A Concrete Categorical Model for the Lambek Calculus

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Abstract: There are concrete constructions like the universe of functions and function composition [Almeida93], which determine a denotational semantics for the Lambek Calculus [Lambek58] and can be regarded as realizing the dynamic features of it. On the other hand, non-symmetric monoidal bi-closed categories determine an abstract model for the calculus [De Paiva91]. We present a categorical/denotational semantics for the Lambek Calculus, indeed for a λ-typed version Curry-Howard isomorphic to it. The main novelty of our approach is an abstract noncommutative construction, so-called *sequential product*, with right and left adjoints which is defined through a hierarchical structure of categories. It is motivated by those concrete constructions and it reflects the implicit permission of sequencing of objects. We show that the Lambek's noncommutative product corresponds to a noncommutative bi-endofunctor into a category enclosing all categories of such hierarquical structure, presenting a soundness property for this semantical framework.

Keywords: Sentence Construction, Noncommutativity, λ-Calculus, Denotational Semantics and Categorical Models.

Resumo: Existem construções concretas, tais como universo de funções e composição de funções [Almeida93], que determinam uma semântica denotacional para o Cálculo Lambek [Lambek58] e que representam os aspectos dinâmicos deste cálculo. Por outro lado, categorias monoidais não-simétricas (bi-)fechadas determinam um modelo abstrato para este cálculo [De Paiva91]. Nós apresentamos uma semântica categórica/denotacional para o Cálculo Lambek, de fato para um λ-cálculo tipado que é ("Curry-Howard") isomórfico a ele. O aspecto original desta abordagem está numa construção abstrata não-comutativa com adjuntos à direita e à esquerda, que chamamos produto sequencial, definida sobre uma estrutura hierárquica de categorias. A motivação para esta contrução surgiu de contruções concretas, objetivando representar abstratamente a permissão implíta necessária para sequenciar objetos. Nós mostramos que o produto não-comutativo do Cálculo Lambek corresponde a um (bi-)endofuntor não-comutativo sobre uma categoria que contém todas as categorias da estrutura hierárquica, apresentando uma propriedade de corretude para este esquema semântico.

Palavras-Chave: Construção de Sentenças, Não-Comutatividade, λ-Cálculo, Semântica Denotacional e Modelos Categóricos.

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1 Introduction

The Lambek Calculus [Lambek58], originally *Syntactic Calculus*, developed in order to formalize the process of sentence construction in a language, characterizes the expressions that compose a sentence and how they are combined for sentence formation. Expressions may be interpreted as types: nouns and sentences are primitive types, and they can be used in order to characterize compound types of expressions, which represent noun/sentence modifiers. The basic process of sentence formation is the sequencing. We call "sequential product" the abstraction of this process. There are two noun/sentence modifiers associated with it: one that retrieves left-elements when put on the right-side of an appropriated noun/sentence and another that works in a symmetric way.

When reading a sentence, it seems clear that there is a mandatory order, namely from left to right (west languages). This dynamic feature can be captured by a less declarative formalism, such as a λ-typed calculus, which provides an operational semantics for the Lambek calculus. Denotational and categorical semantics adequately abstract some low-level features of it.

Examples of denotational semantics for the Lambek Calculus are concrete constructions like *String Semantics* (near original Lambek's motivation), the universe of functions and function composition or the universe of matrices and matrix product. Those kinds of universes can be regarded as realizing the dynamic features of the calculus. On the other hand, monoidal closed categories with left and right adjoints for a noncommutative product functor determine an abstract model for the Lambek Calculus.

We are interested in a semantical framework which join both of these aspects, that is, it is based on abstract constructions of category theory and also realizes the dynamic features of the Lambek Calculus. This categorical/denotational semantics will be developed for a λ-typed version of the Lambek Calculus.

2 The Lambek Calculus

The main motivation of the Lambek Calculus relies on the dynamic of the phrase structure, i.e., some expressions can play distinct *roles* in a phrase, so-called *syntactic types* . They can assume a certain type in a context and a different one in another context. So, meaningful syntactic constructions of sentences obey rigorous rules about types.

The syntactic types are defined on two primitive types: the type of sentences($\mathbf{s}$) and the type of nouns($\mathbf{n}$). Compound types are formed from the primitive ones in the following way:

If A and B are types, then so are AB (combining of A and B), B/A (B **over** A) and $A \setminus B$ (A **under** B).

The two last compound types have the following behaviour: when we have an expression of type B/A, followed by an expression of type A, it is produced an expression of type B, and so does when we have an expression of type A followed by an expression of type $A \setminus B$. Symbolically:

$$(B/A)A \rightarrow B \qquad A(A \setminus B) \rightarrow B$$

In general, if we have expressions x and y of type A and B, respectively, it is assigned to expression xy the type AB. Like this, the type B/A is assigned to all expressions z such that zy has type B for any y of type A, and the type $A \setminus B$ is assigned to all expression z such that xz has type B for any x of type A.

In order to improve understanding of the notions discussed here, we adopt the following convention, presented in [Almeida93] and [De Paiva91]:

- $A \rightharpoonup B$ indicates the type $A \setminus B$
- $A \leftharpoondown B$ indicates the type A/B
- $A \otimes B$ indicates the type AB

The syntactic types are illustrated in [Lambek58] by the following examples:

- **Example 1**

  $$\text{John works}$$

  $$\underbrace{n \quad n \rightharpoonup s}_{s}$$

  In this case, "John" is a name, so we can assign to it the type n. When the name "John" is combined to the verb "works" at left, it produces the sentence "John works". So, we should assign the type $n \rightharpoonup s$ to the verb "works".

- **Example 2**

  $$\text{(Poor} \quad \text{John)} \quad \text{works}$$

  $$\underbrace{\underbrace{n \leftharpoondown n \quad n}_{n} \quad n \rightharpoonup s}_{s}$$

  The adjective "Poor" combined to the name "John", in this order, produces the name "Poor John" (compound name). It means that, a compound name is formed when a name is placed at right of the adjective "poor". So, "poor" has type $n \leftharpoondown n$. The verb "works" here has the same function as the example above.

- **Example 3**

  $$\text{(John works) here}$$

  $$\underbrace{\underbrace{n \quad n \rightharpoonup s}_{s} \quad s \rightharpoonup s}_{s}$$

In this case, when the sentence "John works" is combined to the name "here" at left, it produces the new sentence "John works here". So, we can assign the type $s \rightharpoonup s$ to the name "here".

The Lambek Syntactic Calculus has the following sequent rules:

$$A \Rightarrow A \quad (identity)$$

$$\frac{A, \Gamma \Rightarrow B}{\Gamma \Rightarrow A \rightharpoonup B}(\rightharpoonup_i) \qquad \frac{\Gamma \Rightarrow A \qquad \Delta_1, B, \Delta_2 \Rightarrow C}{\Delta_1, \Gamma, A \rightharpoonup B, \Delta_2 \Rightarrow C}(\rightharpoonup_e)$$

$$\frac{\Gamma, A \Rightarrow B}{\Gamma \Rightarrow B \leftharpoonup A}(\leftharpoonup_i) \qquad \frac{\Gamma \Rightarrow A \qquad \Delta_1, B, \Delta_2 \Rightarrow C}{\Delta_1, B \leftharpoonup A, \Gamma, \Delta_2 \Rightarrow C}(\leftharpoonup_e)$$

$$\frac{\Gamma \Rightarrow A \qquad \Delta \Rightarrow B}{\Gamma, \Delta \Rightarrow A \otimes B}(\otimes_i) \qquad \frac{\Gamma, A, B, \Delta \Rightarrow C}{\Gamma, A \otimes B, \Delta \Rightarrow C}(\otimes_e)$$

$$\frac{\Gamma \Rightarrow A \qquad \Delta_1, A, \Delta_2 \Rightarrow B}{\Delta_1, \Gamma, \Delta_2 \Rightarrow B}(cut)$$

This calculus has no structural rules: permutation, contraction or weakening. In [Abrusci90], it is shown the relationship between the Lambek Calculus and a fragment of the noncommutative intuitionistic linear propositional logic.

3 A Concrete Categorical Model

There are some successful attempts to give semantics to Lambek Calculus. In the so-called concrete models, such as *String Semantics* [Emms90], *Matrices and Semigroups Models* [Buszkowski78][Dosen85], the noncommutativity of the product is given by **construction**. On the other hand, there are more abstract models based on Non-symmetric Monoidal Biclosed Categories, for instance *Dialectica Model* [De Paiva91], in which the noncommutativity of the product is obtained by **constraint**.

We are particularly interested on the abstract treatment provided by categorical approach ([Maclane71][Goldblatt79]), but we want to give a concrete characterization for the noncommutativity of the Lambek's product. So, we aim to obtain a construction for a noncommutative product in categories.

The important features of the Lambek Calculus which should be modelled are: primitive types (atomic formulas); the inductive construction of the compound types $A \rightharpoonup B$ and $B \leftharpoonup A$ and the implicit mention of concatenation process in the sequent rules.

We propose an hierarchical structure of categories as a model for these features. The first level contains the collection of primitive types. At the second level, the sequential product of primitive types is defined and the compound types $A \rightharpoonup B$ and $B \leftharpoonup A$ are obtained as adjoints of this product. The same structure is repeated at the third level with respect to objects of the second level, and so on, in such a way that each level is embedded in the next one. We will begin to present this framework by defining the first and the second levels.

Definition 3.1 *:*

(i) *The category $\mathcal{C}_0$ is defined by:*

- $*$ *its objects are primitive types of expressions, and*
- $*$ *its arrows represent the possibility of combining primitive types (there is an arrow for each pair of them).*

(ii) *$\mathcal{C}_1$ is a category including any $\mathcal{C}_0$-objects and any $\mathcal{C}_0$-arrows. Its objects are either primitive types or compound types obtained from those.*

(iii) *$\mathcal{C}_0 \xrightarrow{\mathcal{F}_0} \mathcal{C}_1$ is an Embedding Faithful Functor, which characterizes $\mathcal{C}_0$ as a subcategory of $\mathcal{C}_1$.*

Turning back our attention to concatenation process on the phrase structure, we remark that there exists an underlying order in that process, which must be captured appropriately in order to distinguish a concatenation at left from one at right. The problem of combining a compound expression A at left of an expression B can be reduced to the problem of combining the second component of A to B in this order. Concatenation at right can be treated in an analogous way. So, a (finite) compound expression can be represented inductively, since each pair of primitive expressions can be combined.

Based on this intuitive idea, concatenation of primitive types is modelled by the "Sequential Product", which is defined as follows:

Definition 3.2 *Let "a" and "b" be objects of the category $\mathcal{C}_1$ and $a \xrightarrow{s} b$ an arrow of $\mathcal{C}_1$. The Sequential Product $a \otimes_1 b$ is a limit to the following diagram*

$$\begin{tikzcd} & a \otimes_1 b \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\ a \arrow[rr, "s"] & & b \end{tikzcd}$$

The arrow s, so-called a Sequencing Arrow, *indicates the possibillity of combining a with b in this order and must satisfy one of the following conditions:*

(i) *There is an arrow $\alpha \xrightarrow{g} \beta$ of $\mathcal{C}_0$, such that $s = \mathcal{F}_0(g)$, $a = \mathcal{F}_0(\alpha)$ and $b = \mathcal{F}_0(\beta)$;*

(ii) *There is a sequencing arrow s', such that $a \equiv a_1 \otimes_1 a_2$ and $s = s' \circ \pi'_2$*

$$\begin{tikzcd} & a \otimes_1 b \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\ a \arrow[rr, "s"] \arrow[dr, "\pi'_2"'] & & b \\ & a_2 \arrow[ur, "s'"'] & \end{tikzcd}$$

(iii) *There is a sequencing arrow s', such that $b \equiv b_1 \otimes_1 b_2$ and $\pi_1' \circ s = s'$,*

$$\begin{tikzcd}
 & a \otimes_1 b \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\
a \arrow[rr, "s"] \arrow[dr, "s'"'] & & b \arrow[dl, "\pi_1'"] \\
 & b_1 &
\end{tikzcd}$$

The $a \rightharpoonup_1 b$ and $b \leftharpoonup_1 a$ objects can be obtained in the category $\mathcal{C}_1$ by adjunction of pairs of particular Hom-Functors, in a similar way that the exponential is obtained in Cartesian Closed Categories. However, we must adopt special evaluation functions $\overrightarrow{ev}^1$ and $\overleftarrow{ev}^1$.

Definition 3.3 *Let a, b and c be $\mathcal{C}_1$-objects, f, g, k and j $\mathcal{C}_1$-arrows and*

(i) *$F : \mathcal{C}_1 \to \mathcal{C}_1$ is a Hom-Functor, such that $F \equiv a \otimes_1 \_$ and*

$F(c) = a \otimes_1 c$

$F(g) = I_a \otimes_1 g$, *where $g : c \to (a \rightharpoonup_1 b)$ is an arrow of $\mathcal{C}_1$.*

$G : \mathcal{C}_1 \to \mathcal{C}_1$ is a Hom-Functor, such that $G \equiv a \rightharpoonup_1 \_$ and

$G(c) = a \rightharpoonup_1 c$

$G(f) = I_a \rightharpoonup_1 f$, *where $f : (a \otimes_1 c) \to b$ is an arrow of $\mathcal{C}_1$.*

By adjunction of F and G, we have

$$\begin{tikzcd}
a \otimes_1 (a \rightharpoonup_1 b) \arrow[r, "\overrightarrow{ev}^1"] & b \\
a \otimes_1 c \arrow[u, "I_a \otimes_1 g"] \arrow[ur, "f"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
a \rightharpoonup_1 b \\
c \arrow[u, "g"]
\end{tikzcd}$$

(ii) *$K : \mathcal{C}_1 \to \mathcal{C}_1$ is a Hom- Functor, such that $K \equiv \_ \otimes_1 a$ and*

$K(c) = c \otimes_1 a$

$K(j) = j \otimes_1 I_a$, *where $j : c \to (b \leftharpoonup_1 a)$ is an arrow of $\mathcal{C}_1$.*

$J : \mathcal{C}_1 \to \mathcal{C}_1$ is a Hom-Functor, such that $J \equiv \_ \leftharpoonup_1 a$ and

$J(c) = c \leftharpoonup_1 a$

$J(k) = k \leftharpoonup_1 I_a$, *where $k : (c \otimes_1 a) \to b$ is an arrow of $\mathcal{C}_1$.*

By adjunction of K and J, we have

$$
\begin{tikzcd}
(b \leftharpoondown_1 a) \otimes_1 a \arrow[r, "\overleftharpoon{ev}^1"] & b \\
c \otimes_1 a \arrow[u, "j \otimes_1 I_a"] \arrow[ur, "k"'] &
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
b \leftharpoondown_1 a \\
c \arrow[u, "j"]
\end{tikzcd}
$$

The adjunction of the Hom-Functors F and G determines the existence of the product $a \otimes_1 (a \rightharpoonup_1 b)$ and the adjunction of the Hom-Functors K and J determines the existence of the product $(b \leftharpoondown_1 a) \otimes_1 a$. Then, it must be considered the existence of sequencing arrows $\overrightharpoon{e}^1$ from a to $a \rightharpoonup_1 b$ and $\overleftharpoon{e}^1$ from $b \leftharpoondown_1 a$ to a in $\mathcal{C}_1$, such that $a \otimes_1 (a \rightharpoonup_1 b)$ and $(b \leftharpoondown_1 a) \otimes_1 a$ are the limit to the following diagrams, respectively:

$$
\begin{tikzcd}
& a \otimes_1 (a \rightharpoonup_1 b) \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\
a \arrow[rr, "\overrightharpoon{e}^1"] & & a \rightharpoonup_1 b
\end{tikzcd}
\qquad
\begin{tikzcd}
& (b \leftharpoondown_1 a) \otimes_1 a \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\
b \leftharpoondown_1 a \arrow[rr, "\overleftharpoon{e}^1"] & & a
\end{tikzcd}
$$

Each category $\mathcal{C}_i$, $i \geq 2$, is defined inductively over the category $\mathcal{C}_1$. The sequential product $\otimes_i$ in the category $\mathcal{C}_i$ is such that the objects $a \rightharpoonup_i b$ and $b \leftharpoondown_i a$ are defined as exponentials related to this product (in the same way that as it was done in category $\mathcal{C}_1$):

Definition 3.4 *:*

(i) *The category $\mathcal{C}_i$ includes all $\mathcal{C}_{i-1}$-objects and all $\mathcal{C}_{i-1}$-arrows, and its objects are either types represented by the $\mathcal{C}_{i-1}$-objects or compound types which can be obtained from those.*

(ii) $\mathcal{C}_{i-1} \stackrel{\mathcal{F}_{i-1}}{\rightarrow} \mathcal{C}_i$ *is an Embedding Faithful Functor, which characterizes $\mathcal{C}_{i-1}$ as a subcategory of $\mathcal{C}_i$.*

Definition 3.5 *Let "a" and "b" be objects of the category $\mathcal{C}_i$, $i \geq 2$, and $a \stackrel{s}{\rightarrow} b$ an arrow of $\mathcal{C}_i$.*

The Sequential Product $a \otimes_i b$ is a limit to the following diagram

$$
\begin{tikzcd}
& a \otimes_i b \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\
a \arrow[rr, "s"] & & b
\end{tikzcd}
$$

The so-called sequencing arrow s must satisfy the following conditions:

(i) *There is a sequencing arrow s_{i-1} of $\mathcal{C}_{i-1}$, such that $s = \mathcal{F}_{i-1}(s_{i-1})$*

(ii) *There is a sequencing arrow s' of $\mathcal{C}_i$, such that* $a \equiv a_1 \otimes_i a_2$ *and* $s = s' \circ \pi'_2$

$$\begin{tikzcd}
& a \otimes_i b \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\
a \arrow[rr, "s"] \arrow[dr, "\pi'_2"'] & & b \\
& a_2 \arrow[ur, "s'"'] &
\end{tikzcd}$$

(iii) *There is a sequencing arrow s' of $\mathcal{C}_i$, such that* $b \equiv b_1 \otimes_i b_2$ *and* $\pi'_1 \circ s = s'$,

$$\begin{tikzcd}
& a \otimes_i b \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\
a \arrow[rr, "s"] \arrow[dr, "s'"'] & & b \arrow[dl, "\pi'_1"] \\
& b_1 &
\end{tikzcd}$$

(iv) *There is a sequencing arrow s_{i-1} of $\mathcal{C}_{i-1}$, such that* $a \equiv \mathcal{F}_{i-1}(\alpha)$ *and* $b \equiv \mathcal{F}_{i-1}(\beta)$, *where* $\beta \equiv \beta_1 \rightharpoonup_{i-1} \beta_2$ *and* $s = \mathcal{F}_{i-1}(\overset{\rightharpoonup}{e}^{\,i-1} \circ s_{i-1})$,

$$\begin{tikzcd}
& & \beta_1 \otimes_{i-1} (\beta_1 \rightharpoonup_{i-1} \beta_2) \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\
\alpha \arrow[r, "s_{i-1}"] & \beta_1 \arrow[rr, "\overset{\rightharpoonup}{e}^{\,i-1}"] & & (\beta_1 \rightharpoonup_{i-1} \beta_2)
\end{tikzcd}$$

(v) *There is a sequencing arrow s_{i-1} of $\mathcal{C}_{i-1}$, such that* $b \equiv \mathcal{F}_{i-1}(\beta)$ *and* $a \equiv \mathcal{F}_{i-1}(\alpha)$, *where* $\alpha \equiv \alpha_2 \leftharpoonup_{i-1} \alpha_1$ *and* $s = \mathcal{F}_{i-1}(s_{i-1} \circ \overset{\leftharpoonup}{e}^{\,i-1})$,

$$\begin{tikzcd}
& \alpha_1 \otimes_{i-1} (\alpha_1 \rightharpoonup_{i-1} \alpha_2) \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & & \\
(\alpha_1 \rightharpoonup_{i-1} \alpha_2) \arrow[rr, "\overset{\leftharpoonup}{e}^{\,i-1}"] & & \alpha_1 \arrow[r, "s_{i-1}"] & \beta
\end{tikzcd}$$

Finally, we define a category enclosing all of this inductive construction.

Definition 3.6 *Let $\mathcal{C}$ be the smallest category containing all objects and arrows of each category $\mathcal{C}_i$, $i \geq 0$, of the hierarquy of categories presented.*

Remarks:

(i) We will use the symbols $\otimes$, $\rightharpoonup$ and $\leftharpoonup$ to the product and the exponentials in the category $\mathcal{C}$, respectively.

(ii) According to the inductive construction of the categories $\mathcal{C}_i$, there exists a $\mathcal{C}$-object $a \otimes b$ to each pair of $\mathcal{C}$-objects a and b.

(iii) Whenever it will be possible, we will abstract some technical details of the definition of the product and consider it just a limit to a special cone determined by a sequencing arrow.

In the following, we are going to characterize the sequential product operator $\otimes$ as a functor on the category $\mathcal{C}$.

Definition 3.7 *Let $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ be a mapping, such that:*

- $\otimes(a, b) = a \otimes b$, *for any $\mathcal{C}$-objects a and b*
- $\otimes(f, g) = f \otimes g : a \otimes b \to a' \otimes b'$, *where $a \xrightarrow{f} a'$ and $b \xrightarrow{f} b'$ are $\mathcal{C}$-arrows.*

In according to the limit notion, there exists an unique arrow $f \otimes g$, such that the following diagram commutes:

$$\begin{tikzcd}
& a' \otimes b' \arrow[dl, "\pi'_1"'] \arrow[dr, "\pi'_2"] & \\
a' \arrow[rr, "s'"] & & b' \\
a \arrow[u, "f"] \arrow[rr, "s"] & & b \arrow[u, "g"'] \\
& a \otimes b \arrow[ul, "\pi_1"] \arrow[ur, "\pi_2"'] \arrow[uuu, "f \otimes g"] &
\end{tikzcd}$$

Then, the definition presented to the mapping $\otimes$ is adequated. Next, we will prove that it is actually a functor.

Theorem 3.1 $\otimes : \mathcal{C}_1 \times \mathcal{C}_1 \to \mathcal{C}_1$ *is a functor.*

Proof:

(i) Let $a \xrightarrow{f} a' \xrightarrow{f'} a''$ and $b \xrightarrow{g} b' \xrightarrow{g'} b''$ be $\mathcal{C}_1$-arrows.

Then, we can construct the following diagram:

$$\begin{tikzcd}
& a'' \otimes b'' \arrow[dl, "\pi''_1"'] \arrow[dr, "\pi''_2"] & & \\
a'' \arrow[rr, "s''"] & & b'' & \\
a' \arrow[u, "f'"] \arrow[rr, "s'"'] & & b' \arrow[u, "g'"] & a' \otimes b' \arrow[uull, "f' \otimes g'"'] \arrow[lll, "\pi'_1"'] \arrow[l, "\pi'_2"] \\
a \arrow[u, "f"] \arrow[rr, "s"] & & b \arrow[u, "g"'] & \\
& a \otimes b \arrow[ul, "\pi_1"] \arrow[ur, "\pi_2"'] \arrow[uuuu, "h"] \arrow[uurr, "f \otimes g"'] & &
\end{tikzcd}$$

where $f \otimes g$ is the unique arrow such that

$$\begin{tikzcd}
& a' \otimes b' \arrow[dl, "\pi'_1"'] \arrow[dr, "\pi'_2"] & \\
a' \arrow[rr, "s'"] & & b' \\
a \arrow[u, "f"] \arrow[rr, "s"] & & b \arrow[u, "g"'] \\
& a \otimes b \arrow[ul, "\pi_1"] \arrow[ur, "\pi_2"'] \arrow[uuu, "f \otimes g"] &
\end{tikzcd}$$

commutes and $f' \otimes g'$ is the unique arrow such that

$$\begin{tikzcd}
& a'' \otimes b'' \arrow[dl, "\pi''_1"'] \arrow[dr, "\pi''_2"] & \\
a'' \arrow[rr, "s''"] & & b'' \\
a' \arrow[u, "f'"] \arrow[rr, "s'"] & & b' \arrow[u, "g'"'] \\
& a' \otimes b' \arrow[ul, "\pi'_1"] \arrow[ur, "\pi'_2"'] \arrow[uuu, "f' \otimes g'"] &
\end{tikzcd}$$

commutes. So, $k = (f' \otimes g') \circ (f \otimes g)$ makes the following diagram commute.

$$\begin{tikzcd}
& a'' \otimes b'' \arrow[dl, "\pi''_1"'] \arrow[dr, "\pi''_2"] & \\
a'' \arrow[rr, "s''"] & & b'' \\
a \arrow[u, "f' \circ f"] \arrow[rr, "s"] & & b \arrow[u, "g' \circ g"'] \\
& a \otimes b \arrow[ul, "\pi_1"] \arrow[ur, "\pi_2"'] \arrow[uuu, "k"'] &
\end{tikzcd}$$

However, $h = (f' \circ f) \otimes (g' \circ g)$ is the unique arrow which does so. Then,

$$(f' \circ f) \otimes (g' \circ g) = (f' \otimes g') \circ (f \otimes g)$$

Therefore,

$$\otimes(f' \circ f, g' \circ g) = \otimes(f', g') \circ \otimes(f, g)$$

(ii) The arrow $1_a \otimes 1_b$ is the unique such that the following diagram commutes:

$$\begin{tikzcd}
& a \otimes b \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\
a \arrow[rr, "s"] & & b \\
a \arrow[u, "1_a"] \arrow[rr, "s"] & & b \arrow[u, "1_b"'] \\
& a \otimes b \arrow[ul, "\pi_1"] \arrow[ur, "\pi_2"'] \arrow[uuu, "1_a \otimes 1_b"] &
\end{tikzcd}$$

However, $1_{a \otimes_1 b}$ also makes this diagram commute. So, $1_a \otimes 1_b = 1_a \otimes 1_b$. Therefore,

$$\otimes(1_a, 1_b) = 1_{\otimes(a,b)}$$

Now, consider the following diagram

$$\begin{tikzcd}
& a \otimes b \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\
a \arrow[rr, shift left, "s"] & & b \arrow[ll, shift left, "s'"] \\
& b \otimes a \arrow[ul, "\pi_2"] \arrow[ur, "\pi_1"'] &
\end{tikzcd}$$

$a \otimes b$ is a limit to the top triangle and $b \otimes a$ to the bottom triangle, respectively. So, in general, there can be no isomorphism between $a \otimes b$ and $b \otimes a$. Therefore, our construction actually determines a **noncommutative product**.

Next, we will prove that the sequential product is associative.

Theorem 3.2 *: Let a, b and c be $\mathcal{C}$-objects. Then*

$$a \otimes (b \otimes c) \cong (a \otimes b) \otimes c$$

Proof: We can construct the following diagram

$$\begin{tikzcd}
a \arrow[rr, "s_1"] & & b \arrow[rr, "s_2"] & & c \\
& a \otimes b \arrow[ul, "\pi_1"'] \arrow[ur, "\pi_2"] & & b \otimes c \arrow[ul, "\pi_1'"'] \arrow[ur, "\pi_2'"] & \\
& a \otimes (b \otimes c) \arrow[uul, "\pi_1''"] \arrow[u, "g_1"] \arrow[urr, "\pi_2''"] \arrow[rr, shift left, "g_2"] & & (a \otimes b) \otimes c \arrow[ull, "\pi_1'''"'] \arrow[u, "f_1"'] \arrow[uur, "\pi_2'''"'] \arrow[ll, shift left, "f_2"] &
\end{tikzcd}$$

where s_1 and s_2 are sequencing arrows and

(i) g_1 is the unique arrow making the following diagram commute

$$\begin{tikzcd}
& a \otimes b \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\
a \arrow[rr, "s_1"] & & b \\
& a \otimes (b \otimes c) \arrow[ul, "\pi_1''"] \arrow[ur, "\pi_1' \circ \pi_2''"'] \arrow[uu, "g_1"'] &
\end{tikzcd}$$

(ii) g_2 is the unique arrow making the following diagram commute

$$\begin{tikzcd}
& (a \otimes b) \otimes c \arrow[dl, "\pi_1'''"'] \arrow[dr, "\pi_2'''"] & \\
a \otimes b \arrow[rr, "s_3"] & & c \\
& a \otimes (b \otimes c) \arrow[ul, "g_1"] \arrow[ur, "\pi_2' \circ \pi_2''"'] \arrow[uu, "g_2"'] &
\end{tikzcd}$$

where $s_3 = s_2 \circ \pi_2$ is a sequencing arrow, by definition.

(iii) f_1 is the unique arrow making the following diagram commute

$$\begin{tikzcd}
& b \otimes c \arrow[dl, "\pi_1'"'] \arrow[dr, "\pi_2'"] & \\
b \arrow[rr, "s_2"] & & c \\
& (a \otimes b) \otimes c \arrow[ul, "\pi_2 \circ \pi_1'''"] \arrow[ur, "\pi_2'''"'] \arrow[uu, "f_1"'] &
\end{tikzcd}$$

(iv) f_2 is the unique arrow making the following diagram commute

$$\begin{tikzcd}
& a \otimes (b \otimes c) \arrow[dl, "\pi_1''"'] \arrow[dr, "\pi_2''"] & \\
a \arrow[rr, "s_4"] & & b \otimes c \\
& (a \otimes b) \otimes c \arrow[ul, "\pi_1 \circ \pi_1'''"] \arrow[ur, "f_1"'] \arrow[uu, "f_2"'] &
\end{tikzcd}$$

onde s_4 is the unique arrow making the following diagram commute

$$\begin{tikzcd}
& b \otimes c \arrow[dl, "\pi_1'"'] \arrow[dr, "\pi_2'"] & \\
b \arrow[rr, "s_2"] & & c \\
& a \arrow[ul, "s_1"] \arrow[ur, "s_2 \circ s_1"'] \arrow[uu, "s_4"'] &
\end{tikzcd}$$

Since $s_1 = \pi_1' \circ s_4$, s_4 is a sequencing arrow, by definition.

By (iii) and (iv), $\pi'_2 \circ (\pi''_2 \circ f_2) = (\pi'_2 \circ \pi''_2) \circ f_2 = \pi'''_2$.
So, by (ii), $(\pi'''_2 \circ g_2) \circ f_2 = \pi'''_2$. Then, $\pi'''_2 \circ (g_2 \circ f_2) = \pi'''_2$.
Therefore, $g_2 \circ f_2 = 1_{(a \otimes b) \otimes c}$. $(*)$

By (i) and (ii), $\pi''_1 = \pi_1 \circ (\pi'''_1 \circ g_2) = (\pi_1 \circ \pi'''_1) \circ g_2$.
So, by (iv), $(\pi''_1 \circ f_2) \circ g_2 = \pi''_1$. Then, $\pi''_1 \circ (f_2 \circ g_2) = \pi''_1$.
Therefore, $f_2 \circ g_2 = 1_{a \otimes (b \otimes c)}$. $(**)$

Then, by $(*)$ and $(**)$, there is an isomorphism g_2 from $a \otimes (b \otimes c)$ to $(a \otimes b) \otimes c$.

4 Soundness of the Model

We presented in the previous section a categorical framework, which represents adequately the formulas (types of expressions) of the Lambek Calculus as objects. So, in this section we will prove that our categorical framework is actually a model for the Lambek Calculus, in the sense that the proofs of the calculus are arrows.

We will adopt the λ-Calculus approach (see [Girard91]). Therefore, we will present a λ-Lambek Calculus Curry-Howard isomorphic to the sequent version and we will construct a denotational semantics for it using our categorical framework. Finally, we will prove the soundness of this semantics.

λ - Lambek Calculus

- Types:
  - $*$ Atomic types: A, B,...
  - $*$ If A and B are types, then A $\otimes$ B, A $\rightharpoonup$ B, A $\leftharpoonup$ B are types
- Terms:

x:A[x:A] is a term.

$$\frac{t : B[x : A, \Gamma]}{\overrightarrow{\lambda}\ x.t : A \rightharpoonup B[\Gamma]} \qquad \frac{t : B[\Gamma, x : A]}{\overleftarrow{\lambda}\ x.t : B \leftharpoonup A[\Gamma]}$$

$$\frac{t_1 : A[\Gamma] \quad t_2 : A \rightharpoonup B[\Delta]}{A\overrightarrow{P}P\ (t_2, t_1) : B[\Gamma, \Delta]} \qquad \frac{t_2 : B \leftharpoonup A[\Delta] \quad t_1 : A[\Gamma]}{A\overleftarrow{P}P\ (t_2, t_1) : B[\Delta, \Gamma]}$$

$$\frac{t_1 : A[\Gamma_1] \quad t_2 : B[\Gamma_2]}{\langle t_1, t_2 \rangle : A \otimes B[\Gamma_1, \Gamma_2]}$$

$$\frac{t : C[\Gamma, x : A, y : B, \Gamma_1] \quad t_1 : A \otimes B[\Delta]}{LET(< x : A, y : B >= t_1)t : C[\Gamma, \Delta, \Gamma_1]}$$

This λ-Lambek Calculus presented in [Almeida93] is proved to be Curry-Howard isomorphic to the Lambek Calculus.

This Calculus has the following *Reduction Rules*:

- $\overrightarrow{APP}\ (\overrightarrow{\lambda}\ x.t, t') : B[\Gamma, \Delta]\ \ \triangleright\ t[x \leftarrow t'] : B[\Gamma, \Delta]$

- $\overleftarrow{APP}\ (\overleftarrow{\lambda}\ x.t, t') : B[\Delta, \Gamma]\ \ \triangleright\ t[x \leftarrow t'] : B[\Delta, \Gamma]$

- $LET(< x : A, y : B > = < t_1, t_2 >)t : C[\Gamma, \Delta_1, \Delta_2, \Gamma']\ \ \triangleright\ t[x \leftarrow t_1, y \leftarrow t_2] : C[\Gamma, \Delta_1, \Delta_2, \Gamma']$

Next, we will define a denotational semantics for it in terms of the categorical framework presented.

Denotational Semantics

- Types are mapped to objects of the category $\mathcal{C}$

  * $[[A]] \in obj(\mathrm{C})$
  * $[[A \otimes B]] = [[A]] \otimes [[B]]$
  * $[[A \rightharpoonup B]] = [[A]] \rightharpoonup [[B]]$
  * $[[B \leftharpoonup A]] = [[B]] \leftharpoonup [[A]]$
  * $[[\Gamma]] = \bigotimes_{(x_i:\gamma_i)\in\Gamma} [[\gamma_i]]$

  i.e., the denotation of the context Γ is the generalized categorical product of the denotations of the types γ_i of variables x_i of Γ. So, Γ is interpreted as an ordered sequence.

- Terms are mapped to arrows of $\mathcal{C}$

  To each term $t : A[\Delta]$ we assign a morphism $[[t]] : [[\Delta]] \rightarrow [[A]]$

  * $[[x : A[x : A]\,]] = I_{[[A]]} : [[A]] \rightarrow [[A]]$
  * $[[< t_1, t_2 >: A \otimes B[\Gamma_1, \Gamma_2]\,]] = [[t_1 : A[\Gamma_1]\,]] \otimes [[t_2 : B[\Gamma_2]\,]]$,
    which is the unique arrow making the following diagram to commute

$$\begin{tikzcd}
& {[[A \otimes B]]} \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\
{[[A]]} \arrow[rr, "s"] & & {[[B]]} \\
{[[\Gamma_1]]} \arrow[u, "{[[t_1]]}"] \arrow[rr, "s'"] & & {[[\Gamma_2]]} \arrow[u, "{[[t_2]]}"'] \\
& {[[\Gamma_1]] \otimes [[\Gamma_2]]} \arrow[ul, "\pi'_1"] \arrow[ur, "\pi'_2"'] \arrow[uuu, "{[[t_1]] \otimes [[t_2]]}"] &
\end{tikzcd}$$

* $[[\overset{\rightharpoonup}{\lambda}\ x.t : A \rightharpoonup B[\Gamma]\,]] = f$, such that $f = \overset{\rightharpoonup}{\widehat{[[t]]}} : [[\Gamma]] \to [[A \rightharpoonup B]]$ and the following diagram commutes

$$\begin{tikzcd} {[[A]] \otimes [[A \rightharpoonup B]]} \arrow[r, "\overset{\rightharpoonup}{ev}"] & {[[B]]} \\ {[[A]] \otimes [[\Gamma]]} \arrow[u, "I_{[[A]]} \otimes f"] \arrow[ur, "{[[t]]}"'] & \end{tikzcd}$$

* $[[\overset{\leftharpoonup}{\lambda}\ x.t : A \leftharpoonup B[\Gamma]\,]] = f$, such that $f = \overset{\leftharpoonup}{\widehat{[[t]]}} : [[\Gamma]] \to [[B \leftharpoonup A]]$ and the following diagram commutes

$$\begin{tikzcd} {[[B \leftharpoonup A]] \otimes [[A]]} \arrow[r, "\overset{\leftharpoonup}{ev}"] & {[[B]]} \\ {[[\Gamma]] \otimes [[A]]} \arrow[u, "f \otimes I_{[[A]]}"] \arrow[ur, "{[[t]]}"'] & \end{tikzcd}$$

* $[[A\overset{\rightharpoonup}{P}P\ (t_2, t_1) : B[\Gamma_1, \Gamma_2]\,]] = f$, such that the following diagram commutes

$$\begin{tikzcd} {[[A]] \otimes [[A \rightharpoonup B]]} \arrow[r, "\overset{\rightharpoonup}{ev}"] & {[[B]]} \\ {[[\Gamma_1]] \otimes [[\Gamma_2]]} \arrow[u, "{[[t_1]] \otimes [[t_2]]}"] \arrow[ur, "f"'] & \end{tikzcd}$$

* $[[A\overset{\leftharpoonup}{P}P\ (t_2, t_1) : A[\Gamma_2, \Gamma_1]\,]] = f$, such that the following diagram commutes

$$\begin{tikzcd} {[[B \leftharpoonup A]] \otimes [[A]]} \arrow[r, "\overset{\leftharpoonup}{ev}"] & {[[B]]} \\ {[[\Gamma_2]] \otimes [[\Gamma_1]]} \arrow[u, "{[[t_2]] \otimes [[t_1]]}"] \arrow[ur, "f"'] & \end{tikzcd}$$

* $[[LET((x : A, y : B) = t_1)t : C[\Gamma, \Delta, \Gamma_1]\,]] = f$, such that the following diagram commutes

$$\begin{tikzcd} {[[\Gamma]] \otimes [[A]] \otimes [[B]] \otimes [[\Gamma_1]]} \arrow[r, "{[[t]]}"] & {[[C]]} \\ {[[\Gamma]] \otimes [[\Delta]] \otimes [[\Gamma_1]]} \arrow[u, "{I_{[[\Gamma]]} \otimes [[t_1]] \otimes I_{[[\Gamma_1]]}}"] \arrow[ur, "f"'] & \end{tikzcd}$$

and $[[t_1]] : [[\Delta]] \to [[A \otimes B]] = [[A]] \otimes [[B]]$

We need to verify that denotation of terms are preserved by the reduction rules in order to assure that this denotational semantics is sound. The reduction rules presented are based on substituitions of terms. So, we present a Substitution Lemma which supports the proof of soundness.

Lemma 4.1 *: Let $t : A[\Gamma_1, x : C, \Gamma_2]$ and $t' : C[\Delta]$ be terms. Then,*

$$[[t : A[\Gamma_1, x : C, \Gamma_2]\,]] \circ (I_{[[\Gamma_1]]} \otimes [[t' : A[\Delta]\,]] \otimes I_{[[\Gamma_2]]})$$
$$=$$
$$[[t[x \leftarrow t'] : C[\Gamma_1, \Delta, \Gamma_2]\,]]$$

where we can substitute t' by x in t and x is a variable of type C.

This lemma is proved by induction on structure of t, considering substitutions in subterms of t.

Proof: Induction on structure of t

Base Step: $t = x : C[x : C]$.

So, $[[t]] = I_{[[C]]}$.

Then, $[[t]] \circ [[t' : C[\Delta]\,]] = [[t' : C[\Delta]\,]] = [[x\,[x \leftarrow t'] : C[\Delta]\,]] =$
$= [[t[x \leftarrow t'] : C[\Delta]\,]]$

Inductive Step: Suppose that a λ-term t_α satisfies the lemma, where x occurs in t_α. The cases in which x occurs in t_β are proved like these following.

(i) $t = < t_\alpha, t_\beta >: A \otimes B[\Gamma_1, x : C, \Gamma_2, \Gamma]$

So, $[[< t_\alpha, t_\beta >]] = [[t_\alpha]] \otimes [[t_\beta]]$ is the unique arrow from $[[\Gamma_1]] \otimes [[C]] \otimes [[\Gamma_2]] \otimes [[\Gamma]]$ to $[[A \otimes B]]$, such that the following diagram commutes

$$\begin{tikzcd}
 & {[[A \otimes B]]} \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\
{[[A]]} \arrow[rr, "s"] & & {[[B]]} \\
{[[\Gamma_1]] \otimes [[C]] \otimes [[\Gamma_2]]} \arrow[u, "{[[t_\alpha]]}"] \arrow[rr, "s'"] & & {[[\Gamma]]} \arrow[u, "{[[t_\beta]]}"'] \\
 & {[[\Gamma_1]] \otimes [[C]] \otimes [[\Gamma_2]] \otimes [[\Gamma]]} \arrow[ul, "\pi'_1"] \arrow[ur, "\pi'_2"'] \arrow[uuu, "{[[t_\alpha]] \otimes [[t_\beta]]}"] & \\
 & {[[\Gamma_1]] \otimes [[\Delta]] \otimes [[\Gamma_2]] \otimes [[\Gamma]]} \arrow[u, "{I_{[[\Gamma_1]]} \otimes [[t']] \otimes I_{[[\Gamma_2]]} \otimes I_{[[\Gamma]]}}"'] &
\end{tikzcd}$$

Moreover, $f = ([[t_\alpha]] \circ (I_{[[\Gamma_1]]} \otimes [[t']] \otimes I_{[[\Gamma_2]]}\,)) \otimes [[t_\beta]]$ is the unique arrow from

$[[\Gamma_1]] \otimes [[\Delta]] \otimes [[\Gamma_2]] \otimes [[\Gamma]]$ to $[[A \otimes B]]$, such that the following diagram commutes

$$\begin{tikzcd}
 & {[[A \otimes B]]} \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & & \\
{[[A]]} \arrow[rr, "s"] & & {[[B]]} & \\
{[[\Gamma_1]] \otimes [[C]] \otimes [[\Gamma_2]]} \arrow[u, "{[[t_\alpha]]}"] & & & {[[\Gamma]]} \arrow[ul, "{[[t_\beta]]}"'] \\
{[[\Gamma_1]] \otimes [[\Delta]] \otimes [[\Gamma_2]]} \arrow[u, "I_{[[\Gamma_1]]} \otimes [[t']] \otimes I_{[[\Gamma_2]]}"] \arrow[rr, "s''"] & & {[[\Gamma]]} \arrow[uu, "{[[t_\beta]]}"] \arrow[ur, "I_{[[\Gamma]]}"'] & \\
 & {[[\Gamma_1]] \otimes [[\Delta]] \otimes [[\Gamma_2]] \otimes [[\Gamma]]} \arrow[ul, "\pi''_1"] \arrow[ur, "\pi''_2"'] \arrow[uuuu, "f"'] & &
\end{tikzcd}$$

So, $[[< t_\alpha, t_\beta >]] \circ (I_{[[\Gamma_1]]} \otimes [[t']] \otimes I_{[[\Gamma_2]]} \otimes I_{[[\Gamma]]}) = f$.

By induction hypothesis,
$[[t_\alpha]] \circ (I_{[[\Gamma_1]]} \otimes [[t']] \otimes I_{[[\Gamma_2]]}) = [[t_\alpha[x \leftarrow t'] : A[\Gamma_1, \Delta, \Gamma_2]\ ]] =$

Then, $f = [[t_\alpha[x \leftarrow t'] : A[\Gamma_1, \Delta, \Gamma_2]\ ]] \otimes [[t_\beta : B[\Gamma]\ ]] =$
$= [[< t_\alpha, t_\beta > [x \leftarrow t'] : A \otimes B[\Gamma_1, \Delta, \Gamma_2, \Gamma]\ ]]$

(ii) $t = A\overrightarrow{P}P\ (t_\beta, t_\alpha) : B[\Gamma_1, x : C, \Gamma_2, \Gamma]$

The following diagrams commute

$$
\begin{tikzcd}
 & {[[A \otimes (A \rightharpoonup B)]]} \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] \arrow[rr, "\vec{ev}"] & & {[[B]]} \\
{[[A]]} \arrow[rr, "s"] & & {[[A \rightharpoonup B]]} & \\
{[[\Gamma_1]] \otimes [[C]] \otimes [[\Gamma_2]]} \arrow[u, "{[[t_\alpha]]}"] \arrow[rr, "s'"] & & {[[\Gamma]]} \arrow[u, "{[[t_\beta]]}"'] & \\
 & {[[\Gamma_1]] \otimes [[C]] \otimes [[\Gamma_2]] \otimes [[\Gamma]]} \arrow[ul, "\pi'_1"] \arrow[ur, "\pi'_2"'] \arrow[uuu, "{[[t_\alpha]] \otimes [[t_\beta]]}"] \arrow[uuurr, "{[[t]]}"'] & & \\
 & {[[\Gamma_1]] \otimes [[\Delta]] \otimes [[\Gamma_2]] \otimes [[\Gamma]]} \arrow[u, "{I_{[[\Gamma_1]]} \otimes [[t']] \otimes I_{[[\Gamma_2]]} \otimes I_{[[\Gamma]]}}"'] & &
\end{tikzcd}
$$

$$
\begin{tikzcd}
 & {[[A \otimes (A \rightharpoonup B)]]} \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & & \\
{[[A]]} \arrow[rr, "s"] & & {[[A \rightharpoonup B]]} & \\
{[[\Gamma_1]] \otimes [[C]] \otimes [[\Gamma_2]]} \arrow[u, "{[[t_\alpha]]}"] & & & {[[\Gamma]]} \arrow[ul, "{[[t_\beta]]}"'] \\
{[[\Gamma_1]] \otimes [[\Delta]] \otimes [[\Gamma_2]]} \arrow[u, "{I_{[[\Gamma_1]]} \otimes [[t']] \otimes I_{[[\Gamma_2]]}}"] \arrow[rr, "s''"] & & {[[\Gamma]]} \arrow[uu, "{[[t_\beta]]}"] \arrow[ur, "I_{[[\Gamma]]}"'] & \\
 & {[[\Gamma_1]] \otimes [[\Delta]] \otimes [[\Gamma_2]] \otimes [[\Gamma]]} \arrow[ul, "\pi''_1"] \arrow[ur, "\pi''_2"'] \arrow[uuuu, "f"'] & &
\end{tikzcd}
$$

where the arrow $f = ([[t_\alpha]] \circ (I_{[[\Gamma_1]]} \otimes [[t']] \otimes I_{[[\Gamma_2]]})) \otimes [[t_\beta]]$ is unique.

So, $([[t_\alpha]] \otimes [[t_\beta]]) \circ (I_{[[\Gamma_1]]} \otimes [[t']] \otimes I_{[[\Gamma_2]]} \otimes I_{[[\Gamma]]}) = f$.

Then, $[[t]] \circ (I_{[[\Gamma_1]]} \otimes [[t']] \otimes I_{[[\Gamma_2]]} \otimes I_{[[\Gamma]]}) = \overrightarrow{ev} \circ f$

Moreover, by induction hypothesis,
$f = [[t_\alpha[x \leftarrow t'] : A[\Gamma_1, \Delta, \Gamma_2]\ ]] \otimes [[t_\beta : B[\Gamma]\ ]]$

Therefore,
$[[\overrightarrow{APP}\,(t_\beta, t_\alpha) : B[\Gamma_1, x : C, \Gamma_2, \Gamma]\ ]] \circ (I_{[[\Gamma_1]]} \otimes [[t']] \otimes I_{[[\Gamma_2]]} \otimes I_{[[\Gamma]]}) =$
$\overrightarrow{ev} \circ ([[t_\alpha[x \leftarrow t'] : A[\Gamma_1, \Delta, \Gamma_2]\ ]] \otimes [[t_\beta : B[\Gamma]\ ]]) =$
$[[\overrightarrow{APP}\,(t_\beta, t_\alpha)[x \leftarrow t'] : B[\Gamma_1, \Delta, \Gamma_2, \Gamma]\ ]]$

(iii) $t = \overleftarrow{APP}\,(t_\beta, t_\alpha) : B[\Gamma, \Gamma_1, x : C, \Gamma_2]$

Analogous to the item (ii).

(iv) $t = \overrightarrow{\lambda}\, y.t_\alpha : A \rightharpoonup B[\Gamma_1, x : C, \Gamma_2]$

The following diagrams commute

$$
\begin{tikzcd}
{[[A \otimes (A \rightharpoonup B)]]} \arrow[r, "\overrightarrow{ev}"] & {[[B]]} \\
{[[A]] \otimes [[\Gamma_1]] \otimes [[C]] \otimes [[\Gamma_2]]} \arrow[u, "I_{[[A]]} \otimes [[t]]"'] \arrow[ur, "{[[t_\alpha]]}"'] & \\
{[[A]] \otimes [[\Gamma_1]] \otimes [[\Delta]] \otimes [[\Gamma_2]]} \arrow[u, "I_{[[A]]} \otimes I_{[[\Gamma_1]]} \otimes [[t']] \otimes I_{[[\Gamma_2]]}"'] &
\end{tikzcd}
$$

$$
\begin{tikzcd}
 & {[[A \otimes (A \rightharpoonup B)]]} \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\
{[[A]]} \arrow[rr, "s"] & & {[[A \rightharpoonup B]]} \\
 & & {[[\Gamma_1]] \otimes [[C]] \otimes [[\Gamma_2]]} \arrow[u, "{[[t]]}"'] \\
{[[A]]} \arrow[uu, "I_{[[A]]}"] \arrow[rr, "s'"] & & {[[\Gamma_1]] \otimes [[\Delta]] \otimes [[\Gamma_2]]} \arrow[u, "I_{[[\Gamma_1]]} \otimes [[t']] \otimes I_{[[\Gamma_2]]}"'] \\
 & {[[A]] \otimes [[\Gamma_1]] \otimes [[\Delta]] \otimes [[\Gamma_2]]} \arrow[ul, "\pi'_1"] \arrow[ur, "\pi'_2"'] \arrow[uuuu, "f"] &
\end{tikzcd}
$$

where $f = I_{[[A]]} \otimes ([[t]] \circ (I_{[[\Gamma_1]]} \otimes [[t']] \otimes I_{[[\Gamma_2]]}))$ is unique. $(*)$

So, $f = (I_{[[A]]} \otimes [[t]]) \circ (I_{[[A]]} \otimes I_{[[\Gamma_1]]} \otimes [[t']] \otimes I_{[[\Gamma_2]]})$.

Then, by induction hypothesis,
$\overrightarrow{ev} \circ f = [[t_\alpha]] \circ (I_{[[A]]} \otimes I_{[[\Gamma_1]]} \otimes [[t']] \otimes I_{[[\Gamma_2]]}) =$
$= [[t_\alpha[x \leftharpoondown t'] : B[y : A, \Gamma_1, \Delta, \Gamma_2]]]$

Thus, $f = I_{[[A]]} \otimes [[\overrightarrow{\lambda}\, y.t_\alpha[x \leftharpoondown t'] : A \rightharpoonup B[\Gamma_1, \Delta, \Gamma_2]]]$. $(**)$

Therefore, by $(*)$ and $(**)$,
$[[\overrightarrow{\lambda}\, y.t_\alpha : A \rightharpoonup B[\Gamma_1, x : C, \Gamma_2]]] \circ (I_{[[\Gamma_1]]} \otimes [[t']] \otimes I_{[[\Gamma_2]]}) =$
$= [[\overrightarrow{\lambda}\, y.t_\alpha[x \leftharpoondown t'] : A \rightharpoonup B[\Gamma_1, \Delta, \Gamma_2]]]$.

(v) $t = \overleftarrow{\lambda}\, y.t_\alpha : B \leftharpoondown A[\Gamma_1, x : C, \Gamma_2]$

Analogous to the item (iv).

(vi) $t = LET(< y : A, z : B >= t_\beta)\, t_\alpha : D[\Gamma_{\alpha 1}, \Sigma, \Gamma_{\alpha 2}, \Gamma_1, x : C, \Gamma_2]$,
where $t_\alpha : D[\Gamma_{\alpha 1}, y : A, z : B, \Gamma_{\alpha 2}, \Gamma_1, x : C, \Gamma_2]$,
$t_\beta : A \otimes B[\Sigma]$, and

y and z do not occur in t'.

So, the following diagram commutes

$$
\begin{tikzcd}
{[[\Gamma_{\alpha 1}]] \otimes [[A]] \otimes [[B]] \otimes [[\Gamma_{\alpha 2}]] \otimes [[\Gamma_1]] \otimes [[C]] \otimes [[\Gamma_2]]} \arrow[r, "{[[t_\alpha]]}"] & {[[D]]} \\
{[[\Gamma_{\alpha 1}]] \otimes [[\Sigma]] \otimes [[\Gamma_{\alpha 2}]] \otimes [[\Gamma_1]] \otimes [[C]] \otimes [[\Gamma_2]]} \arrow[u, "{I_{[[\Gamma_{\alpha 1}]]} \otimes [[t_\beta]] \otimes I_{[[\Gamma_{\alpha 2}]]} \otimes I_{[[\Gamma_1]]} \otimes I_{[[C]]} \otimes I_{[[\Gamma_2]]}}"] \arrow[ur, "{[[t]]}"'] & \\
{[[\Gamma_{\alpha 1}]] \otimes [[\Sigma]] \otimes [[\Gamma_{\alpha 2}]] \otimes [[\Gamma_1]] \otimes [[\Delta]] \otimes [[\Gamma_2]]} \arrow[u, "{I_{[[\Gamma_{\alpha 1}]]} \otimes I_{[[\Sigma]]} \otimes I_{[[\Gamma_{\alpha 2}]]} \otimes I_{[[\Gamma_1]]} \otimes [[t']] \otimes I_{[[\Gamma_2]]}}"] &
\end{tikzcd}
$$

The following diagram also commutes,

$$
\begin{tikzcd}[column sep=small]
& {[[\Gamma_{\alpha 1}]] \otimes [[A]] \otimes [[B]] \otimes [[\Gamma_{\alpha 2}]] \otimes [[\Gamma_1]] \otimes [[C]] \otimes [[\Gamma_2]]} \arrow[dl, "\pi_1'''"'] \arrow[dr, "\pi_2'''"] & \\
{[[\Gamma_{\alpha 1}]] \otimes [[A]] \otimes [[B]] \otimes [[\Gamma_{\alpha 2}]]} \arrow[rr, "s'''"] & & {[[\Gamma_1]] \otimes [[C]] \otimes [[\Gamma_2]]} \\
{[[\Gamma_{\alpha 1}]] \otimes [[A]] \otimes [[B]] \otimes [[\Gamma_{\alpha 2}]]} \arrow[u, "I_{[[\Gamma_{\alpha 1}]]} \otimes I_{[[A]]} \otimes I_{[[B]]} \otimes I_{[[\Gamma_{\alpha 2}]]}"] \arrow[rr, "s''"] & & {[[\Gamma_1]] \otimes [[\Delta]] \otimes [[\Gamma_2]]} \arrow[u, "I_{[[\Gamma_1]]} \otimes [[t']] \otimes I_{[[\Gamma_2]]}"'] \\
& {[[\Gamma_{\alpha 1}]] \otimes [[A]] \otimes [[B]] \otimes [[\Gamma_{\alpha 2}]] \otimes [[\Gamma_1]] \otimes [[\Delta]] \otimes [[\Gamma_2]]} \arrow[ul, "\pi_1''"] \arrow[ur, "\pi_2''"'] \arrow[uuu, "g"] \arrow[dl, "\pi_1'"'] \arrow[dr, "\pi_2'"] & \\
{[[\Gamma_{\alpha 1}]] \otimes [[A]] \otimes [[B]] \otimes [[\Gamma_{\alpha 2}]]} \arrow[rr, "s'"] & & {[[\Gamma_1]] \otimes [[\Delta]] \otimes [[\Gamma_2]]} \\
{[[\Gamma_{\alpha 1}]] \otimes [[\Sigma]] \otimes [[\Gamma_{\alpha 2}]]} \arrow[u, "I_{[[\Gamma_{\alpha 1}]]} \otimes [[t_\beta]] \otimes I_{[[\Gamma_{\alpha 2}]]}"] \arrow[rr, "s"] & & {[[\Gamma_1]] \otimes [[\Delta]] \otimes [[\Gamma_2]]} \arrow[u, "I_{[[\Gamma_1]]} \otimes I_{[[\Delta]]} \otimes I_{[[\Gamma_2]]}"'] \\
& {[[\Gamma_{\alpha 1}]] \otimes [[\Sigma]] \otimes [[\Gamma_{\alpha 2}]] \otimes [[\Gamma_1]] \otimes [[\Delta]] \otimes [[\Gamma_2]]} \arrow[ul, "\pi_1"] \arrow[ur, "\pi_2"'] \arrow[uuu, "f"] &
\end{tikzcd}
$$

such that
$f = (I_{[[\Gamma_{\alpha 1}]]} \otimes [[t_\beta]] \otimes I_{[[\Gamma_{\alpha 2}]]}) \otimes (I_{[[\Gamma_1]]} \otimes I_{[[\Delta]]} \otimes I_{[[\Gamma_2]]})$ and
$g = (I_{[[\Gamma_{\alpha 1}]]} \otimes [[A]] \otimes [[B]] \otimes I_{[[\Gamma_{\alpha 2}]]}) \otimes (I_{[[\Gamma_1]]} \otimes [[t']] \otimes I_{[[\Gamma_2]]})$ are unique.

So,
$[[t_\alpha]] \circ g \circ f = [[t]] \circ (\,(I_{[[\Gamma_{\alpha 1}]]} \otimes I_{[[\Delta]]} \otimes I_{[[\Gamma_{\alpha 2}]]}) \otimes (I_{[[\Gamma_1]]} \otimes [[t']] \otimes I_{[[\Gamma_2]]})\,)$.

By induction hypothesis,
$[[t_\alpha]] \circ g = [[t_\alpha[x \leftarrow t'] : D[\Gamma_{\alpha 1}, y : A, z : B, \Gamma_{\alpha 2}, \Gamma_1, \Delta, \Gamma_2]\,]]$.

Then,
$[[t_\alpha]] \circ g \circ f =$
$= [[LET(< y : A, z : B >= t_\beta)t_\alpha[x \leftarrow t'] : D[\Gamma_{\alpha 1}, \Sigma, \Gamma_{\alpha 2}, \Gamma_1, \Delta, \Gamma_2]\,]]$.

Therefore,

$$[[LET(< y : A, z : B >= t_\beta)t_\alpha : D[\Gamma_{\alpha 1}, \Sigma, \Gamma_{\alpha 2}, \Gamma_1, x : C, \Gamma_2]\,]]$$
$$=$$
$$[[LET(< y : A, z : B >= t_\beta)t_\alpha[x \leftarrow t'] : D[\Gamma_{\alpha 1}, \Sigma, \Gamma_{\alpha 2}, \Gamma_1, \Delta, \Gamma_2]\,]]$$

Theorem 4.1 (Soundness) : *If $M \rhd N$ then we have $[[M]] = [[N]]$.*

Proof:To prove this theorem, it's sufficient to verify that a single application of a reduction rule on a term preserves the denotation of it. This is provided by the Substitution Lemma and by transitivity of the equality.

(i) $\overrightarrow{APP}\,(\overrightarrow{\lambda}\,x.t, t') : B[\Gamma, \Delta] \rhd t[x \leftarrow t'] : B[\Gamma, \Delta]$

The following diagram commutes

$$\begin{tikzcd}
{[[A \otimes (A \rightharpoonup B)]]} \arrow[r, "\overrightarrow{ev}"] & {[[B]]} \\
{[[A]] \otimes [[\Delta]]} \arrow[u, "I_{[[A]]} \otimes [[\overrightarrow{\lambda}\,x.t]]"] \arrow[ur, "{[[t]]}"] & \\
{[[\Gamma]] \otimes [[\Delta]]} \arrow[u, "{[[t']] \otimes I_{[[\Delta]]}}"] \arrow[uur, "f"'] &
\end{tikzcd}$$

So, by Substitution Lemma
$f = [[\overrightarrow{APP}\,(\overrightarrow{\lambda}\,x.t, t') : B[\Gamma, \Delta]\,]] = [[t : B[x : A, \Delta]\,]] \circ ([[t' : A[\Gamma]\,]] \otimes I_{[[\Gamma]]}) =$
$= [[t[x \leftarrow t'] : B[\Gamma, \Delta]\,]]$.

(ii) $A\overleftarrow{P}P\,(\overleftarrow{\lambda}\ x.t, t') : B[\Delta,\Gamma] \ \triangleright t[x \leftharpoondown t'] : B[\Delta,\Gamma]$

Considering the following diagram, the proof is analogous to the item (i).

$$\begin{tikzcd}
{[[(B \leftharpoondown A) \otimes A]]} \arrow[r, "\overleftarrow{ev}"] & {[[B]]} \\
{[[\Delta]] \otimes [[A]]} \arrow[u, "{[[\overleftarrow{\lambda}\, x.t]] \otimes I_{[[A]]}}"] \arrow[ur, "{[[t]]}"] & \\
{[[\Delta]] \otimes [[\Gamma]]} \arrow[u, "{I_{[[\Delta]]} \otimes [[t']]}"] \arrow[uur, "f"'] &
\end{tikzcd}$$

(iii) $LET(< x : A, y : B >=< t_1, t_2 >)t : C[\Gamma,\Delta_1,\Delta_2,\Gamma'] \ \triangleright \ t[x \leftharpoondown t_1, y \leftharpoondown t_2] : C[\Gamma,\Delta_1,\Delta_2,\Gamma']$

The following diagram commutes

$$\begin{tikzcd}
{[[\Gamma]] \otimes [[A]] \otimes [[B]] \otimes [[\Gamma']]} \arrow[r, "{[[t]]}"] & {[[C]]} \\
{[[\Gamma]] \otimes [[\Delta_1]] \otimes [[B]] \otimes [[\Gamma']]} \arrow[u, "{I_{[[\Gamma]]} \otimes [[t_1]] \otimes I_{[[B]]} \otimes I_{[[\Gamma']]}}"] \arrow[ur, "f"] & \\
{[[\Gamma]] \otimes [[\Delta_1]] \otimes [[\Delta_2]] \otimes [[\Gamma']]} \arrow[u, "{I_{[[\Gamma]]} \otimes I_{[[\Delta_1]]} \otimes [[t_2]] \otimes I_{[[\Gamma']]}}"] \arrow[uur, "g"] &
\end{tikzcd}$$

So, applying the Substituition Lemma twice,

$g = [[LET(< x : A, y : B >=< t_1, t_2 >)t : C[\Gamma,\Delta_1,\Delta_2,\Gamma']\ ]] =$

$= [[t : C[\Gamma, x : A, y : B, \Gamma']\ ]] \circ (I_{[[\Gamma]]} \otimes [[t_1 : A[\Delta_1]\ ]] \otimes I_{[[B]]} \otimes I_{[[\Gamma']]})\circ$

$(I_{[[\Gamma]]} \otimes I_{[[\Delta_1]]} \otimes [[t_2 : B[\Delta_2]\ ]] \otimes I_{[[\Gamma']]}) = [[t[x \leftharpoondown t_1, y \leftharpoondown t_2] : C[\Gamma,\Delta_1,\Delta_2,\Gamma']\ ]]$

5 Conclusion

We presented a construction of a noncommutative product in categories and a concrete model based on it. We must remark that we have adopted a denotational/categorical approach, in which the semantics treatment concerns on the operational features of the proof theory of a system. So, we do not have a completeness result like in logical approaches. Instead, it can be inquired whether the model is fully abstract or not, i.e., any morphism is a denotation of a proof. Our model is not fully abstract, since it was necessary to introduce sequencing, projections and evaluations arrows in order to construct a noncommutative product and such arrows do not denote proofs.

Other interesting features of our categorical framework can be pointed out. One of them is the syntactic role played by the Category $\mathcal{C}_0$, which represents the sintactic possibility of combining each pair of primitive types. However, this is particular feature of the Lambek Calculus. If we consider the sequential computational process, for instance, we know that there are pairs of processes which can not be sequenced. So, an another work is to consider a selective sequencing.

Another related work concerns on selective permutation of components of a sequence. One can think of a calculus similar to the Lambek Calculus, in which some formulas of a sequent can be permutated. This can be done by introducing a new operator which when applied to a formula determines that it can be permutated. It can capture the lazy evaluation process, in the sense that a marked formula can be moved in a sequence delaying its "evaluation".

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