlying set Σ of sorts. A structure $\mathcal{A}$ for L assigns to each sort $\sigma \in \Sigma$ a nonempty set $\sigma^{\mathcal{A}}$ realising it. Given a sequence $\overline{\sigma} = \sigma_1 . \ldots . \sigma_n$ of sorts of L, we use $\overline{\sigma}^{\mathcal{A}}$ to denote the cartesian product $\sigma_1^{\mathcal{A}} \times \ldots \times \sigma_n^{\mathcal{A}}$. So, $\overline{\sigma}^{\mathcal{A}}$ consists of those sequences $\overline{a} = a_1 . \ldots . a_n$ of elements with $a_1 \in \sigma_i^{\mathcal{A}}$, for $i = 1, \ldots, n$.

Now, consider a set $\overline{w} = \{w^1, \ldots, w^n\}$ of variables of L, with w^i ranging over sort σ^i, for each $i = 1, \ldots, n$. Given a structure $\mathcal{A}$ let $\overline{a}$ be a sequence of n, not necessariy distinct, elements $a^1, \ldots, a^n$, with $a_1 \in \sigma_i^{\mathcal{A}}$. We use the notation $\overline{w}/\overline{a}$ for the assignment α such that $\alpha(w^i) = a^i$, for each $i = 1, \ldots, n$. As usual this notation in the form $\mathcal{A} \models \varphi(\overline{w})[\overline{w}/\overline{a}]$ is simplified to $\mathcal{A} \models \varphi(\overline{w})[\overline{a}]$ when convenient.

Let also us recall the concept of maximal conservative extension, a property of extensions by classical definitions whose name is perhaps not very well known. Let $T' =< L', Ax' >$ be an extension of $T =< L, Ax >$. T' is a maximal conservative extension of T iff it is conservative and for any extension $T" =< L', Ax" >$ of T' that is a conservative extension of T, we have $T"$ equivalent to T'. In other words, T' is a conservative extension of T, such that no proper extensin of T' with the same language can still be a conservative extension of T.

2 Definability for Product Sorts

Given theory T with language L we wish to add a new sort P to behave as the product of sorts S_1 and S_2 of L.

For this purpose we consider the axiomatisation *Prod* for product intrtoduction given in [MV92, MV94]:

sort	P
operations	$p_1 : P \to S_1$
	$p_2 : P \to S_2$
axioms	

- $\forall x_1 : S_1 \; \forall x_2 : S_2 \;\; \exists y : P \quad p_1(y) = x_1 \wedge p_2(y) = x_2$
- $\forall y, y' : P \quad p_1(y) = p_1(y') \wedge p_2(y) = p_2'(y) \Longrightarrow y = y'$

Call $T' = T \cup Prod$ the extension of T by introduction of the new sort P according to *Prod*. Let L' be the language of theory T'.

Given a model $\mathcal{A} \models T$, consider its expansion $\mathcal{A}^\times$ to a structure of L' with $P^{\mathcal{A}^\times} = S_1^{\mathcal{A}} \times S_2^{\mathcal{A}}$ and its projections. Call $\mathcal{A}^\times$ the **canonical expansion** of $\mathcal{A}$ to L'.

We proved in [MV92, prop.3.1.1] the following adequacy result, characterising the models of T' up to isomorphism.

Theorem 2.1 (Adequacy) *$\mathcal{A}' \models T'$ iff $\mathcal{A}'$ is isomorphic to the canonical expansion $\mathcal{A}^\times$ of the reduct $\mathcal{A}$ of $\mathcal{A}'$ to L.*

In [MV94] we proved that we have a generalised eliminability: eliminability under connection. For this purpose we defined a **(product) connection** translating variables $\overline{z}$ of L' to variables $\overline{w}$ of L. We shall now use a slightly changed version of this connection.

Given a set $\overline{z}$ of variables of L', we can partition it into two sets $\overline{z} = \overline{y} \cup \overline{v}$, where $\overline{y} = \{y^1, \ldots, y^n\}$, say, consists of those variables in $\overline{z}$ ranging over the product sort P, and $\overline{v} = \{v^1, \ldots, v^m\}$, say, consists of those variables in $\overline{z}$ ranging over sorts in L. Now, consider a set $\overline{w}$ of variables of L as follows. For each variable y^k in $\overline{z}$ we have two distinct variables x_1^k of sort S_1 and x_2^k of sort S_2; for each variable v^l in $\overline{v}$ over sort S in L we have a distinct variable u^l over sort S. We thus have a partition $\overline{w} = (\overline{x_1} \cup \overline{x_2}) \cup \overline{u}$, where $\overline{x_1}$ and $\overline{x_2}$ have n variables each and $\overline{u}$ has m variables. We say that this matching of variables forms a **(product) connection** iff $\overline{z} \cap \overline{w} = \emptyset$.

We now wish to code this matching of variables into formulae of language L', as follows. For each $k = 1, \ldots, n$, call $K(y^k, x_1^k, x_2^k)$ the formula $p_1(y^k) = x_1^k \wedge p_2(y^k) = x_2^k$; and for each $l = 1, \ldots, m$, call $K(v^l, u^l)$ the formula $v^l = u^l$. By the **content** of the connection we mean the conjunction $K(\overline{z}, \overline{w})$ of these fomulae $K(y^k, x_1^k, x_2^k)$, for $k = 1, \ldots, n$, and $K(v^l, u^l)$, for $l = 1, \ldots, m$.

Such a (product) connection provides enough information for eliminating formulae of the larger language in favour of formulae of the smaller language. The proof of the next proposition provides a constructive procedure, by induction on the structure of formulae, for such elimination under connection [MV94].

Proposition 2.1 (Eliminability under connection) *Consider a (product) connection matching $\overline{z}$ to $\overline{w}$. For every formula $\rho(\overline{z})$ of L' having as its free variables those in $\overline{z}$, there exists a formula $\varphi(\overline{w})$ of L with free variables in $\overline{w}$, such that $T' \vdash K(\overline{z}, \overline{w}) \rightarrow [\rho(\overline{z}) \leftrightarrow \varphi(\overline{w})]$. Call such a formula ϕ of L the* **reflection** *of ρ into L (via K), and denote it by ρ^K.*

In particular, for a formula without free variables over the product sort P no connection is necessary. We then have the usual eliminability (without connection).

Corollary 2.1 (Eliminability without connection) *For each formula σ of L' without free variables over the product sort P, there exists a formula $\sigma_\times$ of L' (with the same free variables) such that $T' \vdash \sigma \leftrightarrow \sigma_\times$.*

As consequences we have invariance under elementary equivalence ([MV94]), as well as the maximal conservativeness of the product extension.

Corollary 2.2 (Maximal Conservativeness) *$T \subseteq T'$ is a maximal conservative extension.*

We now wish to establish that the mapping $\rho \rightarrow \rho^K$ given by eliminability under connection has the behaviour akin to that of an interpretation, namely that it induces a correspondence between models and assignments preserving satisfaction [End72].

A (product) connection correlates variables of the larger language L' to variables of the smaller language L, mainly by matching variables over the product sort P with pairs of variables over the component sorts S_1 and S_2. This establishes a correspondence between assignments of values to variables. Consider a structure $\mathcal{A}'$ of L' with reduct $\mathcal{A}$ to L, and a connection matching $\overline{z} = \overline{y} \cup \overline{v}$ to $\overline{w} = (\overline{x_1} \cup \overline{x_2}) \cup \overline{u}$. Each assignment η of values in $\mathcal{A}'$ to the variables in $\overline{z}$ induces an assignment η^K for variables of $\overline{w}$ in $\mathcal{A}$ as follows:

- for each $k = 1, \ldots, n$, $\eta^K(x_1^k) = p_1^{\mathcal{A}'}(\eta(y^k))$, and $\eta^K(x_2^k) = p_2^{\mathcal{A}'}(\eta(y^k))$;

- for each $l = 1, \ldots, m$, $\eta^K(u^l) = \eta(v^l)$.

The connection content actually codes this induction of assignments, as stated in the next result, whcih characterises induced assignments.

Lemma 2.1 (Connection content and assignment induction) *Consider a structure $\mathcal{A}'$ of L' with reduct $\mathcal{A}$ to L, and a connection matching $\overline{z} = \overline{y} \cup \overline{v}$ to $\overline{w} = (\overline{x_1} \cup \overline{x_2}) \cup \overline{u}$, with content $K(\overline{z}, \overline{w})$.*

Given an assignment η for the variables of $\overline{z}$ in $\mathcal{A}'$, let η^K be the assignment for the variables of $\overline{w}$ in $\mathcal{A}$ induced by η, i.e. $\eta^K(x_1^k) = p_1^{\mathcal{A}'}(\eta(y^k))$, $\eta^K(x_2^k) = p_2^{\mathcal{A}'}(\xi(y^k))$ and $\eta^K(u^l) = \eta(v^l)$.

- *(a) Given an assignment ξ for the variables of $\overline{w}$ in $\mathcal{A}$, $\xi = \eta^K$ iff $\mathcal{A}' \models K(\overline{z}, \overline{w})[\eta \cup \xi]$.*
- *(b) The mapping $\eta \to \eta^K$ is a bijection between assignments for $\overline{z}$ in $\mathcal{A}'$ and assignments for $\overline{w}$ in $\mathcal{A}$.*

The next proposition now follows from eliminability under connection because product structures satisfy the connection content with the assignments constructed.

Proposition 2.2 (Generalised interpretation) *Consider a structure $\mathcal{A}'$ of L' with reduct $\mathcal{A}$ to L, and a connection matching $\overline{z}$ to $\overline{w}$.*

For every formula $\rho(\overline{z})$ of L' with free variables in $\overline{z}$ and with K-reflection $\rho^K(\overline{w})$ and for every assignment η for variables of $\overline{z}$ in $\mathcal{A}'$ with induced assignment η^K:
$\mathcal{A}' \models \rho(\overline{z})\ [\eta]$ iff $\mathcal{A} \models \rho^K(\overline{w})\ [\eta^K]$

As a consequence we have that the mapping $\sigma \to \sigma_\times$ is a faithful interpretation of T' into T.

Corollary 2.3 (Faithful interpretation) *For every sentence σ of L' with reflection $\sigma_\times$ into L':*
$T' \vdash \sigma$ iff $T \vdash \sigma_\times$.

The canonical expansion of a model $\mathcal{A} \models T$ is a model $\mathcal{A}^\times \models T'$ where sort P is realised by $P^{\mathcal{A}^\times} = S_1^{\mathcal{A}} \times S_2^{\mathcal{A}}$. We now wish to internalise relations involving the product sort to relations where P is unfolded into its components.

For any sets M, A_1, A_2, N we have a one-to-one correspondence between $M \times (A_1 \times A_2) \times N$ and $M \times A_1 \times A_2 \times N$, which maps a ternary relation $R \subseteq M \times (A_1 \times A_2) \times N$ to a quaternary relation $R_\times \subseteq M \times A_1 \times A_2 \times N$. This is the basic idea behind internalisation of a relation involving a product sort.

We first define how sorts are to be translated for internalisation.
Let Σ be the set of sorts of L. Then the set of sorts of L' is $\Sigma \cup \{P\}$. We now define a mapping $m : (\Sigma \cup \{P\})^* \to \Sigma^*$ as follows $m(\lambda) = \lambda$, and, given $\overline{\tau} \in (\Sigma \cup \{P\})^*$, for $S \in \Sigma$ $m(\overline{\tau}.S) = m(\overline{\tau}).S$, and $m(\overline{\tau}.P) = m(\overline{\tau}).S_1.S_2$. Notice that if $\overline{\tau}$ is a sequence of lenth k with p occurrences of sort P, then $m(\overline{\tau})$ has length $k + p$.

This mapping m induces natural mappings from sequences of elements of $\mathcal{A}'$ to sequences of elements of $\mathcal{A}$ as follows. Given $\overline{\tau} \in (\Sigma \cup \{P\})^*$, we define the mapping

$\overline{\tau}^{\times}_{\mathcal{A}'} : \overline{\tau}^{\mathcal{A}'} \to m(\overline{\tau})^{\mathcal{A}}$ so that $[\overline{\tau}.\tau]^{\times}_{\mathcal{A}'}(\overline{c}.a) = \overline{\tau}^{\times}_{\mathcal{A}'}(\overline{c}).a$ if $\tau \in \Sigma$, and $[\overline{\tau}.\tau]^{\times}_{\mathcal{A}'}(\overline{c}.b) = \overline{\tau}^{\times}_{\mathcal{A}'}(\overline{c}).p_1^{\mathcal{A}'}(b).p_2^{\mathcal{A}'}(b)$ if $\tau = P$.

Finally, given a relation R over sorts $\overline{\tau}$ in $\mathcal{A}'$, its **internalised version** is the relation $R_{\times}$ over sorts $m(\overline{\tau})$ in $\mathcal{A}$ defined by $R_{\times} = \{\overline{\tau}^{\times}_{\mathcal{A}'}(\overline{b})/\overline{b} \in R\}$.

We wish to show that a relation R is definable in a model $\mathcal{A}' \models T'$ iff its internalised version $R_{\times}$ is definable in its reduct $\mathcal{A}$. For this purpose it is convenient to examine how internalisation and ref l ection are linked. The next lemma, indicating how the elements of a relation and those of its internalised version are linked, is a step in this direction, following from the definitions of internalisation.

Lemma 2.2 (Internalisation and connection content) *Consider a structure $\mathcal{A}'$ of L' with reduct $\mathcal{A}$ to L. Let R be a relation over sorts $\overline{\tau} = \tau_1.\ldots.\tau_k$ in $\mathcal{A}'$ with internalised version $R_{\times}$ over sorts $m(\overline{\tau}) = \sigma_1.\ldots.\sigma_l$ in $\mathcal{A}$. Let $\overline{z}$ be a set of k variables over $\overline{\tau}$ and $\overline{w}$ be a set of l variables over $m(\overline{\tau})$ such that $\overline{z} \cap \overline{w} = \emptyset$. Consider any connection matching $\overline{z}$ to $\overline{w}$, with content $K(\overline{z}, \overline{w})$.*

- *(a) Given $\overline{b} \in \overline{\tau}^{\mathcal{A}'}$ and $\overline{a} \in m(\overline{\tau})^{\mathcal{A}}$, $\overline{a} = \overline{\tau}^{\times}_{\mathcal{A}'}(\overline{b})$ iff $\mathcal{A}' \models K(\overline{z}, \overline{w})[\overline{z}/\overline{b}, \overline{w}/\overline{a}]$.*
- *(b) Given $\overline{a} \in m(\overline{\tau})^{\mathcal{A}}$, $\overline{a} \in R_{\times}$ iff for some $\overline{b} \in R$, $\mathcal{A}' \models K(\overline{z}, \overline{w})[\overline{z}/\overline{b}, \overline{w}/\overline{a}]$.*
- *(c) Given $\overline{b} \in \overline{\tau}^{\mathcal{A}'}$, $\overline{b} \in R$ iff for some $\overline{a} \in R_{\times}$, $\mathcal{A}' \models K(\overline{z}, \overline{w})[\overline{z}/\overline{b}, \overline{w}/\overline{a}]$.*

A simple consequence of elimination under connection, which will be useful, is given in the next result.

Corollary 2.4 (Internalisation of connection) *Consider a (product) connection matching $\overline{z}$ to $\overline{w}$ with content $K(\overline{z}, \overline{w})$.*

For every formula $\rho(\overline{z})$ of L' having as its free variables those in $\overline{z}$, we have that $T' \vdash (\exists \overline{z})[K(\overline{z}, \overline{w}) \wedge \rho(\overline{z})] \leftrightarrow \rho^K(\overline{w})$.

We are now ready for linking internalisation to ref l ection. The next result shows that ref l ection is a syntactic version of internalisation. It follows from the preceding results (lemma 2.2 and corollary 2.4).

Proposition 2.3 (Ref l ection and internalisation) *Consider a model $\mathcal{A}' \models T'$ with reduct $\mathcal{A}$ to L and a connection matching $\overline{z}$ over $\overline{w}$.*

Given a formula $\rho(\overline{z})$ with K-ref l ection $\rho^K(\overline{w})$, the relation $(\rho^K)^{\mathcal{A}}$ defined by ρ^K on $\mathcal{A}$ is the internalised version of the relation $\rho^{\mathcal{A}'}$ defined by ρ on $\mathcal{A}'$, i.e., $(\rho^{\mathcal{A}'})_{\times} = (\rho^K)^{\mathcal{A}}$.

Finally we can show that a relation in a model of the larger theory is definable iff its internalisation is definable in the reduct to the smaller language.

Theorem 2.2 (Definability) *Consider a model $\mathcal{A}' \models T'$ with reduct $\mathcal{A} = \mathcal{A}'/L$. A relation R is definable in $\mathcal{A}'$ iff its internalised version $R_{\times}$ is definable in $\mathcal{A}$.*

We thus have the following results for product extensions:

- eliminability under connection;
- maximal conservativeness;
- internalisation of connection;
- generalised interpretation adjunction connecting models, formulae and assignments;
- ref l ection as a syntactic version of internalisation
- no new definable relations, up to internalisation.

These results generalise the analogous properties presented by extensions by definition of predicate or function symbols to the case of product sort.

3 Definability for Subsorts

Consider a theory T with language L, where $T \vdash \exists x\ r(x)$. We wish to add a new sort N to behave as the subsort of sort S of L according to a predicate r.

For this purpose we consider the axiomatisation *Sbsrt* for subsort introduction given in [MV92, MV94]:

sort	N
operations	$j : N \to S$
axioms	

- $\forall x : S \quad r(x) \iff \exists y : N \ \ x = j(y)$
- $\forall y, y' : N \ \ j(y) = j(y') \Longrightarrow y = y'$

Call $T' = T \cup Sbsrt$ the extension of T by introduction of the new sort N according to *Sbsrt*. Let L' be the language of theory T'.

Given a model $\mathcal{A} \models T$, $r^{\mathcal{A}}$ is nonempty. So, consider its expansion $\mathcal{A}^r$ to a structure of L' with $N^{\mathcal{A}^r} = r^{\mathcal{A}}$ and its inclusion. Call $\mathcal{A}^r$ the **canonical expansion** of $\mathcal{A}$ to L'. We proved in [MV92, prop.3.2.1] the following adequacy result, characterising the models of T' up to isomorphism.

Theorem 3.1 (Adequacy) *$\mathcal{A}' \models T'$ iff $\mathcal{A}'$ is isomorphic to the canonical expansion $\mathcal{A}'$ of the reduct $\mathcal{A}$ of $\mathcal{A}'$ to L.*

In [MV94] we proved that we have a generalised eliminability: eliminability under connection. In this case, as for product, we have new variables over the subsort. The ideia is similar: we use a connection, which indicates how to translate variables ranging over the sort N to variables over sort S. For this purpose we defined a **(subsort) connection** for variables $\overline{y}$ over variables $\overline{v}$. We shall now use a slightly changed version of such subsort connection.

Given a set $\overline{z}$ of variables of L', we can partition it into two sets $\overline{z} = \overline{y} \cup \overline{v}$, where $\overline{y} = \{y^1, \ldots, y^n\}$, say, consists of those variables in $\overline{z}$ ranging over the new subsort N, and $\overline{v} = \{v^1, \ldots, v^m\}$, say, consists of those variables in $\overline{z}$ ranging over sorts in L. Now, consider a set $\overline{w}$ of variables of L as follows. For each variable y^k in $\overline{z}$ we have a distinct variable x^k of sort S; for each variable v^l in $\overline{v}$ over sort E in L we have a distinct variable u^l over sort E. We thus have a partition $\overline{w} = \overline{x} \cup \overline{u}$, where $\overline{x}$ has n variables and $\overline{u}$ has m variables. We say that this matching of variables forms a **(subsort) connection** iff $\overline{z} \cap \overline{w} = \emptyset$.

We now wish to code this matching of variables into formulae of language L', as follows. For each $k = 1, \ldots, n$, call $K(y^k, x^k)$ the formula $j(y^k) = x^k$; and for each $l = 1, \ldots, m$, call $K(v^l, u^l)$ the formula $v^l = u^l$. By the **content** of the connection we mean the conjunction $K(\overline{z}, \overline{w})$ of these fomulae $K(y^k, x^k)$, for $k = 1, \ldots, n$, and $K(v^l, u^l)$, for $l = 1, \ldots, m$.

Such a (subsort) connection provides enough information for eliminating formulae of the larger language in favour of formulae of the smaller language, as stated in the next proposition, whose proof proceeds by induction on the structure of formulae [MV94].

Proposition 3.1 (Eliminability under connection) *Consider a (subsort) connection matching $\overline{z}$ to $\overline{w}$ with content $K(\overline{z}, \overline{w})$. For every formula $\rho(\overline{z})$ of L' having as its free variables those in $\overline{z}$, there exists a formula $\varphi(\overline{w})$ of L with free variables in $\overline{w}$, such that $T' \vdash K(\overline{z}, \overline{w}) \to [\rho(\overline{z}) \leftrightarrow \varphi(\overline{w})]$. Call such a formula φ of L the* **reflection of** *ρ into L (via K), and denote it by ρ^K.*

In particular, for a formula without free variables over the new subsort N no connection is necessary. We then have the usual eliminability (without connection).

Corollary 3.1 (Eliminability without connection) *For each formula σ of L' without free variables over the new subsort N, there exists a formula σ_r of L' (with the same free variables) such that $T' \vdash \sigma \leftrightarrow \sigma_r$.*

As in the previous case, we have as consequences invariance under elementary equivalence ([MV94]), as well as the maximal conservativeness of the subsort extension.

Corollary 3.2 (Maximal Conservativeness) *$T \subseteq T'$ is a maximal conservative extension.*

Another simple consequence of elimination under connection, related to relativisation, is given in the next result.

Lemma 3.1 (Image of connection) *Consider a (subsort) connection matching $\overline{z} = \overline{y} \cup \overline{v}$ to $\overline{w} = \overline{x} \cup \overline{u}$ with content $K(\overline{z}, \overline{w})$. Let $\overline{r(x)}$ be the conjunction $r(x^1) \wedge \ldots \wedge r(x^n)$. Then, $T' \vdash (\exists \overline{z}) K(\overline{z}, \overline{w}) \leftrightarrow \overline{r(x)}$.*

As in the case of product, we now wish to establish that the mapping $\rho \to \rho^K$ given by eliminability under connection exhibits a behaviour akin to that of an interpretation, namely that it induces a correspondence between models and assignments preserving satisfaction [End72].

A (subsort) connection correlates variables of the larger language L' to variables of the smaller language L, mainly by matching variables over the subsort N with variables over the originating sort S. This establishes a correspondence between assignments of values to variables. Consider a structure $\mathcal{A}'$ for L' with reduct $\mathcal{A}$ to L, and a connection matching $\overline{z} = \overline{y} \cup \overline{v}$ to $\overline{w} = \overline{x} \cup \overline{u}$. Each assignment η for the variables of $\overline{z}$ in $\mathcal{A}'$ induces an assignment η^K for the variables of $\overline{u}$ in $\mathcal{A}$ as follows:

- for each $k = 1, \ldots, n$, $\eta^K(x^k) = j^{\mathcal{A}'}(\eta(y^k))$;
- for each $l = 1, \ldots, m$, $\eta^K(u^l) = \eta(v^l)$.

The connection content actually codes this induction of assignments, as stated in the next result, which also characterises the induced assignments.

Lemma 3.2 (Connection content and assignment induction) *Consider a model $\mathcal{A}' \models T'$ with reduct $\mathcal{A}$ to L, and a connection matching $\overline{z} = \overline{y} \cup \overline{v}$ to $\overline{w} = \overline{x} \cup \overline{u}$ with content $K(\overline{z}, \overline{w})$.*

Given an assignment η for the variables of $\overline{z}$ in $\mathcal{A}'$, let η^K be the assignment for the variables of $\overline{w}$ in $\mathcal{A}$ induced by η, i.e. $\eta^K(x^k) = j^{\mathcal{A}'}(\eta(y^k))$ and $\eta^K(u^l) = \eta(v^l)$.

- *(a) Given an assignment ξ for the variables of $\overline{w}$ in $\mathcal{A}$, $\xi = \eta^K$ iff $\mathcal{A}' \models K(\overline{z}, \overline{w})[\eta \cup \xi]$.*
- *(b) Let $\overline{r(x)}$ be the conjunction $r(x^1) \wedge \ldots \wedge r(x^n)$. Then, $\xi = \eta^K$, for some assignment η for the variables of $\overline{z}$ in $\mathcal{A}'$, iff $\mathcal{A} \models \overline{r(x)}[\xi]$.*

The next result now follows from eliminabililty under connection because the subsort structures satisfy the connection content with the assignments constructed.

Proposition 3.2 (Generalised interpretation) *Consider a model $\mathcal{A}' \models T'$ with reduct $\mathcal{A}$ to L, and a connection matching $\overline{z}$ to $\overline{w}$.*

For every formula $\rho(\overline{z})$ of L' with free variables in $\overline{z}$ and with K-reflection $\rho^K(\overline{w})$ and for every assignment η for variables of $\overline{z}$ in $\mathcal{A}'$ with induced assignment η^K:
$\mathcal{A}' \models \rho(\overline{z})\ [\eta]$ *iff* $\mathcal{A} \models \rho^K(\overline{w})\ [\eta^K]$

As a consequence we have that the mapping $\sigma \to \sigma_r$ is a faithful interpretation of T' into T.

Corollary 3.3 (Faithful interpretation) *For every sentence σ of L' with reflection σ_r into L':*
$T' \vdash \sigma$ *iff* $T \vdash \sigma_r$.

The canonical expansion of a model $\mathcal{A} \models T$ is a model $\mathcal{A}^r \models T'$, where sort N is realised by $r^{\mathcal{A}} \subseteq S^{\mathcal{A}}$. We now wish to internalise relations in $\mathcal{A}^r$ involving the new subsort to relations over the original sort S.

As in the case of product, we first define how sorts are to be translated for internalisation.
Let Σ be the set of sorts of L, then the set of sorts of L' is $\Sigma \cup \{N\}$. We define $n : (\Sigma \cup \{N\})^* \to \Sigma^*$ by $n(\lambda) = \lambda$, and, given $\overline{\tau} \in (\Sigma \cup \{N\})^*$, $n(\overline{\tau}.E) = n(\overline{\tau}).E$ for $E \in \Sigma$ and $n(\overline{\tau}.N) = n(\overline{\tau}).S$. Notice that $\overline{\tau}$ and $n(\overline{\tau})$ have the same length.

This mapping n induces natural mappings from sequences of elements of $\mathcal{A}'$ to sequences of elements of $\mathcal{A}$ as follows. Given $\overline{\tau} \in (\Sigma \cup \{N\})^*$, we define the mapping $\overline{\tau}^r_{\mathcal{A}'} : \overline{\tau}^{\mathcal{A}'} \to n(\overline{\tau})^{\mathcal{A}}$ so that $[\overline{\tau}.\tau]^r_{\mathcal{A}'}(\overline{c}.a) = \overline{\tau}^r_{\mathcal{A}'}(\overline{c}).a$ if $\tau \in \Sigma$, and $[\overline{\tau}.\tau]^r_{\mathcal{A}'}(\overline{c}.b) = \overline{\tau}^r_{\mathcal{A}'}(\overline{c}).j^{\mathcal{A}'}(b)$ if $\tau = N$.

Finally, given a relation R over sorts $\overline{\tau}$ in $\mathcal{A}'$, its **internalised version** is the relation R_r over sorts $n(\overline{\tau})$ in $\mathcal{A}$ defined by $R_r = \{\overline{\tau}^r_{\mathcal{A}'}(\overline{b})/\overline{b} \in R\}$.

We wish to show that a relation R is definable in a model $\mathcal{A}' \models T'$ iff its internalised version R_r is definable in its reduct $\mathcal{A}$. For this purpose it is convenient to examine how internalisation and reflection are linked. The next lemma is a step in this direction, following from the definition of internalised relation and indicating how the elements of a relation and those of its internalised version are linked.

Lemma 3.3 (Internalisation and connection content) *Consider a structure $\mathcal{A}'$ of L' with reduct $\mathcal{A}$ to L. Let R be a relation over sorts $\overline{\tau} = \tau_1.\ldots.\tau_k$ in $\mathcal{A}'$ with internalised version R_r over sorts $n(\overline{\tau}) = \sigma_1.\ldots.\sigma_k$ in $\mathcal{A}$. Let $\overline{z}$ be a set of k variables over $\overline{\tau}$ and $\overline{w}$ be a set of k variables over $n(\overline{\tau})$ such that $\overline{z} \cap \overline{w} = \emptyset$. Consider any connection matching $\overline{z}$ to $\overline{w}$, with content $K(\overline{z}, \overline{w})$.*

- *(a) Given $\overline{b} \in \overline{\tau}^{\mathcal{A}'}$ and $\overline{a} \in n(\overline{\tau})^{\mathcal{A}}$, $\overline{a} = \overline{\tau}^r_{\mathcal{A}'}(\overline{b})$ iff $\mathcal{A}' \models K(\overline{z}, \overline{w})[\overline{z}/\overline{b}, \overline{w}/\overline{a}]$.*
- *(b) Given $\overline{a} \in n(\overline{\tau})^{\mathcal{A}}$, $\overline{a} \in R_r$ iff for some $\overline{b} \in R$, $\mathcal{A}' \models K(\overline{z}, \overline{w})[\overline{z}/\overline{b}, \overline{w}/\overline{a}]$.*
- *(c) Given $\overline{b} \in \overline{\tau}^{\mathcal{A}'}$, $\overline{b} \in R$ iff for some $\overline{a} \in R_r$, $\mathcal{A}' \models K(\overline{z}, \overline{w})[\overline{z}/\overline{b}, \overline{w}/\overline{a}]$.*

A simple consequence of elimination under connection, which will be useful, is given in the next result.

Corollary 3.4 (Internalisation of connection) *Consider a (subsort) connection matching $\overline{z} = \overline{y} \cup \overline{v}$ to $\overline{w} = \overline{x} \cup \overline{u}$ with content $K(\overline{z}, \overline{w})$. Let $\overline{r(x)}$ be the conjunction $r(x^1) \wedge \ldots \wedge r(x^n)$.*

For every formula $\rho(\overline{z})$ of L' having as its free variables those in $\overline{z}$, we have that $T' \vdash (\exists \overline{z})[K(\overline{z}, \overline{w}) \wedge \rho(\overline{z})] \leftrightarrow [\rho^K(\overline{w}) \wedge \overline{r(x)}]$.

We now wish to link internalisation to reflection. The next result shows that reflection provides the basis for a syntactic version of internalisation. It follows from the preceding results (lemma 3.3 and corollary 3.4).

Proposition 3.3 (Reflection and internalisation) *Consider a model $\mathcal{A}' \models T'$ with reduct $\mathcal{A}$ to L and a (subsort) connection matching $\overline{z} = \overline{y} \cup \overline{v}$ to $\overline{w} = \overline{x} \cup \overline{u}$ with content $K(\overline{z}, \overline{w})$. Let $\overline{r(x)}$ be the conjunction $r(x^1) \wedge \ldots \wedge r(x^n)$. Given a formula $\rho(\overline{z})$ of L' with K-reflection $\rho^K(\overline{w})$, the internalised version of the relation $\rho^{\mathcal{A}'}$ defined by ρ in $\mathcal{A}'$ is the relation defined in $\mathcal{A}$ by the formula $\rho^K(\overline{v}, \overline{x}) \wedge \overline{r(x)}$ i.e., $(\rho^{\mathcal{A}'})_r = (\rho^K \wedge \overline{r})^{\mathcal{A}}$.*

We can now finally show that a relation in a model of the larger theory is definable iff its internalisation is definable in the reduct to the smaller language.

Theorem 3.2 (Definability) *Consider a model $\mathcal{A}' \models T'$ with reduct $\mathcal{A} = \mathcal{A}'/L$. A relation R is definable in $\mathcal{A}'$ iff its internalised version R_r is definable in $\mathcal{A}$.*

We thus have the following results for subsort extensions:

- eliminability under connection;
- maximal conservativeness;
- internalisation of connection;
- generalised interpretation adjunction connecting models, formulae and assignments;
- reflection as a syntactic version of internalisation
- no new definable relations, up to internalisation.

These results generalise the analogous properties presented by extensions by definition of predicate or function symbols to the case of subsort.

4 Definability for Quotient Sorts

Consider a theory T with language L, where we have a binary predicate symbol q, over sort S, which is provable in T to be reflexive, symmetric and transitive. We wish to add a new sort Q to behave as the quotient of sort S of L according to equivalence relation q.

For this purpose we consider the axiomatisation *Quot* for quotient introduction given in [MV92, MV94]:

sort	Q
operations	$p : S \to Q$
axioms	

- $\forall x, x' : S \;\; q(x, x') \Longleftrightarrow p(x) = p(x')$
- $\forall y : Q \;\; \exists x : S \;\; p(x) = y$

Call $T' = T \cup Quot$ the extension of T by introduction of the new sort Q according to $Quot$. Let L' be the language of T'.

Consider a model $\mathcal{A} \models T$; so $q^{\mathcal{A}}$ is an equivalence relation over $S^{\mathcal{A}}$. Consider its expansion $\mathcal{A}^q$ to a structure of L' with $Q^{\mathcal{A}^q} = S^{\mathcal{A}}/q^{\mathcal{A}}$ together with its natural projection onto the quotient. Call $\mathcal{A}^q$ the **canonical expansion** of $\mathcal{A}$ to L'. We proved in [MV92, prop.3.4.1] the following adequacy result, characterising the models of T' up to isomorphism.

Theorem 4.1 (Adequacy) *$\mathcal{A}' \models T'$ iff $\mathcal{A}'$ is isomorphic to the canonical expansion $\mathcal{A}^q$ of the reduct $\mathcal{A}$ of $\mathcal{A}'$ to L.*

In [MV94] we proved that we have a generalised eliminability: eliminability under connection. As in the case of subsort the connection indicates how to translate variables ranging over the sort Q to variables over sort S. For this purpose we defined a **(quotient) connection** for variables $\overline{y}$ over variables $\overline{v}$. We shall now use a slightly changed version of this connection.

Given a set $\overline{z}$ of variables of L', we can partition it into two sets $\overline{z} = \overline{y} \cup \overline{v}$, where $\overline{y} = \{y^1, \ldots, y^n\}$, say, consists of those variables in $\overline{z}$ ranging over the new quotient sort Q, and $\overline{v} = \{v^1, \ldots, v^m\}$, say, consists of those variables in $\overline{z}$ ranging over sorts in L. Now, consider a set $\overline{w}$ of variables of L as follows. For each variable y^k in $\overline{z}$ we have a distinct variable x^k of sort S; for each variable v^l in $\overline{v}$ over sort E in L we have a distinct variable u^l over sort E. We thus have a partition $\overline{w} = \overline{x} \cup \overline{u}$, where $\overline{x}$ has n variables and $\overline{u}$ has m variables. We say that this matching of variables forms a **(quotient) connection** iff $\overline{z} \cap \overline{w} = \emptyset$.

We now wish to code this matching of variables into formulae of language L', as follows. For each $k = 1, \ldots, n$, call $K(y^k, x^k)$ the formula $p(x^k) = y^k$; and for each $l = 1, \ldots, m$, call $K(v^l, u^l)$ the formula $v^l = u^l$. By the **content** of the connection we mean the conjunction $K(\overline{z}, \overline{w})$ of these fomulae $K(y^k, x^k)$, for $k = 1, \ldots, n$, and $K(v^l, u^l)$, for $l = 1, \ldots, m$.

As in the previous cases, such a (quotient) connection provides enough information for eliminating formulae of the larger language in favour of formulae of the smaller language. The (constructive) proof of the next proposition, by induction on the structure of formulae, provides such a elimination procedure [MV94].

Proposition 4.1 (Eliminability under connection) *Consider a (quotient) connection matching $\overline{z}$ to $\overline{w}$ with content $K(\overline{z}, \overline{w})$. For every formula $\rho(\overline{z})$ of L' having as its free variables those in $\overline{z}$, there exists a formula $\varphi(\overline{w})$ of L with free variables in $\overline{w}$, such that $T' \vdash K(\overline{z}, \overline{w}) \to [\rho(\overline{z}) \leftrightarrow \varphi(\overline{w})]$. Call such a formula φ of L the* **reflection of** *ρ into L (via K), and denote it by ρ^K.*

In particular, for a formula without free variables over the quotient sort Q no connection is necessary. We then have the usual eliminability (without connection).

Corollary 4.1 (Eliminability without connection) *For each formula σ of L' without free variables over the quotient sort Q, there exists a formula σ_q of L' (with the same free variables) such that $T' \vdash \sigma \leftrightarrow \sigma_q$.*

As in the previous case, we have as consequences invariance under elementary equivalence ([MV94]), as well as the maximal conservativeness of the subsort extension.

Corollary 4.2 (Maximal Conservativeness) *$T \subseteq T'$ is a maximal conservative extension.*

As in the previous cases, we now wish to establish that the mapping $\rho \to \rho^K$ given by eliminability under connection behaves like an interpretation, namely that it induces a correspondence between models and assignments preserving satisfaction [End72].

A (quotient) connection correlates variables of the larger language L' to variables of the smaller language L, mainly by matching variables over the quotient sort Q with variables over the originating sort S. This establishes a correspondence between assignments of values to variables. Consider a structure $\mathcal{A}'$ for L' with reduct $\mathcal{A}$ to L, and a connection matching $\overline{z} = \overline{y} \cup \overline{v}$ to $\overline{w} = \overline{x} \cup \overline{u}$. Each assignment ξ for the variables of $\overline{u}$ in $\mathcal{A}$ induces an assignment ξ^K for the variables of $\overline{z}$ in $\mathcal{A}'$ as follows:

- for each $k = 1, \ldots, n$, $\xi^K(y^k) = p^{\mathcal{A}'}(\xi(x^k))$;
- for each $l = 1, \ldots, m$, $\xi^K(u^l) = \xi(v^l)$.

The connection content actually codes this induction of assignments, as stated in the next result, which characterises induced assignments.

Lemma 4.1 (Connection content and assignment induction) *Consider a model $\mathcal{A}' \models T'$ with reduct $\mathcal{A}$ to L, and a connection matching $\overline{z} = \overline{y} \cup \overline{v}$ to $\overline{w} = \overline{x} \cup \overline{u}$ with content $K(\overline{z}, \overline{w})$.*

Given an assignment ξ for the variables of $\overline{u}$ in $\mathcal{A}$, let ξ^K be the assignment for the variables of $\overline{z}$ in $\mathcal{A}'$ induced by ξ, i.e. $\xi^K(y^k) = p^{\mathcal{A}'}(\xi(x^k))$ and $\xi^K(u^l) = \xi(v^l)$.

- *(a) Given an assignment η for the variables of $\overline{z}$ in $\mathcal{A}'$, $\eta = \xi^K$ iff $\mathcal{A}' \models K(\overline{z}, \overline{w})[\eta \cup \xi]$.*
- *(b) The mapping $\xi \to \xi^K$ is a surjection from assignments for $\overline{w}$ in $\mathcal{A}$ onto assignments for $\overline{z}$ in $\mathcal{A}'$.*

The next result now follows from eliminability under connection because quotient structures satisfy the connection content with the assignments constructed.

Proposition 4.2 (Generalised interpretation) *Consider a model $\mathcal{A}' \models T'$ with reduct $\mathcal{A}$ to L, and a connection matching $\overline{z}$ to $\overline{w}$.*

For every formula $\rho(\overline{z})$ of L' with free variables in $\overline{z}$ and with K-reflection $\rho^K(\overline{w})$ and for every assignment ξ for variables of $\overline{w}$ in $\mathcal{A}$ with induced assignment ξ^K:
$\mathcal{A}' \models \rho(\overline{z})\ [\xi^K]$ iff $\mathcal{A} \models \rho^K(\overline{w})\ [\xi]$

As a consequence we have that the mapping $\sigma \to \sigma_q$ is a faithful interpretation of T' into T.

Corollary 4.3 (Faithful interpretation) *For every sentence σ of L' with reflection σ_q into L':*
$T' \vdash \sigma$ iff $T \vdash \sigma_q$.

The canonical expansion of a model $\mathcal{A} \models T$ is a model $\mathcal{A}^q \models T'$ where sort Q is realised by the set of equivalence classes of the elements of $S^{\mathcal{A}}$ under equivalence relation $q^{\mathcal{A}}$.

We now wish to internalise relations in $\mathcal{A}'$ involving the new quotient sort to relations involving the original sort S.

As before, we first define how sorts are to be translated for internalisation. Let Σ be the set of sorts of L, so the set of sorts of L' is $\Sigma \cup \{Q\}$. We define a mapping $n : (\Sigma \cup \{Q\})^* \to \Sigma^*$ as in the case of subsort.

This mapping n induces natural mappings from sequences of elements of $\mathcal{A}$ to sequences of elements of $\mathcal{A}'$ as follows. Given $\overline{\tau} \in (\Sigma \cup \{Q\})^*$, we define the mapping $\overline{\tau}^q_{\mathcal{A}'} : n(\overline{\tau})^{\mathcal{A}} \to \overline{\tau}^{\mathcal{A}'}$ so that $[\overline{\tau}.\tau]^q_{\mathcal{A}'}(\overline{c}.a) = \overline{\tau}^q_{\mathcal{A}'}(\overline{c}).a$ if $\tau \in \Sigma$, and $[\overline{\tau}.\tau]^q_{\mathcal{A}'}(\overline{c}.b) = \overline{\tau}^q_{\mathcal{A}'}(\overline{c}).p^{\mathcal{A}'}(a_k)$ if $\tau = Q$.

Finally, given a relation R over sorts $\overline{\tau}$ in $\mathcal{A}'$, its **internalised version** is the relation R_q over sorts $n(\overline{\tau})$ in $\mathcal{A}$ defined by $R_q = \{\overline{a}/\overline{\tau}^q_{\mathcal{A}'}(\overline{a}) \in R\}$.

As before, we now wish to show that a relation R is definable in a model $\mathcal{A}' \models T'$ iff its internalised version R_q is definable in its reduct $\mathcal{A}$. For this purpose it is convenient to examine how internalisation and reflection are linked. The next lemma is a step in this direction.

Lemma 4.2 (Internalisation and connection content) *Consider a structure $\mathcal{A}'$ of L' with reduct $\mathcal{A}$ to L. Let R be a relation over sorts $\overline{\tau} = \tau_1.\dots.\tau_k$ in $\mathcal{A}'$ with internalised version R_q over sorts $n(\overline{\tau}) = \sigma_1.\dots.\sigma_k$ in $\mathcal{A}$. Let $\overline{z}$ be a set of k variables over $\overline{\tau}$ and $\overline{w}$ be a set of k variables over $n(\overline{\tau})$ such that $\overline{z} \cap \overline{w} = \emptyset$. Consider any connection matching $\overline{z}$ to $\overline{w}$, with content $K(\overline{z}, \overline{w})$.*

- *(a) Given $\overline{b} \in \overline{\tau}^{\mathcal{A}'}$ and $\overline{a} \in n(\overline{\tau})^{\mathcal{A}}$, $\overline{b} = \overline{\tau}^q_{\mathcal{A}'}(\overline{a})$ iff $\mathcal{A}' \models K(\overline{z}, \overline{w})[\overline{z}/\overline{b}, \overline{w}/\overline{a}]$.*
- *(b) Given $\overline{a} \in n(\overline{\tau})^{\mathcal{A}}$, $\overline{a} \in R_q$ iff for some $\overline{b} \in R$, $\mathcal{A}' \models K(\overline{z}, \overline{w})[\overline{z}/\overline{b}, \overline{w}/\overline{a}]$.*
- *(c) Given $\overline{b} \in \overline{\tau}^{\mathcal{A}'}$, $\overline{b} \in R$ iff for some $\overline{a} \in R_q$, $\mathcal{A}' \models K(\overline{z}, \overline{w})[\overline{z}/\overline{b}, \overline{w}/\overline{a}]$.*

A simple consequence of elimination under connection, which will be useful, is given in the next result.

Corollary 4.4 (Internalisation of connection) *Consider a (quotient) connection matching $\overline{z}$ to $\overline{w}$ with content $K(\overline{z}, \overline{w})$.*

For every formula $\rho(\overline{z})$ of L' having as its free variables those in $\overline{z}$, we have that $T' \vdash (\exists \overline{z})[K(\overline{z}, \overline{w}) \wedge \rho(\overline{z})] \leftrightarrow \rho^K(\overline{w})$.

We now wish to link internalisation to reflection. The next result shows that reflection provides the basis for a syntactic version of internalisation. It follows from the preceding results (lemma 4.2 and corollary 4.4).

Proposition 4.3 (Reflection and internalisation) *Consider a model $\mathcal{A}' \models T'$ with reduct $\mathcal{A}$ to L and a connection matching $\overline{z}$ over $\overline{w}$.*

Consider a formula formula $\rho(\overline{z})$ of L' with K-reflection $\rho^K(\overline{w})$. Then, the internalised version of the relation $\rho^{\mathcal{A}'}$ defined by ρ in $\mathcal{A}'$ is the relation defined in $\mathcal{A}$ by the reflection ρ^K, i.e., $(\rho^{\mathcal{A}'})_q = (\rho^K)^{\mathcal{A}}$.

We can now finally show that a relation in a model of the larger theory is definable iff its internalisation is definable in the reduct to the smaller language.

Theorem 4.2 (Definability) *Consider a model $\mathcal{A}' \models T'$ with reduct $\mathcal{A} = \mathcal{A}'/L$. A relation R is definable in $\mathcal{A}'$ iff its internalised version R_q is definable in $\mathcal{A}$.*

We thus have the following results for quotient extensions:

- eliminability under connection;
- maximal conservativeness;
- internalisation of connection;
- generalised interpretation adjunction connecting models, formulae and assignments;
- reflection as a syntactic version of internalisation
- no new definable relations, up to internalisation.

These results generalise the analogous properties presented by extensions by definition of predicate or function symbols to the case of quotient sort.

5 Definability for Discriminated Union Sorts

Given theory T with language L, we wish to add a new sort U to behave as the discriminated union of sorts S_1 and S_2 of L.

For this purpose we consider the axiomatisation Uni for union introduction given in [MV92, MV94]:

sort	U
operations	$i_1 : S_1 \to U$
	$i_2 : S_2 \to U$
axioms	

- $\forall u : U \ \ (\exists x_1 : S_1 \quad u = i_1(x_1)) \vee (\exists x_2 : S_2 \quad u = i_2(x_2))$
- $\forall x_1 : S_1 \, \forall x_2 : S_2 \, \neg \, (i_1(x_1) = i_2(x_2))$
- $\forall x, x' : S_1 \ \ i_1(x) = i_1(x') \Rightarrow x = x'$
- $\forall y, y' : S_2 \ \ i_2(y) = i_2(y') \Rightarrow y = y'$

Call $T' = T \cup D\_Uni$ the extension of T by introduction of the new sort U according to D_Uni. Let L' be the language of T'.

Given a model $\mathcal{A} \models T$, consider its expansion $\mathcal{A}^+$ to a structure of L' with $U^{\mathcal{A}^+} = S_1^{\mathcal{A}} + S_2^{\mathcal{A}}$ (i.e. $U^{\mathcal{A}^+}$ is the discriminated union of $S_1^{\mathcal{A}}$ and $S_2^{\mathcal{A}}$) and its insertions. Call $\mathcal{A}^+$ the **canonical expansion** of $\mathcal{A}$ to L'.

We proved in [MV92, prop.3.3.1] the following adequacy result, characterising the models of T' up to isomorphism.

Theorem 5.1 (Adequacy) *$\mathcal{A}' \models T'$ iff $\mathcal{A}'$ is isomorphic to the canonical expansion $\mathcal{A}^+$ of the reduct $\mathcal{A}$ of $\mathcal{A}'$ to L.*

In [MV94] we proved a generalised version of the concept of eliminability similar to the case of product: a connection indicates how to translate variables ranging over the sort U as either variables over sort S_1 or over sort S_2. For this purpose we defined a **(union) connection** for variables $\overline{y}$ over variables $\overline{v}$. We shall now use a slightly changed version of such connection.

Given a set $\overline{z}$ of variables of L', we can partition it into two sets $\overline{z} = \overline{y} \cup \overline{v}$, where $\overline{y} = \{y^1, \ldots, y^n\}$, say, consists of those variables in $\overline{z}$ ranging over the union sort U, and $\overline{v} = \{v^1, \ldots, v^m\}$, say, consists of those variables in $\overline{z}$ ranging over sorts in L. Now, consider a set $\overline{w}$ of variables of L as follows. For each variable y^k in $\overline{z}$ we have two distinct variables x_1^k of sort S_1 and x_2^k of sort S_2; for each variable v^l in $\overline{v}$ over sort S in L we have a distinct variable u^l over sort S. We thus have a partition $\overline{w} = (\overline{x_1} \cup \overline{x_2}) \cup \overline{u}$, where $\overline{x_1}$ and $\overline{x_2}$ have n variables each and $\overline{u}$ has m variables. We say that this matching of variables forms a **(union) connection** iff $\overline{z} \cap \overline{w} = \emptyset$.

Consider a function $\delta : \{1, \ldots, n\} \to \{1, 2\}$. We wish to code this matching of variables into formulae of language L', as follows. For each $k = 1, \ldots, n$, call $K^\delta(y^k, x^k_{\delta(k)})$ the formula $y^k = i_{\delta(k)}(x^k_{\delta(k)})$; and for each $l = 1, \ldots, m$, call $K(v^l, u^l)$ the formula $v^l = u^l$. By the δ-**content** of the connection we mean the conjunction $K^\delta(\overline{z}, \overline{w})$ of these fomulae $K^\delta(y^k, x^k_{\delta(k)})$, for $k = 1, \ldots, n$, and $K(v^l, u^l)$, for $l = 1, \ldots, m$.

As in the previous cases, such a (union) connection provides enough information for eliminating formulae of the larger language in favour of formulae of the smaller language. The (constructive) proof of the next proposition, by induction on the structure of formulae, provides such a elimination procedure [MV94].

Proposition 5.1 (Eliminability under connection) *Consider a function $\delta : \{1, \ldots, n\} \to \{1, 2\}$ and a (union) connection matching $\overline{z}$ to $\overline{w}$ with δ-content $K^\delta(\overline{z}, \overline{w})$.*

*Given a formula $\rho(\overline{z})$ of L' having as its free variables those of $\overline{z}$, there exist a formula $\varphi(\overline{w})$ of L with free variables in $\overline{w}$, such that $T' \vdash K^\delta(\overline{z}, \overline{w}) \to [\rho(\overline{z}) \leftrightarrow \varphi(\overline{w})]$. Call such a formula φ of L the δ-**reflection of** ρ into L (via K), and denote it by ρ^δ.*

In particular, for a formula without free variables over the union sort U no connection is necessary. We then have the usual eliminability (without connection).

Corollary 5.1 (Eliminability without connection) *For each formula σ of L' without free variables over the union sort U, there exists a formula σ_+ of L' (with the same free variables) such that $T' \vdash \sigma \leftrightarrow \sigma_+$.*

As consequences we have invariance under elementary equivalence ([MV94]), as well as the maximal conservativeness of the product extension.

Corollary 5.2 (Maximal Conservativeness) *$T \subseteq T'$ is a maximal conservative extension.*

We now wish establish that the mapping $\rho \to \rho^\delta$ given by eliminability under connection has the behaviour akin to that of an interpretation, namely that it induces a correspondence between models and assignments preserving satisfaction [End72].

A (union) connection correlates variables of the larger language L' to variables of the smaller language L, mainly by matching variables over the union sort U with variables over the originating sorts S_1 or S_2 according to a function δ. This establishes a correspondence between assignments of values to variables. Consider a structure $\mathcal{A}'$ for L' with reduct $\mathcal{A}$ to L. Consider also a function $\delta : \{1, \ldots, n\} \to \{1, 2\}$ and a (union) connection matching $\overline{z} = \overline{y} \cup \overline{v}$ to $\overline{w} = (\overline{x_1} \cup \overline{x_2}) \cup \overline{u}$, with δ-content $K^\delta(\overline{z}, \overline{w})$. Each assignment ξ of values in $\mathcal{A}$ for the variables of $\overline{w}$ induces an assignment ξ^δ in $\mathcal{A}'$ for the variables $\overline{w}$ as follows:

- for each $k = 1, \ldots, n$, $\xi^\delta(y_1^k) = i^{\mathcal{A}'}_{\delta(k)}(\xi(x^k_{\delta(k)}))$;
- for each $l = 1, \ldots, m$, $\xi^\delta(u^l) = \xi(v^l)$.

The connection content actually codes this induction of assignments, as stated in the next result, characterising induced assignments.

Lemma 5.1 (Connection content and assignment induction) *Consider a model $\mathcal{A}' \models T'$ with reduct $\mathcal{A}$ to L. Consider also a function $\delta : \{1, \ldots, n\} \to \{1, 2\}$ and a (union) connection matching $\overline{z} = \overline{y} \cup \overline{v}$ to $\overline{w} = (\overline{x_1} \cup \overline{x_2}) \cup \overline{u}$, with δ-content $K^\delta(\overline{z}, \overline{w})$. Given an assignment ξ of values in $\mathcal{A}$ for the variables of $\overline{w}$, let ξ^δ be the assignment in $\mathcal{A}'$ induced by ξ, i.e. $\xi^\delta(y^k) = i^{\mathcal{A}'}_{\delta(k)}(\xi(x^k_{\delta(k)}))$ and $\xi^\delta(u^l) = \xi(v^l)$.*

- *(a) For an assignment η for the variables of $\overline{z}$ in $\mathcal{A}'$, $\eta = \xi^\delta$ iff $\mathcal{A}' \models K^\delta(\overline{z}, \overline{w})[\eta \cup \xi]$.*
- *(b) The mapping $\xi \to \xi^\delta$ is an injection from assignments for $\overline{w}$ in $\mathcal{A}$ to assignments for $\overline{z}$ in $\mathcal{A}'$.*

The next result now follows from eliminability under connection because union structures satisfy the connection content with the assignments constructed.

Proposition 5.2 (Generalised interpretation) *Let $\mathcal{A}'$ be a structure for L' with reduct $\mathcal{A}$ to L. Consider a function $\delta : \{1, \ldots, n\} \to \{1, 2\}$ and a connection matching $\overline{z}$ to $\overline{w}$.*

For every formula $\rho(\overline{z})$ of L' with free variables in $\overline{z}$ and with δ-ref l ection $\rho^\delta(w)$ and for every assignment ξ for variables of $\overline{x}$ in $\mathcal{A}$ with induced assignment ξ^δ:
$\mathcal{A}' \models \rho(\overline{z})$ $[\xi^\delta]$ iff $\mathcal{A} \models \rho^\delta(w)$ $[\xi]$

As a consequence,the mapping $\sigma \to \sigma_+$ is a faithful interpretation of T' into T.

Corollary 5.3 (Faithful interpretation) *For every sentence σ of L' with reflection σ_+ into L':*
$T' \vdash \sigma$ iff $T \vdash \sigma_+$.

The canonical expansion of a model $\mathcal{A} \models T$ is a model $\mathcal{A}^+ \models T'$ where sort U is realised by the (disjoint) union of the elements of the sets $S_1^{\mathcal{A}}$ and $S_2^{\mathcal{A}}$. We now wish to internalise relations in $\mathcal{A}^+$ involving the new union sort to relations involving the original sorts S_1, S_2.

For any sets M, A_1, A_2, N we have $M \times (A_1 \cup A_2) \times N = (M \times A_1 \times N) \cup (M \times A_2 \times N)$. Thus a ternary relation $R \subseteq M \times (A_1 \cup A_2) \times N$ is the union of its views $R_1 \subseteq M \times A_1 \times N$ and $R_2 \subseteq M \times A_2 \times N$. This is the basic idea behind internalisation of a relation involving a (discriminated) union sort.

As before, we first define how sorts are to be translated for internalisation: we define a mapping internalising sequences of sorts to their views.
Let Σ be the set of sorts of L; so the set of sorts of L' is $\Sigma \cup \{U\}$. Given $\overline{\tau} = \tau_1 \ldots \tau_l \in (\Sigma \cup \{U\})^+$, let $J = \{j_1, \ldots, j_n\}$ be the set of indices h such that $\tau_h = U$. Given a function $\delta : \{1, \ldots, n\} \to \{1, 2\}$, we define $\overline{\tau}_\delta = \sigma_1 \ldots \sigma_n \in \Sigma^+$ where $\sigma_h = \tau_h$ if $h \notin J$, and $\sigma_h = S_{\delta(h)}$ if $h \in J$. Notice that $\overline{\tau}$ and $\overline{\tau}_\delta$ have the same length.

This mapping induces natural mappings from sequences of elements of $\mathcal{A}$ to sequences of elements of $\mathcal{A}'$ as follows. Given $\overline{\tau} = \tau_1 \ldots \tau_l \in (\Sigma \cup \{N\})^*$, we define the mapping $\overline{\tau}^{\delta}_{\mathcal{A}'} : (\overline{\tau}_\delta)^{\mathcal{A}} \to \overline{\tau}^{\mathcal{A}'}$ by $\overline{\tau}^{\delta}_{\mathcal{A}'}(a_1 \ldots a_l) = b_1 \ldots b_l$, where $b_1 = a_1$ if $h \notin J$, and $b_1 = i^{\mathcal{A}'}_{\delta(h)}(a_h)$ if $h \in J$.

Finally, given a relation R over sorts $\overline{\tau}$ in $\mathcal{A}'$ its δ-**internalisation** is the relation R_δ over sorts $\overline{\tau}_\delta$ in $\mathcal{A}$ defined by $R_\delta = \{\overline{a} / \overline{\tau}^{\delta}_{\mathcal{A}'}(\overline{a}) \in R\}$.

We wish to show that a relation R is definable in a model $\mathcal{A}' \models T'$ iff every δ-internalisation R_δ is definable in its reduct $\mathcal{A}$. For this purpose it is convenient to examine how internalisation and ref l ection are linked. The next lemma is a step in this direction.

Lemma 5.2 (Internalisation and connection content) *Consider a structure $\mathcal{A}'$ of L' with reduct $\mathcal{A}$ to L. Let R be a relation over sorts $\overline{\tau} = \tau_1 \ldots . \tau_k$ in $\mathcal{A}'$ with δ-internalisation R_δ over sorts $\overline{\tau}_\delta = \sigma_1 \ldots . \sigma_k$ in $\mathcal{A}$. Let $\overline{z}$ be a set of k variables over $\overline{\tau}$ and $\overline{w}$ be a set of k variables over $\overline{\tau}^\delta$ such that $\overline{z} \cap \overline{w} = \emptyset$. Consider any connection matching $\overline{z}$ to $\overline{w}$, with content $K(\overline{z}, \overline{w})$.*

- *(a) Given $\overline{b} \in \overline{\tau}_{\mathcal{A}'}$ and $\overline{a} \in \overline{\tau}^{\mathcal{A}}_\delta$, $\overline{b} = \overline{\tau}^{\delta}_{\mathcal{A}'}(\overline{a})$ iff $\mathcal{A}' \models K^\delta(\overline{z}, \overline{w})[\overline{z}/\overline{b}, \overline{w}/\overline{a}]$.*
- *(b) Given $\overline{a} \in \overline{\tau}^{\mathcal{A}}_\delta$, $\overline{a} \in R_\delta$ iff for some $\overline{b} \in R$, $\mathcal{A}' \models K^\delta(\overline{z}, \overline{w})[\overline{z}/\overline{b}, \overline{w}/\overline{a}]$.*
- *(c) Given $\overline{b} \in \overline{\tau}^{\mathcal{A}'}$, $\overline{b} \in \overline{\tau}^{\delta}_{\mathcal{A}'}(R_\delta)$ iff for some $\overline{a} \in R_\delta$, $\mathcal{A}' \models K^\delta(\overline{z}, \overline{w})[\overline{z}/\overline{b}, \overline{w}/\overline{a}]$.*
- *(d) $R = \bigcup_{\delta \in \Delta} \overline{\tau}^{\delta}_{\mathcal{A}'}(R_\delta)$, where Δ be the set of all 2^n functions $\delta : \{1, \ldots, n\} \to \{1, 2\}$.*

A simple consequence of elimination under connection, which will be useful, is given in the next result.

Corollary 5.4 (Internalisation of connection) *Consider a function $\delta : \{1, \ldots, n\} \to \{1, 2\}$ and a (union) connection matching $\overline{z}$ to $\overline{w}$ with δ-content $K^\delta(\overline{z}, \overline{w})$.*

For every formula $\rho(\overline{z})$ of L' having as its free variables those in $\overline{z}$, we have that $T' \vdash (\exists \overline{z})[K^\delta(\overline{z}, \overline{w}) \wedge \rho(\overline{z})] \leftrightarrow \rho^\delta(\overline{w})$.

We are now ready for linking internalisation to reflection. The next result shows that reflection is a syntactic version of internalisation. It follows from the preceding results (lemma 5.2 and corollary 5.4).

Lemma 5.3 (Reflection and internalisation) *Consider a model $\mathcal{A}' \models T'$ with reduct $\mathcal{A}'$ to L as well as a function $\delta : \{1, \ldots, n\} \to \{1, 2\}$ and a (union) connection matching $\overline{z}$ to $\overline{w}$.*

Given a formula $\rho(\overline{z})$ with δ-reflection $\rho^\delta(\overline{w})$, the relation $(\rho^\delta)^{\mathcal{A}}$ defined by ρ^δ on $\mathcal{A}$ is the δ-internalisation of the relation $\rho^{\mathcal{A}'}$ defined by ρ on $\mathcal{A}'$, i.e., $(\rho^{\mathcal{A}'})_\delta = (\rho^\delta)^{\mathcal{A}}$.

Finally we can show that a relation in a model of the larger theory is definable iff its internalisation is definable in the reduct to the smaller language.

Theorem 5.2 (Definability) *Consider a model $\mathcal{A}' \models T'$ with reduct $\mathcal{A} = \mathcal{A}'/L$. A relation R is definable in $\mathcal{A}'$ iff every δ-internalisation R_δ is definable in $\mathcal{A}$.*

We thus have the following results for union extensions:

- eliminability under connection;
- maximal conservativeness;
- internalisation of connection;
- generalised interpretation adjunction connecting models, formulae and assignments;
- reflection as a syntactic version of internalisation;
- no new definable relations, up to internalisation.

These results generalise the analogous properties presented by extensions by definition of predicate or function symbols to the case of union sort.

6 Conclusions

We have examined four sort introducing constructs: cartesian product, subsort, discriminated union and quotient.

Introduction of new sorts is of interest because it occurs often in implementing formal specifications. In constrast to the introduction of new function or predicate symbols [Sho67], the case of new sorts is not extensively studied.

The four constructs examined correspond to natural constructions of new sets. Three of them have found their way as data structuring constructs in programming languages [Hoa74]. The quotient construct is of importance in developing and verifying correct implementations [TM87, Vel87].

In [MV92] we proved that, in each case, the new sort is unique, but only up to isomorphism. This contrasts with the case of introduction of new function or predicate symbols by definitions, when we have unique expandability of models. Also, these more familiar extensions by definitions exhibit eliminability: any formula of the extended language is equivalent to some formula in the original language. In our case of introduction of new sorts one should not expect full eliminability, because of the new variables ranging over the sort introduced. Nevertheless, a kind of generalised eliminability is still possible: eliminability under connection [MV94].

Actually, along this paper we have established, for each sort introduction construct, the following properties:

1. expression of the content of the connection into the theory.
2. generalised interpretation to formulae with new free variables;
3. relationship between reflection and internalisation;
4. completeness of definability.

Other properties, like the independence of the order of introduction and the completeness of the set of the constructs, are object of recently started work and they will be presented in forthcoming papers.

References

[End72] H.B. Enderton. *A Mathematical Introduction to Logic.* Academic Press, Inc., New York, 1972.

[Eri82] H.D. Erich. On the theory of specification, implementation and parametrization of abstract data types. *Journal of the Association for Computing Machinery*, 29(1):206–227, January 1982.

[Hoa74] C.A.R. Hoare. Notes on data structuring. In O.J. Dahl, E.W. Dijkstra, and C.A.R. Hoare, editors, *Structured Programming*, pages 83–174. Academic Press, 1974.

[MV92] M. C. Meré and P. A. Veloso. On Extensions by Sorts. *Monografias em Ciências da Computação, D.I., PUC-Rio*, 38/92, December 1992.

[MV94] M.C. Meré and P.A. Veloso. Definition-like extensions by sorts. *Monografias em Ciências da Computação, D.I., PUC-Rio*, xx/94, October 1994. Also abstract in Annals of Workshop on Logic, Language, Information and Computation '94, Julho de 1994, Recife, Brasil.

[VM94] P.A. Veloso and M.C. Meré. Construtores Parametrizados de Sortes. *Monografias em Ciências da Computação, D.I., PUC-Rio*, 14/94, Julho 1992.

[Sho67] J.R. Shoenfield. *Mathematical Logic.* Addison-Wesley Publishing Company, 1967.

[TM87] W.M. Turski and T.S.E. Maibaum. *The Specification of Computer Programs.* Addison-Wesley Publishing Company, 1987.

[Vel87] P.A.S Veloso. *Estruturação e Verificação de Programas com Tipos de Dados.* Editora E. Blücher ltda., 1987.

A APPENDIX

Definability Results for Product Sort

In this appendix we prove the definability results for extensions by introduction of product sorts.

Recall the concept of (product) connection.
Given a set $\overline{z}$ of variables of L', we can partition it into two sets $\overline{z} = \overline{y} \cup \overline{v}$, where $\overline{y} = \{y^1, \ldots, y^n\}$, say, consists of those variables in $\overline{z}$ ranging over the product sort P, and $\overline{v} = \{v^1, \ldots, v^m\}$, say, consists of those variables in $\overline{z}$ ranging over sorts in L. Now, consider a set $\overline{w}$ of variables of L as follows. For each variable y^k in $\overline{z}$ we have two distinct variables x_1^k of sort S_1 and x_2^k of sort S_2; for each variable v^l in $\overline{v}$ over sort S in L we have a distinct variable u^l over sort S. We thus have a partition $\overline{w} = (\overline{x_1} \cup \overline{x_2}) \cup \overline{u}$, where $\overline{x_1}$ and $\overline{x_2}$ have n variables each and $\overline{u}$ has m variables. We say that this matching of variables forms a **(product) connection** iff $\overline{z} \cap \overline{w} = \emptyset$.
We code this matching of variables into formulae of language L', as follows. For each $k = 1, \ldots, n$, call $K(y^k, x_1^k, x_2^k)$ the formula $p_1(y^k) = x_1^k \wedge p_2(y^k) = x_2^k$; and for each $l = 1, \ldots, m$, call $K(v^l, u^l)$ the formula $v^l = u^l$. By the **content** of the connection we mean the conjunction $K(\overline{z}, \overline{w})$ of these fomulae $K(y^k, x_1^k, x_2^k)$, for $k = 1, \ldots, n$, and $K(v^l, u^l)$, for $l = 1, \ldots, m$.

Corollary A.1 (Eliminability without connection) *For each formula σ of L' without free variables over the product sort P, there exists a formula $\sigma_\times$ of L' (with the same free variables) such that $T' \vdash \sigma \leftrightarrow \sigma_\times$.*

Proof.
Let $\overline{v} = \{v^1, \ldots, v^m\}$ be the set of variables with free occurrences in σ. Given $\overline{u} = \{u^1, \ldots, u^m\}$ with $\overline{v} \cap \overline{u} = \emptyset$, the connection content $K(\overline{z}, \overline{w})$ reduces to $\wedge_{j=1}^n v^l = u^l$. From eliminability under connection, we have a formula $\tau(\overline{u})$, with free variables in $\overline{u}$, such that $T' \vdash \wedge_{j=1}^n v^l = u^l \rightarrow [\sigma(\overline{v}) \leftrightarrow \tau(\overline{u})]$. If we take $\sigma_\times$ as the result of replacing in $\tau(\overline{u})$ each u^l by the corresponding v^l, we have the desired result: $T' \vdash \sigma(\overline{v}) \leftrightarrow \sigma_\times(\overline{v})$.
QED

Corollary A.2 (Maximal Conservativeness) *$T \subseteq T'$ is a maximal conservative extension.*

Proof.
Consider an extension T" of T' with language L' that extends T conservatively. Let σ be a sentence of L' such that T" $\vdash \sigma$. By eliminability without connection, we know that $T' \vdash \sigma \leftrightarrow \sigma_\times$. So, since $T' \subseteq T$", T" $\vdash \sigma_\times$. Thus, by conservativeness, $T \vdash \sigma_\times$. So, since $T \subseteq T'$, $T' \vdash \sigma_\times$; whence $T' \vdash \sigma$.
QED

Lemma A.1 (Connection content and assignment induction) *Consider a structure $\mathcal{A}'$ of L' with reduct $\mathcal{A}$ to L, and a connection matching $\overline{z} = \overline{y} \cup \overline{v}$ to $\overline{w} = (\overline{x_1} \cup \overline{x_2}) \cup \overline{u}$, with content $K(\overline{z}, \overline{w})$.*

Given an assignment η for the variables of $\overline{z}$ in $\mathcal{A}'$, let η^K be the assignment for the variables of $\overline{w}$ in $\mathcal{A}$ induced by η, i.e. $\eta^K(x_1^k) = p_1^{\mathcal{A}'}(\eta(y^k))$, $\eta^K(x_2^k) = p_2^{\mathcal{A}'}(\xi(y^k))$ and $\eta^K(u^l) = \eta(v^l)$.

- *(a) Given an assignment ξ for the variables of $\overline{w}$ in $\mathcal{A}$, $\xi = \eta^K$ iff $\mathcal{A}' \models K(\overline{z}, \overline{w})[\eta \cup \xi]$.*
- *(b) The mapping $\eta \to \eta^K$ is a bijection between assignments for $\overline{z}$ in $\mathcal{A}'$ and assignments for $\overline{w}$ in $\mathcal{A}$.*

Proof.

- (a) This follows from the definitions of connection content and of assignment induction. Indeed,
  - for a variable y^k in $\overline{y}$, $\xi(x_1^k) = p_1^{\mathcal{A}'}(\eta(y^k))$ and $\xi(x_2^k) = p_2^{\mathcal{A}'}(\xi(y^k))$ iff $\mathcal{A}' \models p_1(y^k) = x_1^k \wedge p_2(y^k) = x_2^k[\eta \cup \xi]$;
  - for a variable v^l in $\overline{v}$, $\xi(u^l) = \eta(v^l)$ iff $\mathcal{A}' \models v^l = u^l[\eta \cup \xi]$.
- (b) This follows from the fact that the mapping $b \to < p_1^{\mathcal{A}'}(b), p_2^{\mathcal{A}'}(b) >$ is a bijection between $P^{\mathcal{A}'^\times}$ and $S_1^{\mathcal{A}} \times S_2^{\mathcal{A}}$.

QED

Proposition A.1 (Generalised interpretation) *Consider a structure $\mathcal{A}'$ of L' with reduct $\mathcal{A}$ to L, and a connection matching $\overline{z}$ to $\overline{w}$.*

For every formula $\rho(\overline{z})$ of L' with free variables in $\overline{z}$ and with K-reflection $\rho^K(\overline{w})$ and for every assignment η for variables of $\overline{z}$ in $\mathcal{A}'$ with induced assignment η^K:
$\mathcal{A}' \models \rho(\overline{z})\ [\eta]$ iff $\mathcal{A} \models \rho^K(\overline{w})\ [\eta^K]$

Proof.
This follows from connection content and assignment induction by means of eliminability under connection. Indeed, from connection content and assignment induction, we have that $\mathcal{A}' \models K(\overline{z}, \overline{w})[\eta \cup \eta^K]$;
so, since $\mathcal{A}' \models T'$, from eliminability under connection, we have
$\mathcal{A}' \models \rho(\overline{z})\ [\eta]$ iff $\mathcal{A} \models \rho^K(\overline{w})\ [\eta^K]$.
QED

Corollary A.3 (Faithful interpretation) *For every sentence σ of L' with reflection $\sigma_\times$ into L': $T' \vdash \sigma$ iff $T \vdash \sigma_\times$.*

Proof.
From generalised interpretation, we have, for every structure $\mathcal{A}'$ of L' with reduct $\mathcal{A}$ to L, $\mathcal{A}' \models \sigma$ iff $\mathcal{A} \models \sigma_\times$.

Now, if $T' \vdash \sigma$, given any $\mathcal{A} \models T$, we have the canonical expansion $\mathcal{A}^{\times} \models T'$; thus $\mathcal{A}^{\times} \models \sigma$, whence $\mathcal{A} \models \sigma_{\times}$.
Also, if $T \vdash \sigma_{\times}$, given any $\mathcal{A}' \models T'$ with reduct $\mathcal{A} \models T$, we have $\mathcal{A} \models \sigma_{\times}$, whence $\mathcal{A}' \models \sigma$.
QED

Let us recall the concept of internalised version of a relation.
Let Σ be the set of sorts of L. Then the set of sorts of L' is $\Sigma \cup \{P\}$. We define a mapping $m : (\Sigma \cup \{P\})^* \to \Sigma^*$ as follows $m(\lambda) = \lambda$, and, given $\overline{\tau} \in (\Sigma \cup \{P\})^*$, for $S \in \Sigma$ $m(\overline{\tau}.S) = m(\overline{\tau}).S$, and $m(\overline{\tau}.P) = m(\overline{\tau}).S_1.S_2$. Notice that if $\overline{\tau}$ is a sequence of lenth k with p occurrences of sort P, then $m(\overline{\tau})$ has length $k + p$.
This mapping m induces natural mappings from sequences of elements of $\mathcal{A}'$ to sequences of elements of $\mathcal{A}$ as follows. Given $\overline{\tau} \in (\Sigma \cup \{P\})^*$, we define the mapping $\overline{\tau}^{\times}_{\mathcal{A}'} : \overline{\tau}^{\mathcal{A}'} \to m(\overline{\tau})^{\mathcal{A}}$ so that $[\overline{\tau}.\tau]^{\times}_{\mathcal{A}'}(\overline{c}.a) = \overline{\tau}^{\times}_{\mathcal{A}'}(\overline{c}).a$ if $\tau \in \Sigma$, and $[\overline{\tau}.\tau]^{\times}_{\mathcal{A}'}(\overline{c}.b) = \overline{\tau}^{\times}_{\mathcal{A}'}(\overline{c}).p_1^{\mathcal{A}'}(b).p_2^{\mathcal{A}'}(b)$ if $\tau = P$.
Finally, given a relation R over sorts $\overline{\tau}$ in $\mathcal{A}'$, its **internalised version** is the relation $R_{\times}$ over sorts $m(\overline{\tau})$ in $\mathcal{A}$ defined by $R_{\times} = \{\overline{\tau}^{\times}_{\mathcal{A}'}(\overline{b}) / \overline{b} \in R\}$.

Lemma A.2 (Internalisation and connection content) *Consider a structure $\mathcal{A}'$ of L' with reduct $\mathcal{A}$ to L. Let R be a relation over sorts $\overline{\tau} = \tau_1.\ldots.\tau_k$ in $\mathcal{A}'$ with internalised version $R_{\times}$ over sorts $m(\overline{\tau}) = \sigma_1.\ldots.\sigma_l$ in $\mathcal{A}$. Let $\overline{z}$ be a set of k variables over $\overline{\tau}$ and $\overline{w}$ be a set of l variables over $m(\overline{\tau})$ such that $\overline{z} \cap \overline{w} = \emptyset$. Consider any connection matching $\overline{z}$ to $\overline{w}$, with content $K(\overline{z}, \overline{w})$.*

- *(a) Given $\overline{b} \in \overline{\tau}^{\mathcal{A}'}$ and $\overline{a} \in m(\overline{\tau})^{\mathcal{A}}$, $\overline{a} = \overline{\tau}^{\times}_{\mathcal{A}'}(\overline{b})$ iff $\mathcal{A}' \models K(\overline{z}, \overline{w})[\overline{z}/\overline{b}, \overline{w}/\overline{a}]$.*
- *(b) Given $\overline{a} \in m(\overline{\tau})^{\mathcal{A}}$, $\overline{a} \in R_{\times}$ iff for some $\overline{b} \in R$, $\mathcal{A}' \models K(\overline{z}, \overline{w})[\overline{z}/\overline{b}, \overline{w}/\overline{a}]$.*
- *(c) Given $\overline{b} \in \overline{\tau}^{\mathcal{A}'}$, $\overline{b} \in R$ iff for some $\overline{a} \in R_{\times}$, $\mathcal{A}' \models K(\overline{z}, \overline{w})[\overline{z}/\overline{b}, \overline{w}/\overline{a}]$.*

Proof.

- (a) In view of the definitions, we may consider each element of the sequence at a time.
  If $\tau \in \Sigma$ then $\overline{\tau}^{\times}_{\mathcal{A}'}(b) = b$; so $a = \overline{\tau}^{\times}_{\mathcal{A}'}(b)$ iff $\mathcal{A}' \models v = u[v/b, u/a]$.
  If $\tau = P$ then $\overline{\tau}^{\times}_{\mathcal{A}'}(b) = p_1^{\mathcal{A}'}(b).p_2^{\mathcal{A}'}(b)$ and $a =< a_1, a_1 >$; so $a = \overline{\tau}^{\times}_{\mathcal{A}'}(b)$ iff $\mathcal{A}' \models p_1(y) = x_1 \wedge p_2(y) = x_2[y/b, x_1/a_1, x_2/a_2]$.
- (b) and (c) now follow from the definition of internalised version in view of item (a)

QED

Corollary A.4 (Internalisation of connection) *Consider a (product) connection matching $\overline{z}$ to $\overline{w}$ with content $K(\overline{z}, \overline{w})$.*

For every formula $\rho(\overline{z})$ of L' having as its free variables those in $\overline{z}$, we have that $T' \vdash (\exists \overline{z})[K(\overline{z}, \overline{w}) \wedge \rho(\overline{z})] \leftrightarrow \rho^K(\overline{w})$.

Proof.
First, notice that $T' \vdash (\exists\overline{z})K(\overline{z},\overline{w})$.
Because, for $k = 1,\dots,n$, $K(y^k, x_1^k, x_2^k)$ is $p_1(y^k) = x_1^k \wedge p_2(y^k) = x_2^k$, and
$T' \vdash (\exists y^k : P)[p_1(y^k) = x_1^k \wedge p_2(y^k) = x_2^k]$;
and, for each $l = 1,\dots,m$, $K(v^l, u^l)$ is $v^l = u^l$, and $\vdash (\exists v^l : S)v^l = u^l$.
Now, from eliminability under connection we have
$T' \vdash K(\overline{z},\overline{w}) \rightarrow [\rho(\overline{z}) \leftrightarrow \rho^K(\overline{w})]$, thus
$T' \vdash [K(\overline{z},\overline{w}) \wedge \rho(\overline{z})] \leftrightarrow [\rho^K(\overline{w}) \wedge K(\overline{z},\overline{w})]$, whence
$T' \vdash (\exists\overline{z})[K(\overline{z},\overline{w}) \wedge \rho(\overline{z})] \leftrightarrow (\exists\overline{z})[\rho^K(\overline{w}) \wedge K(\overline{z},\overline{w})]$.
Hence, since no variable in $\overline{z}$ occurs free in ρ^K (as $\overline{z} \cap \overline{w} = \emptyset$), we obtain
$T' \vdash (\exists\overline{z})[K(\overline{z},\overline{w}) \wedge \rho(\overline{z})] \leftrightarrow [\rho^K(\overline{w}) \wedge (\exists\overline{z})K(\overline{z},\overline{w})]$,
whence, by the above remark, we can conclude:
$T' \vdash (\exists\overline{z})[K(\overline{z},\overline{w}) \wedge \rho(\overline{z})] \leftrightarrow \rho^K(\overline{w})$.
QED

Proposition A.2 (Reflection and internalisation) *Consider a model $\mathcal{A}' \models T'$ with reduct $\mathcal{A}$ to L and a connection matching $\overline{z}$ over $\overline{w}$.*

Given a formula $\rho(\overline{z})$ with K-reflection $\rho^K(\overline{w})$, the relation $(\rho^K)^{\mathcal{A}}$ defined by ρ^K on $\mathcal{A}$ is the internalised version of the relation $\rho^{\mathcal{A}'}$ defined by ρ on $\mathcal{A}'$, i.e., $(\rho^{\mathcal{A}'})_\times = (\rho^K)^{\mathcal{A}}$.

Proof.
If $\overline{z}$ is over sorts $\overline{\tau}$, then $\overline{w}$ is over sorts $m(\overline{\tau})$.
Given a sequence $\overline{a} \in m(\overline{\tau})^{\mathcal{A}}$, we have: $\overline{a} \in (\rho^{\mathcal{A}'})_\times$
iff {by connection content and internalisation, item (b)}
for some $\overline{b} \in \overline{\tau}^{\mathcal{A}'}$, $\mathcal{A}' \models \rho(\overline{z})[\overline{z}/\overline{b}]$ and $\mathcal{A}' \models K(\overline{z},\overline{w})[\overline{z}/\overline{b},\overline{w}/\overline{a}]$
iff $\mathcal{A}' \models (\exists\overline{z})[K(\overline{z},\overline{w}) \wedge \rho(\overline{z})][\overline{w}/\overline{a}]$
iff {by internalisation of connection}
$\mathcal{A}' \models \rho^K(\overline{w})[\overline{w}/\overline{a}]$
iff $\overline{a} \in (\rho^K)^{\mathcal{A}}$.
QED

Theorem A.1 (Definability) *Consider a model $\mathcal{A}' \models T'$ with reduct $\mathcal{A} = \mathcal{A}'/L$. A relation R is definable in $\mathcal{A}'$ iff its internalised version $R_\times$ is definable in $\mathcal{A}$.*

Proof.
Consider R be over sorts $\overline{\tau}$ in $\mathcal{A}'$ and its internalised version $R_\times$ over sorts $m(\overline{\tau})$ in $\mathcal{A}$.

- If R is defined in $\mathcal{A}'$ by ρ, then by the preceding proposition, ρ^K defines $R_\times$ in $\mathcal{A}$.
- If $R_\times$ is defined in $\mathcal{A}$ by φ, let $\overline{w}$ be the set of variables, over sorts $m(\overline{\tau})$, with free occurrences in φ. Then, we can select a set $\overline{z}$ of new variables, over sorts

$\overline{\tau}$, matching $\overline{w}$. Thus R will be defined in $\mathcal{A}'$ by
$(\exists\overline{w})[K(\overline{z},\overline{w}) \wedge \varphi(\overline{w})]$.
To see this it suffices to use the lemma on connection content and internalisation, item (c).
Indeed, given a sequence $\overline{b} \in \overline{\tau}^{\mathcal{A}'}$, we have: $\overline{b} \in R$
iff {by connection content and internalisation, item (c)}
for some $\overline{a} \in m(\overline{\tau})^{\mathcal{A}}$, $\overline{a} \in R_{\times}$ and and $\mathcal{A}' \models K(\overline{z},\overline{w})[\overline{z}/\overline{b},\overline{w}/\overline{a}]$
iff for some $\overline{a} \in m(\overline{\tau})^{\mathcal{A}}$, $\mathcal{A} \models \varphi(\overline{w})[\overline{w}/\overline{a}]$
iff $\mathcal{A}' \models (\exists\overline{w})[K(\overline{z},\overline{w}) \wedge \varphi(\overline{w})][\overline{z}/\overline{b}]$
iff $\overline{b} \in ((\exists\overline{w})[K(\overline{z},\overline{w}) \wedge \varphi(\overline{w})])^{\mathcal{A}'}$.

QED

B APPENDIX

Definability Results for Subsort

In this appendix we prove the definability results for extensions by introduction of subsorts.

Recall the concept of (subsort) connection.
Given a set $\overline{z}$ of variables of L', we can partition it into two sets $\overline{z} = \overline{y} \cup \overline{v}$, where $\overline{y} = \{y^1, \ldots, y^n\}$, say, consists of those variables in $\overline{z}$ ranging over the new subsort N, and $\overline{v} = \{v^1, \ldots, v^m\}$, say, consists of those variables in ranging over sorts in L. Now, consider a set $\overline{w}$ of variables of L as follows. For each variable y^k in $\overline{z}$ we have a distinct variable x^k of sort S; for each variable v^l in $\overline{v}$ over sort E in L we have a distinct variable u^l over sort E. We thus have a partition $\overline{w} = \overline{x} \cup \overline{u}$, where $\overline{x}$ has n variables and $\overline{u}$ has m variables. We say that this matching of variables forms a **(subsort) connection** iff $\overline{z} \cap \overline{w} = \emptyset$.
We wish to code this matching of variables into formulae of language L', as follows. For each $k = 1, \ldots, n$, call $K(y^k, x^k)$ the formula $j(y^k) = x^k$; and for each $l = 1, \ldots, m$, call $K(v^l, u^l)$ the formula $v^l = u^l$. By the **content** of the connection we mean the conjunction $K(\overline{z}, \overline{w})$ of these fomulae $K(y^k, x^k)$, for $k = 1, \ldots, n$, and $K(v^l, u^l)$, for $l = 1, \ldots, m$.

Corollary B.1 (Eliminability without connection) *For each formula σ of L' without free variables over the new subsort N, there exists a formula σ_r of L' (with the same free variables) such that $T' \vdash \sigma \leftrightarrow \sigma_r$.*

Proof.{As in the case of product}
Let $\overline{v} = \{v^1, \ldots, v^m\}$ be the set of variables with free occurrences in σ. Given $\overline{u} = \{u^1, \ldots, u^m\}$ with $\overline{v} \cap \overline{u} = \emptyset$, the connection content $K(\overline{z}, \overline{w})$ reduces to $\wedge_{j=1}^{n} v^l = u^l$. From eliminability under connection, we have a formula $\tau(\overline{u})$, with free variables in $\overline{u}$, such that $T' \vdash \wedge_{j=1}^{n} v^l = u^l \to [\sigma(\overline{v}) \leftrightarrow \tau(\overline{u})]$. If we take σ_r as the result of replacing in $\tau(\overline{u})$ each u^l by the corresponding v^l, we have the desired result: $T' \vdash \sigma(\overline{v}) \leftrightarrow \sigma_r(\overline{v})$.
QED

Corollary B.2 (Maximal Conservativeness) *$T \subseteq T'$ is a maximal conservative extension.*

Proof.{As in the case of product}
Consider an extension T" of T' with language L' that extends T conservatively. Let σ be a sentence of L' such that $T" \vdash \sigma$. By eliminability without connection, we know that $T' \vdash \sigma \leftrightarrow \sigma_r$. So, since $T' \subseteq T"$, $T" \vdash \sigma_r$. Thus, by conservativeness, $T \vdash \sigma_r$. So, since $T \subseteq T'$, $T' \vdash \sigma_r$; whence $T' \vdash \sigma$.
QED

Lemma B.1 (Image of connection) *Consider a (subsort) connection matching $\overline{z} = \overline{y} \cup \overline{v}$ to $\overline{w} = \overline{x} \cup \overline{u}$ with content $K(\overline{z}, \overline{w})$. Let $\overline{r(x)}$ be the conjunction $r(x^1) \wedge \ldots \wedge r(x^n)$.*

Then, $T' \vdash (\exists \overline{z}) K(\overline{z}, \overline{w}) \leftrightarrow \overline{r(x)}$.

Proof.
First, notice that for $k = 1, \ldots, n$, $K(y^k, x^k)$ is $j(y^k) = x^k$, and
$T' \vdash (\exists y^k : N) j(y^k) = x^k \leftrightarrow r(x^k)$;
and, for each $l = 1, \ldots, m$, $K(v^l, u^l)$ is $v^l = u^l$, and $\vdash (\exists v^l : S) v^l = u^l$.
Hence, $T' \vdash (\exists \overline{z}) K(\overline{z}, \overline{w}) \leftrightarrow \overline{r(x)}$.
QED

Lemma B.2 (Connection content and assignment induction) *Consider a model $\mathcal{A}' \models T'$ with reduct $\mathcal{A}$ to L, and a connection matching $\overline{z} = \overline{y} \cup \overline{v}$ to $\overline{w} = \overline{x} \cup \overline{u}$ with content $K(\overline{z}, \overline{w})$.*

Given an assignment η for the variables of $\overline{z}$ in $\mathcal{A}'$, let η^K be the assignment for the variables of $\overline{w}$ in $\mathcal{A}$ induced by η, i.e. $\eta^K(x^k) = j^{\mathcal{A}'}(\eta(y^k))$ and $\eta^K(u^l) = \eta(v^l)$.

- *(a) Given an assignment ξ for the variables of $\overline{w}$ in $\mathcal{A}$, $\xi = \eta^K$ iff $\mathcal{A}' \models K(\overline{z}, \overline{w})[\eta \cup \xi]$.*
- *(b) Let $\overline{r(x)}$ be the conjunction $r(x^1) \wedge \ldots \wedge r(x^n)$. Then, $\xi = \eta^K$ for some assignment η for the variables of $\overline{z}$ in $\mathcal{A}'$ iff $\mathcal{A} \models \overline{r(x)}[\xi]$.*

Proof.

- (a) This follows from the definitions of connection content and of assignment induction. Indeed,
  - for a variable y^k in $\overline{y}$, $\xi(x^k) = j^{\mathcal{A}'}(\eta(y^k))$ iff $\mathcal{A}' \models j(y^k) = x^k[\eta \cup \xi]$;
  - for a variable v^l in $\overline{v}$, $\xi(u^l) = \eta(v^l)$ iff $\mathcal{A}' \models v^l = u^l[\eta \cup \xi]$.
- (b) From item (a), we have:
  $\xi = \eta^K$ for some assignment η for the variables of $\overline{z}$ in $\mathcal{A}'$ iff
  for some assignment η for the variables of $\overline{z}$ in $\mathcal{A}'$, $\mathcal{A}' \models K(\overline{z}, \overline{w})[\eta \cup \xi]$ iff
  $\mathcal{A}'$, $\mathcal{A}' \models (\exists \overline{z}) K(\overline{z}, \overline{w})$ iff
  $\mathcal{A} \models \overline{r(x)}[\xi]$ (by the lemma on image of connection).

QED

Proposition B.1 (Generalised interpretation) *Consider a model $\mathcal{A}' \models T'$ with reduct $\mathcal{A}$ to L, and a connection matching $\overline{z}$ to $\overline{w}$.*

For every formula $\rho(\overline{z})$ of L' with free variables in $\overline{z}$ and with K-reflection $\rho^K(\overline{w})$ and for every assignment η for variables of $\overline{z}$ in $\mathcal{A}'$ with induced assignment η^K:
$\mathcal{A}' \models \rho(\overline{z})\ [\eta]$ iff $\mathcal{A} \models \rho^K(\overline{w})\ [\eta^K]$

Proof.
As in the case of product, this follows from connection content and assignment induction in view of eliminability under connection. Indeed, from connection content and assignment induction, we have that $\mathcal{A}' \models K(\overline{z},\overline{w})[\eta \cup \eta^K]$;
so, since $\mathcal{A}' \models T'$, from eliminability under connection, we have
$\mathcal{A}' \models \rho(\overline{z})\ [\eta]$ iff $\mathcal{A} \models \rho^K(\overline{w})\ [\eta^K]$.
QED

Corollary B.3 (Faithful interpretation) *For every sentence σ of L' with reflection σ_r into L': $T' \vdash \sigma$ iff $T \vdash \sigma_r$.*

Proof.{Similar to the case of product}
From generalised interpretation, we have, for every structure $\mathcal{A}'$ of L' with reduct $\mathcal{A}$ to L, $\mathcal{A}' \models \sigma$ iff $\mathcal{A} \models \sigma_r$.
Now, if $T' \vdash \sigma$, given any $\mathcal{A} \models T$, we have the canonical expansion $\mathcal{A}^r \models T'$; thus $\mathcal{A}^r \models \sigma$, whence $\mathcal{A} \models \sigma_r$.
Also, if $T \vdash \sigma_r$, given any $\mathcal{A}' \models T'$ with reduct $\mathcal{A} \models T$, we have $\mathcal{A} \models \sigma_r$, whence $\mathcal{A}' \models \sigma$.
QED

Now, let us recall the concept of internalised relation for subsort.
Let Σ be the set of sorts of L, then the set of sorts of L' is $\Sigma \cup \{N\}$. We define $n : (\Sigma \cup \{N\})^* \to \Sigma^*$ by $n(\lambda) = \lambda$, and, given $\overline{\tau} \in (\Sigma \cup \{N\})^*$, $n(\overline{\tau}.E) = n(\overline{\tau}).E$ for $E \in \Sigma$ and $n(\overline{\tau}.N) = n(\overline{\tau}).S$. Notice that $\overline{\tau}$ and $n(\overline{\tau})$ have the same length.
This mapping n induces natural mappings from sequences of elements of $\mathcal{A}'$ to sequences of elements of $\mathcal{A}$ as follows. Given $\overline{\tau} \in (\Sigma \cup \{N\})^*$, we define the mapping $\overline{\tau}^r_{\mathcal{A}'} : \overline{\tau}^{\mathcal{A}'} \to n(\overline{\tau})^{\mathcal{A}}$ so that $[\overline{\tau}.\tau]^r_{\mathcal{A}'}(\overline{c}.a) = \overline{\tau}^r_{\mathcal{A}'}(\overline{c}).a$ if $\tau \in \Sigma$, and $[\overline{\tau}.\tau]^r_{\mathcal{A}'}(\overline{c}.b) = \overline{\tau}^r_{\mathcal{A}'}(\overline{c}).j^{\mathcal{A}'}(b)$ if $\tau = N$.
Finally, given a relation R over sorts $\overline{\tau}$ in $\mathcal{A}'$, its **internalised version** is the relation R_r over sorts $n(\overline{\tau})$ in $\mathcal{A}$ defined by $R_r = \{\overline{\tau}^r_{\mathcal{A}'}(\overline{b})/\overline{b} \in R\}$.

Lemma B.3 (Internalisation and connection content) *Consider a structure $\mathcal{A}'$ of L' with reduct $\mathcal{A}$ to L. Let R be a relation over sorts $\overline{\tau} = \tau_1.\dots.\tau_k$ in $\mathcal{A}'$ with internalised version R_r over sorts $n(\overline{\tau}) = \sigma_1.\dots.\sigma_k$ in $\mathcal{A}$. Let $\overline{z}$ be a set of k variables over $\overline{\tau}$ and $\overline{w}$ be a set of k variables over $n(\overline{\tau})$ such that $\overline{z} \cap \overline{w} = \emptyset$. Consider any connection matching $\overline{z}$ to $\overline{w}$, with content $K(\overline{z},\overline{w})$.*

- *(a) Given $\overline{b} \in \overline{\tau}^{\mathcal{A}'}$ and $\overline{a} \in n(\overline{\tau})^{\mathcal{A}}$, $\overline{a} = \overline{\tau}^r_{\mathcal{A}'}(\overline{b})$ iff $\mathcal{A}' \models K(\overline{z},\overline{w})[\overline{z}/\overline{b},\overline{w}/\overline{a}]$.*
- *(b) Given $\overline{a} \in n(\overline{\tau})^{\mathcal{A}}$, $\overline{a} \in R_r$ iff for some $\overline{b} \in R$, $\mathcal{A}' \models K(\overline{z},\overline{w})[\overline{z}/\overline{b},\overline{w}/\overline{a}]$.*
- *(c) Given $\overline{b} \in \overline{\tau}^{\mathcal{A}'}$, $\overline{b} \in R$ iff for some $\overline{a} \in R_r$, $\mathcal{A}' \models K(\overline{z},\overline{w})[\overline{z}/\overline{b},\overline{w}/\overline{a}]$.*

Proof.

- (a) In view of the definitions, we may consider each element of the sequence at a time.
  If $\tau \in \Sigma$ then $\overline{\tau}^r_{\mathcal{A}'}(b) = b$; so $a = \overline{\tau}^r_{\mathcal{A}'}(b)$ iff $\mathcal{A}' \models v = u[v/b, u/a]$.
  If $\tau = N$ then $\overline{\tau}^r_{\mathcal{A}'}(b) = j^{\mathcal{A}'}(b)$; so $a = \overline{\tau}^r_{\mathcal{A}'}(b)$ iff $\mathcal{A}' \models j(y) = x[y/b, x/a]$.
- (b) and (c) now follow from the definition of internalised version in view of item (a)

QED

Corollary B.4 (Internalisation of connection) *Consider a (subsort) connection matching $\overline{z} = \overline{y} \cup \overline{v}$ to $\overline{w} = \overline{x} \cup \overline{u}$ with content $K(\overline{z}, \overline{w})$. Let $\overline{r(x)}$ be the conjunction $r(x^1) \wedge \ldots \wedge r(x^n)$.*

For every formula $\rho(\overline{z})$ of L' having as its free variables those in $\overline{z}$, we have that $T' \vdash (\exists \overline{z})[K(\overline{z}, \overline{w}) \wedge \rho(\overline{z})] \leftrightarrow [\rho^K(\overline{w}) \wedge \overline{r(x)}]$.

Proof.
From eliminability under connection we have
$T' \vdash K(\overline{z}, \overline{w}) \to [\rho(\overline{z}) \leftrightarrow \rho^K(\overline{w})]$, thus
$T' \vdash [K(\overline{z}, \overline{w}) \wedge \rho(\overline{z})] \leftrightarrow [\rho^K(\overline{w}) \wedge K(\overline{z}, \overline{w})]$, whence
$T' \vdash (\exists \overline{z})[K(\overline{z}, \overline{w}) \wedge \rho(\overline{z})] \leftrightarrow (\exists \overline{z})[\rho^K(\overline{w}) \wedge K(\overline{z}, \overline{w})]$.
Hence, since $\overline{z} \cap \overline{w} = \emptyset$, we obtain
$T' \vdash (\exists \overline{z})[K(\overline{z}, \overline{w}) \wedge \rho(\overline{z})] \leftrightarrow [\rho^K(\overline{w}) \wedge (\exists \overline{z}) K(\overline{z}, \overline{w})]$,
whence, by the lemma on image of connection, we can conclude:
$T' \vdash (\exists \overline{z})[K(\overline{z}, \overline{w}) \wedge \rho(\overline{z})] \leftrightarrow [\rho^K(\overline{w}) \wedge \overline{r(x)}]$. QED

Proposition B.2 (Reflection and internalisation) *Consider a model $\mathcal{A}' \models T'$ with reduct $\mathcal{A}$ to L and a (subsort) connection matching $\overline{z} = \overline{y} \cup \overline{v}$ to $\overline{w} = \overline{x} \cup \overline{u}$ with content $K(\overline{z}, \overline{w})$. Let $\overline{r(x)}$ be the conjunction $r(x^1) \wedge \ldots \wedge r(x^n)$. Given a formula $\rho(\overline{z})$ of L' with K-reflection $\rho^K(\overline{w})$, the internalised version of the relation $\rho^{\mathcal{A}'}$ defined by ρ in $\mathcal{A}'$ is the relation defined in $\mathcal{A}$ by the formula $\rho^K(\overline{v}, \overline{x}) \wedge \overline{r(x)}$ i.e., $(\rho^{\mathcal{A}'})_r = (\rho^K \wedge \overline{r})^{\mathcal{A}}$.*

Proof.
If $\overline{z}$ is over sorts $\overline{\tau}$, then $\overline{w}$ is over sorts $n(\overline{\tau})$.
Given a sequence $\overline{a} \in n(\overline{\tau})^{\mathcal{A}}$, we have: $\overline{a} \in (\rho^{\mathcal{A}'})_r$
iff {by connection content and internalisation, item (b)}
for some $\overline{b} \in \overline{\tau}^{\mathcal{A}'}$, $\mathcal{A}' \models \rho(\overline{z})[\overline{z}/\overline{b}]$ and $\mathcal{A}' \models K(\overline{z}, \overline{w})[\overline{z}/\overline{b}, \overline{w}/\overline{a}]$
iff $\mathcal{A}' \models (\exists \overline{z})[K(\overline{z}, \overline{w}) \wedge \rho(\overline{z})][\overline{w}/\overline{a}]$
iff {by internalisation of connection}
$\mathcal{A}' \models \rho^K(\overline{w}) \wedge \overline{r(x)}[\overline{w}/\overline{a}]$
iff $\overline{a} \in (\rho^K \wedge \overline{r})^{\mathcal{A}}$.
QED

Theorem B.1 (Definability) *Consider a model $\mathcal{A}' \models T'$ with reduct $\mathcal{A} = \mathcal{A}'/L$. A relation R is definable in $\mathcal{A}'$ iff its internalised version R_r is definable in $\mathcal{A}$.*

Proof.
Consider R be over sorts $\overline{\tau}$ in $\mathcal{A}'$ and its internalised version R_r over sorts $n(\overline{\tau})$ in $\mathcal{A}$.

- If R is defined in $\mathcal{A}'$ by ρ, then by the preceding proposition, $\rho^K \wedge \overline{r}$ defines R_r in $\mathcal{A}$.
- If R_r is defined in $\mathcal{A}$ by φ, let $\overline{w}$ be the set of variables, over sorts $n(\overline{\tau})$, with free occurrences in φ. Then, we can select a set $\overline{z}$ of new variables, over sorts $\overline{\tau}$, matching $\overline{w}$. Thus R will be defined in $\mathcal{A}'$ by
  $(\exists \overline{w})[K(\overline{z}, \overline{w}) \wedge \varphi(\overline{w})]$.
  To see this it suffices to use the lemma on connection content and internalisation, item (c).
  Indeed, given a sequence $\overline{b} \in \overline{\tau}^{\mathcal{A}'}$, we have: $\overline{b} \in R$
  iff {by connection content and internalisation, item (c)}
  for some $\overline{a} \in n(\overline{\tau})^{\mathcal{A}}$, $\overline{a} \in R_r$ and and $\mathcal{A}' \models K(\overline{z}, \overline{w})[\overline{z}/\overline{b}, \overline{w}/\overline{a}]$
  iff for some $\overline{a} \in n(\overline{\tau})^{\mathcal{A}}$, $\mathcal{A} \models \varphi(\overline{w})[\overline{w}/\overline{a}]$ iff
  $\mathcal{A}' \models (\exists \overline{w})[K(\overline{z}, \overline{w}) \wedge \varphi(\overline{w})][\overline{z}/\overline{b}]$
  iff $\overline{b} \in ((\exists \overline{w})[K(\overline{z}, \overline{w}) \wedge \varphi(\overline{w})])^{\mathcal{A}'}$.

QED

C APPENDIX

Definability Results for Quotient Sort

In this appendix we prove the definability results for extensions by introduction of quotient sorts.

Recall the concept of (quotient) connection.
Given a set $\overline{z}$ of variables of L', we can partition it into two sets $\overline{z} = \overline{y} \cup \overline{v}$, where $\overline{y} = \{y^1, \ldots, y^n\}$, say, consists of those variables in $\overline{z}$ ranging over the new quotient sort Q, and $\overline{v} = \{v^1, \ldots, v^m\}$, say, consists of those variables in $\overline{z}$ ranging over sorts in L. Now, consider a set $\overline{w}$ of variables of L as follows. For each variable y^k in $\overline{z}$ we have a distinct variable x^k of sort S; for each variable v^l in $\overline{v}$ over sort E in L we have a distinct variable u^l over sort E. We thus have a partition $\overline{w} = \overline{x} \cup \overline{u}$, where $\overline{x}$ has n variables and $\overline{u}$ has m variables. We say that this matching of variables forms a **(quotient) connection** iff $\overline{z} \cap \overline{w} = \emptyset$.
We code this matching of variables into formulae of language L', as follows. For each $k = 1, \ldots, n$, call $K(y^k, x^k)$ the formula $p(x^k) = y^k$; and for each $l = 1, \ldots, m$, call $K(v^l, u^l)$ the formula $v^l = u^l$. By the **content** of the connection we mean the conjunction $K(\overline{z}, \overline{w})$ of these fomulae $K(y^k, x^k)$, for $k = 1, \ldots, n$, and $K(v^l, u^l)$, for $l = 1, \ldots, m$.

Corollary C.1 (Eliminability without connection) *For each formula σ of L' without free variables over the quotient sort Q, there exists a formula σ_q of L' (with the same free variables) such that $T' \vdash \sigma \leftrightarrow \sigma_q$.*

Proof.{As in the previous cases}
Let $\overline{v} = \{v^1, \ldots, v^m\}$ be the set of variables with free occurrences in σ. Given $\overline{u} = \{u^1, \ldots, u^m\}$ with $\overline{v} \cap \overline{u} = \emptyset$, the connection content $K(\overline{z}, \overline{w})$ reduces to $\wedge_{j=1}^{n} v^l = u^l$. From eliminability under connection, we have a formula $\tau(\overline{u})$, with free variables in $\overline{u}$, such that $T' \vdash \wedge_{j=1}^{n} v^l = u^l \rightarrow [\sigma(\overline{v}) \leftrightarrow \tau(\overline{u})]$. If we take σ_q as the result of replacing in $\tau(\overline{u})$ each u^l by the corresponding v^l, we have the desired result: $T' \vdash \sigma(\overline{v}) \leftrightarrow \sigma_q(\overline{v})$.
QED

Corollary C.2 (Maximal Conservativeness) *$T \subseteq T'$ is a maximal conservative extension.*

Proof.{As in the previous cases}
Consider an extension T" of T' with language L' that extends T conservatively. Let σ be a sentence of L' such that T" $\vdash \sigma$. By eliminability without connection, we know that $T' \vdash \sigma \leftrightarrow \sigma_q$. So, since $T' \subseteq T$", T" $\vdash \sigma_q$. Thus, by conservativeness, $T \vdash \sigma_q$. So, since $T \subseteq T'$, $T' \vdash \sigma_q$; whence $T' \vdash \sigma$.
QED

Lemma C.1 (Connection content and assignment induction) *Consider a model $\mathcal{A}' \models T'$ with reduct $\mathcal{A}$ to L, and a connection matching $\overline{z} = \overline{y} \cup \overline{v}$ to $\overline{w} = \overline{x} \cup \overline{u}$ with content $K(\overline{z}, \overline{w})$.*

Given an assignment ξ for the variables of $\overline{u}$ in $\mathcal{A}$, let ξ^K be the assignment for the variables of $\overline{z}$ in $\mathcal{A}'$ induced by ξ, i.e. $\xi^K(y^k) = p^{\mathcal{A}'}(\xi(x^k))$ and $\xi^K(u^l) = \xi(v^l)$.

- *(a) Given an assignment η for the variables of $\overline{z}$ in $\mathcal{A}'$, $\eta = \xi^K$ iff $\mathcal{A}' \models K(\overline{z}, \overline{w})[\eta \cup \xi]$.*
- *(b) The mapping $\xi \to \xi^K$ is a surjection from assignments for $\overline{w}$ in $\mathcal{A}$ onto assignments for $\overline{z}$ in $\mathcal{A}'$.*

Proof.

- (a) This follows from the definitions of connection content and of assignment induction. Indeed,
  - for a variable y^k in $\overline{y}$, $\eta(y^k) = p^{\mathcal{A}'}(\xi(x^k))$ iff $\mathcal{A}' \models p(x^k) = y^k[\eta \cup \xi]$;
  - for a variable v^l in $\overline{v}$, $\xi(u^l) = \eta(v^l)$ iff $\mathcal{A}' \models v^l = u^l[\eta \cup \xi]$.
- (b) This follows from the fact that the mapping $a \to p^{\mathcal{A}'}(a)$ is a surjection from $S^{\mathcal{A}}$ onto $Q^{\mathcal{A}'}$.

QED

Proposition C.1 (Generalised interpretation) *Consider a model $\mathcal{A}' \models T'$ with reduct $\mathcal{A}$ to L, and a connection matching $\overline{z}$ to $\overline{w}$.*

For every formula $\rho(\overline{z})$ of L' with free variables in $\overline{z}$ and with K-ref l ection $\rho^K(\overline{w})$ and for every assignment ξ for variables of $\overline{w}$ in $\mathcal{A}$ with induced assignment ξ^K:
$\mathcal{A}' \models \rho(\overline{z})$ $[\xi^K]$ iff $\mathcal{A} \models \rho^K(\overline{w})$ $[\xi]$

Proof.{Similar to the the previous cases}
This follows from connection content and assignment induction, in view of eliminability under connection. Indeed, from connection content and assignment induction, we have that $\mathcal{A}' \models K(\overline{z}, \overline{w})[\xi^K \cup \xi]$;
so, since $\mathcal{A}' \models T'$, from eliminability under connection, we have
$\mathcal{A}' \models \rho(\overline{z})$ $[\xi^K]$ iff $\mathcal{A} \models \rho^K(\overline{w})$ $[\xi]$.
QED

Corollary C.3 (Faithful interpretation) *For every sentence σ of L' with reflection σ_q into L': $T' \vdash \sigma$ iff $T \vdash \sigma_q$.*

Proof.{Similar to the the previous cases}
From generalised interpretation, we have, for every structure $\mathcal{A}'$ of L' with reduct $\mathcal{A}$ to L, $\mathcal{A}' \models \sigma$ iff $\mathcal{A} \models \sigma_q$.
Now, if $T' \vdash \sigma$, given any $\mathcal{A} \models T$, we have the canonical expansion $\mathcal{A}^q \models T'$; thus $\mathcal{A}^r \models \sigma$, whence $\mathcal{A} \models \sigma_q$.

Also, if $T \vdash \sigma_q$, given any $\mathcal{A}' \models T'$ with reduct $\mathcal{A} \models T$, we have $\mathcal{A} \models \sigma_q$, whence $\mathcal{A}' \models \sigma$.
QED

Now, let us recall the idea of internalised relation for quotient sort.
Let Σ be the set of sorts of L, so the set of sorts of L' is $\Sigma \cup \{Q\}$. We define a mapping $n : (\Sigma \cup \{Q\})^* \to \Sigma^*$ as in the case of subsort.
This mapping n induces natural mappings from sequences of elements of $\mathcal{A}$ to sequences of elements of $\mathcal{A}'$ as follows. Given $\overline{\tau} \in (\Sigma \cup \{Q\})^*$, we define the mapping $\overline{\tau}^q_{\mathcal{A}'} : n(\overline{\tau})^{\mathcal{A}} \to \overline{\tau}^{\mathcal{A}'}$ so that $[\overline{\tau}.\tau]^q_{\mathcal{A}'}(\overline{c}.a) = \overline{\tau}^q_{\mathcal{A}'}(\overline{c}).a$ if $\tau \in \Sigma$, and $[\overline{\tau}.\tau]^q_{\mathcal{A}'}(\overline{c}.b) = \overline{\tau}^q_{\mathcal{A}'}(\overline{c}).p^{\mathcal{A}'}(a_k)$ if $\tau = Q$.
Finally, given a relation R over sorts $\overline{\tau}$ in $\mathcal{A}'$, its **internalised version** is the relation R_q over sorts $n(\overline{\tau})$ in $\mathcal{A}$ defined by $R_q = \{\overline{a}/\overline{\tau}^q_{\mathcal{A}'}(\overline{a}) \in R\}$.

Lemma C.2 (Internalisation and connection content) *Consider a structure $\mathcal{A}'$ of L' with reduct $\mathcal{A}$ to L. Let R be a relation over sorts $\overline{\tau} = \tau_1.\ldots.\tau_k$ in $\mathcal{A}'$ with internalised version R_q over sorts $n(\overline{\tau}) = \sigma_1.\ldots.\sigma_k$ in $\mathcal{A}$. Let $\overline{z}$ be a set of k variables over $\overline{\tau}$ and $\overline{w}$ be a set of k variables over $n(\overline{\tau})$ such that $\overline{z} \cap \overline{w} = \emptyset$. Consider any connection matching $\overline{z}$ to $\overline{w}$, with content $K(\overline{z}, \overline{w})$.*

- *(a) Given $\overline{b} \in \overline{\tau}^{\mathcal{A}'}$ and $\overline{a} \in n(\overline{\tau})^{\mathcal{A}}$, $\overline{b} = \overline{\tau}^q_{\mathcal{A}'}(\overline{a})$ iff $\mathcal{A}' \models K(\overline{z}, \overline{w})[\overline{z}/\overline{b}, \overline{w}/\overline{a}]$.*
- *(b) Given $\overline{a} \in n(\overline{\tau})^{\mathcal{A}}$, $\overline{a} \in R_q$ iff for some $\overline{b} \in R$, $\mathcal{A}' \models K(\overline{z}, \overline{w})[\overline{z}/\overline{b}, \overline{w}/\overline{a}]$.*
- *(c) Given $\overline{b} \in \overline{\tau}^{\mathcal{A}'}$, $\overline{b} \in R$ iff for some $\overline{a} \in R_q$, $\mathcal{A}' \models K(\overline{z}, \overline{w})[\overline{z}/\overline{b}, \overline{w}/\overline{a}]$.*

Proof.

- (a) In view of the definitions, we may consider each element of the sequence at a time.
  If $\tau \in \Sigma$ then $\overline{\tau}^q_{\mathcal{A}'}(a) = a$; so $b = \overline{\tau}^q_{\mathcal{A}'}(a)$ iff $\mathcal{A}' \models v = u[v/b, u/a]$.
  If $\tau = Q$ then $\overline{\tau}^q_{\mathcal{A}'}(a) = p^{\mathcal{A}'}(a)$; so $b = \overline{\tau}^q_{\mathcal{A}'}(a)$ iff $\mathcal{A}' \models p(x) = y[y/b, x/a]$.
- (b) and (c) are seen to follow from the definition of internalised version in view of item (a)

QED

Corollary C.4 (Internalisation of connection) *Consider a (quotient) connection matching $\overline{z}$ to $\overline{w}$ with content $K(\overline{z}, \overline{w})$.*

For every formula $\rho(\overline{z})$ of L' having as its free variables those in $\overline{z}$, we have that $T' \vdash (\exists \overline{z})[K(\overline{z}, \overline{w}) \wedge \rho(\overline{z})] \leftrightarrow \rho^K(\overline{w})$.

Proof.{Similar to the case of product}
First, notice that $T' \vdash (\exists \overline{z})K(\overline{z}, \overline{w})$.
Because, for $k = 1, \ldots, n$, $K(y^k, x^k)$ is $p(x^k) = y^k$, and
$\vdash (\exists y^k : Q)p(x^k) = y^k$;

and, for each $l = 1, \ldots, m$, $K(v^l, u^l)$ is $v^l = u^l$, and $\vdash (\exists v^l : S) v^l = u^l$.
Now, from eliminability under connection we have
$T' \vdash K(\overline{z}, \overline{w}) \to [\rho(\overline{z}) \leftrightarrow \rho^K(\overline{w})]$, thus
$T' \vdash [K(\overline{z}, \overline{w}) \wedge \rho(\overline{z})] \leftrightarrow [\rho^K(\overline{w}) \wedge K(\overline{z}, \overline{w})]$, whence
$T' \vdash (\exists \overline{z})[K(\overline{z}, \overline{w}) \wedge \rho(\overline{z})] \leftrightarrow (\exists \overline{z})[\rho^K(\overline{w}) \wedge K(\overline{z}, \overline{w})]$.
Hence, since $\overline{z} \cap \overline{w} = \emptyset$, we obtain
$T' \vdash (\exists \overline{z})[K(\overline{z}, \overline{w}) \wedge \rho(\overline{z})] \leftrightarrow [\rho^K(\overline{w}) \wedge (\exists \overline{z}) K(\overline{z}, \overline{w})]$,
whence, by the above remark, we can conclude:
$T' \vdash (\exists \overline{z})[K(\overline{z}, \overline{w}) \wedge \rho(\overline{z})] \leftrightarrow \rho^K(\overline{w})$.
QED

Proposition C.2 (Reflection and internalisation) *Consider a model $\mathcal{A}' \models T'$ with reduct $\mathcal{A}$ to L and a connection matching $\overline{z}$ over $\overline{w}$.*

Consider a formula formula $\rho(\overline{z})$ of L' with K-reflection $\rho^K(\overline{w})$. Then, the internalised version of the relation $\rho^{\mathcal{A}'}$ defined by ρ in $\mathcal{A}'$ is the relation defined in $\mathcal{A}$ by the reflection ρ^K, i.e., $(\rho^{\mathcal{A}'})_q = (\rho^K)^{\mathcal{A}}$.

Proof.{Similar to the case of product}
If $\overline{z}$ is over sorts $\overline{\tau}$, then $\overline{w}$ is over sorts $n(\overline{\tau})$.
Given a sequence $\overline{a} \in m(\overline{\tau})^{\mathcal{A}}$, we have $\overline{a} \in (\rho^{\mathcal{A}'})_q$
iff {by connection content and internalisation, item (b)}
for some $\overline{b} \in \overline{\tau}^{\mathcal{A}'}$, $\mathcal{A}' \models \rho(\overline{z})[\overline{z}/\overline{b}]$ and $\mathcal{A}' \models K(\overline{z}, \overline{w})[\overline{z}/\overline{b}, \overline{w}/\overline{a}]$
iff $\mathcal{A}' \models (\exists \overline{z})[K(\overline{z}, \overline{w}) \wedge \rho(\overline{z})][\overline{w}/\overline{a}]$
iff {by internalisation of connection}
$\mathcal{A}' \models \rho^K(\overline{w})[\overline{w}/\overline{a}]$
iff $\overline{a} \in (\rho^K)^{\mathcal{A}}$.
QED

Theorem C.1 (Definability) *Consider a model $\mathcal{A}' \models T'$ with reduct $\mathcal{A} = \mathcal{A}'/L$.*

A relation R is definable in $\mathcal{A}'$ iff its internalised version R_q is definable in $\mathcal{A}$.

Proof.
Consider R be over sorts $\overline{\tau}$ in $\mathcal{A}'$ and its internalised version R_q over sorts $n(\overline{\tau})$ in $\mathcal{A}$.

- If R is defined in $\mathcal{A}'$ by ρ, then by the preceding proposition, ρ^K defines R_q in $\mathcal{A}$.
- If R_q is defined in $\mathcal{A}$ by φ, let $\overline{w}$ be the set of variables, over sorts $n(\overline{\tau})$, with free occurrences in φ. Then, we can select a set $\overline{z}$ of new variables, over sorts $\overline{\tau}$, matching $\overline{w}$. Thus R will be defined in $\mathcal{A}'$ by
  $(\exists \overline{w})[K(\overline{z}, \overline{w}) \wedge \varphi(\overline{w})]$.
  To see this it suffices to use the lemma on connection content and internalisation, item (c).

Indeed, given a sequence $\overline{b} \in \overline{\tau}^{\mathcal{A}'}$, we have $\overline{b} \in R$
iff {by connection content and internalisation, item (c)}
for some $\overline{a} \in n(\overline{\tau})^{\mathcal{A}}$, $\overline{a} \in R_q$ and and $\mathcal{A}' \models K(\overline{z}, \overline{w})[\overline{z}/\overline{b}, \overline{w}/\overline{a}]$
if for some $\overline{a} \in n(\overline{\tau})^{\mathcal{A}}$, $\mathcal{A} \models \varphi(\overline{w})[\overline{w}/\overline{a}]$
iff $\mathcal{A}' \models (\exists\overline{w})[K(\overline{z}, \overline{w}) \wedge \varphi(\overline{w})][\overline{z}/\overline{b}]$
iff $\overline{b} \in ((\exists\overline{w})[K(\overline{z}, \overline{w}) \wedge \varphi(\overline{w})])^{\mathcal{A}'}$.

QED

D APPENDIX

Definability Results for Union Sort

In this appendix we prove the definability results for extensions by introduction of union sorts.

Recall the concept of (union) connection.
Given a set $\overline{z}$ of variables of L', we can partition it into two sets $\overline{z} = \overline{y} \cup \overline{v}$, where $\overline{y} = \{y^1, \ldots, y^n\}$, say, consists of those variables in $\overline{z}$ ranging over the union sort U, and $\overline{v} = \{v^1, \ldots, v^m\}$, say, consists of those variables in $\overline{z}$ ranging over sorts in L. Now, consider a set $\overline{w}$ of variables of L as follows. For each variable y^k in $\overline{z}$ we have two distinct variables x_1^k of sort S_1 and x_2^k of sort S_2; for each variable v^l in $\overline{v}$ over sort S in L we have a distinct variable u^l over sort S. We thus have a partition $\overline{w} = (\overline{x_1} \cup \overline{x_2}) \cup \overline{u}$, where $\overline{x_1}$ and $\overline{x_2}$ have n variables each and $\overline{u}$ has m variables. We say that this matching of variables forms a **(union) connection** iff $\overline{z} \cap \overline{w} = \emptyset$.
Consider a function $\delta : \{1, \ldots, n\} \to \{1, 2\}$. We wish to code this matching of variables into formulae of language L', as follows. For each $k = 1, \ldots, n$, call $K^\delta(y^k, x^k_{\delta(k)})$ the formula $y^k) = i_{\delta(k)}(x^k_{\delta(k)})$; and for each $l = 1, \ldots, m$, call $K(v^l, u^l)$ the formula $v^l = u^l$. By the δ-**content** of the connection we mean the conjunction $K^\delta(\overline{z}, \overline{w})$ of these fomulae $K^\delta(y^k, x^k_{\delta(k)})$, for $k = 1, \ldots, n$, and $K(v^l, u^l)$, for $l = 1, \ldots, m$.

Corollary D.1 (Eliminability without connection) *For each formula σ of L' without free variables over the union sort U, there exists a formula σ_+ of L' (with the same free variables) such that $T' \vdash \sigma \leftrightarrow \sigma_+$.*

Proof.{As in the the previous cases}
Let $\overline{v} = \{v^1, \ldots, v^m\}$ be the set of variables with free occurrences in σ. Given $\overline{u} = \{u^1, \ldots, u^m\}$ with $\overline{v} \cap \overline{u} = \emptyset$, the connection content $K(\overline{z}, \overline{w})$ reduces to $\wedge_{j=1}^n v^l = u^l$. From eliminability under connection, we have a formula $\tau(\overline{u})$, with free variables in $\overline{u}$, such that $T' \vdash \wedge_{j=1}^n v^l = u^l \to [\sigma(\overline{v}) \leftrightarrow \tau(\overline{u})]$. If we take σ_+ as the result of replacing in $\tau(\overline{u})$ each u^l by the corresponding v^l, we have the desired result: $T' \vdash \sigma(\overline{v}) \leftrightarrow \sigma_+(\overline{v})$.
QED

Corollary D.2 (Maximal Conservativeness) *$T \subseteq T'$ is a maximal conservative extension.*

Proof.{As in the the previous cases}
Consider an extension T" of T' with language L' that extends T conservatively. Let σ be a sentence of L' such that T" $\vdash \sigma$. By eliminability without connection, we know that $T' \vdash \sigma \leftrightarrow \sigma_+$. So, since $T' \subseteq T$", T" $\vdash \sigma_+$. Thus, by conservativeness, $T \vdash \sigma_+$. So, since $T \subseteq T'$, $T' \vdash \sigma_+$; whence $T' \vdash \sigma$.
QED

Lemma D.1 (Connection content and assignment induction) *Consider a model $\mathcal{A}' \models T'$ with reduct $\mathcal{A}$ to L. Consider also a function $\delta : \{1, \ldots, n\} \to \{1, 2\}$ and a (union) connection matching $\overline{z} = \overline{y} \cup \overline{v}$ to $\overline{w} = (\overline{x_1} \cup \overline{x_2}) \cup \overline{u}$, with δ-content $K^\delta(\overline{z}, \overline{w})$. Given an assignment ξ of values in $\mathcal{A}$ for the variables of $\overline{w}$, let ξ^δ be the assignment in $\mathcal{A}'$ induced by ξ, i.e. $\xi^\delta(y^k) = i^{\mathcal{A}'}_{\delta(k)}(\xi(x^k_{\delta(k)}))$ and $\xi^\delta(u^l) = \eta(v^l)$.*

- *(a) For an assignment η for the variables of $\overline{z}$ in $\mathcal{A}'$, $\eta = \xi^\delta$ iff $\mathcal{A}' \models K^\delta(\overline{z}, \overline{w})[\eta \cup \xi]$.*
- *(b) The mapping $\xi \to \xi^\delta$ is an injection from assignments for $\overline{w}$ in $\mathcal{A}$ to assignments for $\overline{z}$ in $\mathcal{A}'$.*

Proof.

- (a) This follows from the definitions of connection content and of assignment induction. Indeed,
  - for a variable y^k in $\overline{y}$, $\eta(y^k) = i^{\mathcal{A}'}_{\delta(k)}(\xi(x^k_{\delta(k)}))$ iff $\mathcal{A}' \models y^k = i_{\delta(k)}(x^k_{\delta(k)})[\eta \cup \xi]$;
  - for a variable v^l in $\overline{v}$, $\eta(v^l) = \xi(u^l)$ iff $\mathcal{A}' \models v^l = u^l[\eta \cup \xi]$.
- (b) This follows from the fact that the mapping $a \to i^{\mathcal{A}'}_{\delta(k)}(a)$ is an injection from $S^{\mathcal{A}}_{\delta(k)}$ to $U^{\mathcal{A}'}$.

QED

Proposition D.1 (Generalised interpretation) *Let $\mathcal{A}'$ be a structure for L' with reduct $\mathcal{A}$ to L. Consider a function $\delta : \{1, \ldots, n\} \to \{1, 2\}$ and a connection matching $\overline{z}$ to $\overline{w}$.*

For every formula $\rho(\overline{z})$ of L' with free variables in $\overline{z}$ and with δ-refl ection $\rho^\delta(w)$ and for every assignment ξ for variables of $\overline{x}$ in $\mathcal{A}$ with induced assignment ξ^δ:
$\mathcal{A}' \models \rho(\overline{z})$ $[\xi^\delta]$ iff $\mathcal{A} \models \rho^\delta(w)$ $[\xi]$

Proof.{Similar to the the previous cases}
This follows from connection content and assignment induction by means of eliminability under connection. Indeed, from connection content and assignment induction, we have that $\mathcal{A}' \models K^\delta(\overline{z}, \overline{w})[\xi^\delta \cup \xi]$;
so, since $\mathcal{A}' \models T'$, from eliminability under connection, we have
$\mathcal{A} \models \rho^\delta(w)$ $[\xi]$ iff $\mathcal{A}' \models \rho(\overline{z})$ $[\xi^\delta]$.
QED

Corollary D.3 (Faithful interpretation) *For every sentence σ of L' with reflection σ_+ into L': $T' \vdash \sigma$ iff $T \vdash \sigma_+$.*

Proof.{As in the the previous cases}
From generalised interpretation, we have, for every structure $\mathcal{A}'$ of L' with reduct

$\mathcal{A}$ to L, $\mathcal{A}' \models \sigma$ iff $\mathcal{A} \models \sigma_+$.
Now, if $T' \vdash \sigma$, given any $\mathcal{A} \models T$, we have the canonical expansion $\mathcal{A}^+ \models T'$; thus $\mathcal{A}^+ \models \sigma$, whence $\mathcal{A} \models \sigma_+$.
Also, if $T \vdash \sigma_+$, given any $\mathcal{A}' \models T'$ with reduct $\mathcal{A} \models T$, we have $\mathcal{A} \models \sigma_+$, whence $\mathcal{A}' \models \sigma$.
QED

Now, let us recall the concept of internalised view of a relation.
Let Σ be the set of sorts of L; so the set of sorts of L' is $\Sigma \cup \{U\}$. Given $\overline{\tau} = \tau_1 \ldots \tau_l \in (\Sigma \cup \{U\})^+$, let $J = \{j_1, \ldots, j_n\}$ be the set of indices h such that $\tau_h = U$. Given a function $\delta : \{1, \ldots, n\} \rightarrow \{1, 2\}$, we define $\overline{\tau}_\delta = \sigma_1 \ldots \sigma_l \in \Sigma^+$ where $\sigma_h = \tau_h$ if $h \notin J$ and $\sigma_h = S_{\delta(h)}$ if $h \in J$. Notice that $\overline{\tau}$ and $\overline{\tau}_\delta$ have the same length.
This mapping induces natural mappings from sequences of elements of $\mathcal{A}$ to sequences of elements of $\mathcal{A}'$ as follows. Given $\overline{\tau} = \tau_1 \ldots \tau_l \in (\Sigma \cup \{N\})^*$, we define the mapping $\overline{\tau}^\delta_{\mathcal{A}'} : (\overline{\tau}_\delta)^{\mathcal{A}} \rightarrow \overline{\tau}^{\mathcal{A}'}$ by $\overline{\tau}^\delta_{\mathcal{A}'}(a_1 \ldots a_l) = b_1 \ldots b_l$, where $b_1 = a_1$ if $h \notin J$, and $b_1 = i^{\mathcal{A}'}_{\delta(h)}(a_h)$ if $h \in J$.
Finally, given a relation R over sorts $\overline{\tau}$ in $\mathcal{A}'$ its δ-**internalisation** is the relation R_δ over sorts $\overline{\tau}_\delta$ in $\mathcal{A}$ defined by $R_\delta = \{\overline{a}/\overline{\tau}^\delta_{\mathcal{A}'}(\overline{a}) \in R\}$.

Lemma D.2 (Internalisation and connection content) *Consider a structure $\mathcal{A}'$ of L' with reduct $\mathcal{A}$ to L. Let R be a relation over sorts $\overline{\tau} = \tau_1 \ldots . \tau_k$ in $\mathcal{A}'$ with δ-internalisation R_δ over sorts $\overline{\tau}_\delta = \sigma_1 \ldots . \sigma_k$ in $\mathcal{A}$. Let $\overline{z}$ be a set of k variables over $\overline{\tau}$ and $\overline{w}$ be a set of k variables over $\overline{\tau}^\delta$ such that $\overline{z} \cap \overline{w} = \emptyset$. Consider any connection matching $\overline{z}$ to $\overline{w}$, with content $K(\overline{z}, \overline{w})$.*

- *(a) Given $\overline{b} \in \overline{\tau}_{\mathcal{A}'}$ and $\overline{a} \in \overline{\tau}^{\mathcal{A}}_\delta$, $\overline{b} = \overline{\tau}^\delta_{\mathcal{A}'}(\overline{a})$ iff $\mathcal{A}' \models K^\delta(\overline{z}, \overline{w})[\overline{z}/\overline{b}, \overline{w}/\overline{a}]$.*
- *(b) Given $\overline{a} \in \overline{\tau}^{\mathcal{A}}_\delta$, $\overline{a} \in R_\delta$ iff for some $\overline{b} \in R$, $\mathcal{A}' \models K^\delta(\overline{z}, \overline{w})[\overline{z}/\overline{b}, \overline{w}/\overline{a}]$.*
- *(c) Given $\overline{b} \in \overline{\tau}^{\mathcal{A}'}$, $\overline{b} \in \overline{\tau}^\delta_{\mathcal{A}'}(R_\delta)$ iff for some $\overline{a} \in R_\delta$, $\mathcal{A}' \models K^\delta(\overline{z}, \overline{w})[\overline{z}/\overline{b}, \overline{w}/\overline{a}]$.*
- *(d) $R = \bigcup_{\delta \in \Delta} \overline{\tau}^\delta_{\mathcal{A}'}(R_\delta)$, where Δ be the set of all 2^n functions $\delta : \{1, \ldots, n\} \rightarrow \{1, 2\}$.*

Proof.

- (a) In view of the definitions, we may consider each element of the sequence at a time.
If $\tau_h \in \Sigma$ then $\overline{\tau}^\delta_{\mathcal{A}'}(a_h) = a_h$; so $b_h = \overline{\tau}^\delta_{\mathcal{A}'}(a_h)$ iff $\mathcal{A}' \models v^h = u^h[v^h/b_h, u^h/a_h]$.
If $\tau_h = U$ then $\overline{\tau}^\delta_{\mathcal{A}'}(a_h) = i^{\mathcal{A}'}_{\delta(h)}(a_h)$; so $b_h = \overline{\tau}^\delta_{\mathcal{A}'}(a_h)$ iff
$\mathcal{A}' \models y^h) = i_{\delta(h)}(x^h_{\delta(h)})[y^h/b_h, x^h_{\delta(h)}/a_h]$.
- (b) and (c) are now seen to follow from the definition of internalised version in view of item (a)
- (d) Clearly, $\bigcup_{\delta \in \Delta} \overline{\tau}^\delta_{\mathcal{A}'}(R_\delta) \subseteq R$.
The converse inclusion follows from the fact that, for each $\overline{b} \in \overline{\tau}^{\mathcal{A}'}$ there exist $\delta \in \Delta$ and $\overline{a} \in (\overline{\tau}_\delta)^{\mathcal{A}}$ such that $\overline{b} = \overline{\tau}^\delta_{\mathcal{A}'}(\overline{a})$; this latter fact following by induction from $U^{\mathcal{A}'} = S^{\mathcal{A}}_1 \cup S^{\mathcal{A}}_2$.

QED

Corollary D.4 (Internalisation of connection) *Consider a function $\delta : \{1, \ldots, n\} \to \{1, 2\}$ and a (union) connection matching $\overline{z}$ to $\overline{w}$ with δ-content $K^\delta(\overline{z}, \overline{w})$.*

For every formula $\rho(\overline{z})$ of L' having as its free variables those in $\overline{z}$, we have that $T' \vdash (\exists \overline{z})[K^\delta(\overline{z}, \overline{w}) \wedge \rho(\overline{z})] \leftrightarrow \rho^\delta(\overline{w})$.

Proof.
First, notice that $T' \vdash (\exists \overline{z}) K^\delta(\overline{z}, \overline{w})$.
Because, for $k = 1, \ldots, n$, $K^\delta(y^k, x^k_{\delta(k)})$ is $y^k = i_{\delta(k)}(x^k_{\delta(k)})$, and
$T' \vdash (\exists y^k : U)[p_1 y^k = i_{\delta(k)}(x^k_{\delta(k)})$;
and, for each $l = 1, \ldots, m$, $K(v^l, u^l)$ is $v^l = u^l$, and $\vdash (\exists v^l : S) v^l = u^l$.
Now, from eliminability under connection we have
$T' \vdash K^\delta(\overline{z}, \overline{w}) \to [\rho(\overline{z}) \leftrightarrow \rho^\delta(\overline{w})]$, thus
$T' \vdash [K^\delta(\overline{z}, \overline{w}) \wedge \rho(\overline{z})] \leftrightarrow [\rho^\delta(\overline{w}) \wedge K^\delta(\overline{z}, \overline{w})]$, whence
$T' \vdash (\exists \overline{z})[K^\delta(\overline{z}, \overline{w}) \wedge \rho(\overline{z})] \leftrightarrow (\exists \overline{z})[\rho^\delta(\overline{w}) \wedge K^\delta(\overline{z}, \overline{w})]$.
Hence, since no variable in $\overline{z}$ occurs free in ρ^δ (as $\overline{z} \cap \overline{w} = \emptyset$), we obtain
$T' \vdash (\exists \overline{z})[K^\delta(\overline{z}, \overline{w}) \wedge \rho(\overline{z})] \leftrightarrow [rho^\delta(\overline{w}) \wedge (\exists \overline{z}) K^\delta(\overline{z}, \overline{w})]$,
whence, by the above remark, we can conclude:
$T' \vdash (\exists \overline{z})[K^\delta(\overline{z}, \overline{w}) \wedge \rho(\overline{z})] \leftrightarrow \rho^\delta(\overline{w})$.
QED

Lemma D.3 (Reflection and internalisation) *Consider a model $\mathcal{A}' \models T'$ with reduct $\mathcal{A}'$ to L as well as a function $\delta : \{1, \ldots, n\} \to \{1, 2\}$ and a (union) connection matching $\overline{z}$ to $\overline{w}$.*

Given a formula $\rho(\overline{z})$ with δ-reflection $\rho^\delta(\overline{w})$, the relation $(\rho^\delta)^{\mathcal{A}}$ defined by ρ^δ on $\mathcal{A}$ is the δ-internalisation of the relation $\rho^{\mathcal{A}'}$ defined by ρ on $\mathcal{A}'$, i.e., $(\rho^{\mathcal{A}'})_\delta = (\rho^\delta)^{\mathcal{A}}$.

Proof.
If $\overline{z}$ is over sorts $\overline{\tau}$, then $\overline{w}$ is over sorts $\overline{\tau}_\delta$.
Given a sequence $\overline{a} \in \overline{\tau}_\delta^{\mathcal{A}'}$, we have: $\overline{a} \in (\rho^{\mathcal{A}'})_\delta$
iff {by connection content and internalisation, item (b)}
for some $\overline{b} \in \overline{\tau}^{\mathcal{A}'}$, $\mathcal{A}' \models \rho(\overline{z})[\overline{z}/\overline{b}]$ and $\mathcal{A}' \models K^\delta(\overline{z}, \overline{w})[\overline{z}/\overline{b}, \overline{w}/\overline{a}]$
iff $\mathcal{A}' \models (\exists \overline{z})[K^\delta(\overline{z}, \overline{w}) \wedge \rho(\overline{z})][\overline{w}/\overline{a}]$
iff {by internalisation of connection}
$\mathcal{A}' \models \rho^\delta(\overline{w})[\overline{w}/\overline{a}]$
iff $\overline{a} \in (\rho^\delta)^{\mathcal{A}}$.
QED

Theorem D.1 (Definability) *Consider a model $\mathcal{A}' \models T'$ with reduct $\mathcal{A} = \mathcal{A}'/L$. A relation R is definable in $\mathcal{A}'$ iff every δ-internalisation R_δ is definable in $\mathcal{A}$.*

Proof.

Consider R be over sorts $\overline{\tau}$ in $\mathcal{A}'$ and its internalised version R_δ over sorts $\overline{\tau}_\delta$ in $\mathcal{A}$.

- If R is defined in $\mathcal{A}'$ by ρ, then by the preceding proposition, ρ^δ defines R_δ in $\mathcal{A}$.
- Conversely, let Δ be the set of all 2^n functions $\delta : \{1, \ldots, n\} \to \{1, 2\}$. Then, by the lemma on connection content and internalisation, item (d), we have $R = \bigcup_{\delta \in \Delta} \sigma_\delta^{\mathcal{A}}(R_\delta)$. This makes R definable if each R_δ is so.

QED