Some Classes Containing a Fork Algebra Equivalent Variety Involving Projections and their Use to Specify Abstract Data Types

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Abstract. Some varieties that are extensions of relational algebras with two constants that play the role of projections are studied. They are obtained by weakening some laws valid in abstract fork algebra. Applications in the literature and in the specification of abstract data types are exhibited. For the subvarieties formed with the models that have a repressentable relational reduct, a repressentation theorem is proved. For them the finitization problem is studied. Next the varieties presented are compared by means of the inclusion order. The existence of equivalent classes with a binary operation like fork is studied. Simple models and elementary properties in the classes are studied.

Keywords: relational algebras, fork algebras, projection, relational specification, abstract data type, representation problem, finitization problem.

Resumo. Algumas variedades que são extensões de álgebras relacionais com duas constantes que desempenham o papel de projeções são estudadas. As variedades definidas são obtidas enfraquecendo algumas propriedades das fork álgebras abstratas (**AFA**). Se mostram aplicações na literatura e na especificação de tipos abstratos de dados. Se prova um teorema de representação para as subvariedades das classes onde os modelos tem reduto relacional representable. O problema de finitização para elas é estudado. As classes apresentadas são comparadas em termos da relação de inclusão. Se mostra que existem classes equivalentes que usam operações binarias com um comportamento parecido ao fork das **AFA**. Modelos simples e propriedades elementares são estudados nas classes.

Palavras chave: álgebras relacionais, fork álgebras abstratas, projeção, especificações relacionais, tipos abstratos de dados, problema de representação, problema de finitização.

1 Introduction

A solution of the representation problem for fork algebras is presented in [Gyu95, FHV95]. The defined class of proper fork algebras (**PFA**) is finitely axiomatizable using equations. The variety defined by these axioms is called abstract fork algebras (**AFA**). In section 3.2 is proved that the class **AFA** is equivalent to a variety with relational reduct included in **RA** and with projection elements.

Some varieties that are extensions of relational algebras with two constants (π *and* ρ) that play the role of projections are studied. The classes have as a subvariety the abstract fork algebra (**AFA**) equivalent variety involving projections. They are obtained by either weakening or eliminating some laws valid in **AFA**.

In the literature, examples of classes of algebras of interest in this paper, are defined in a language that contains the type of **RA** and their relational reducts are subsets of **RA**. In all of them the models contain "projection elements" (for example Q-relation algebras [TG87]). In some of these examples the language has two constants that play the role of "projection elements" (i.e.: pairing algebras [Mad89, AJN91, SN94], true pairing algebras [Mad81, Mad89, SN94, MSS92], full products [Mad93] and direct products [Mad93]).

There are also examples of heterogeneous relational specifications of structures that involve "projection elements". (i.e.: (heterogeneous) direct products [Sch81, BZ86, Ber91, Gri91, Zie91, SS93, BHSV93] and specification of stacks [Ber91, BS93]). In general these examples can be translated to homogeneous algebras with relational reduct in **RA** and with "projection elements". In [BHSV93] a comparison is made between the heterogeneous direct products and fork algebras. Section 3.2 shows a translation of the specification of heterogeneous direct products to full products [Mad93].

The property of **AFA** that $\pi and \rho$ are quasiprojections is weakened, resulting in the following five cases. $\breve{\pi};\rho$ could be:

1. 1 (π *and* ρ are quasiprojections)
2. an ideal
3. a rectangle
4. a homogeneous rectangle
5. an equivalence relation

It is also considered the case where nothing is said about $\breve{\pi};\rho$.

In addition it is assumed that π *and* ρ are functional elements.

The other two properties valid on **AFA** considered for the classes in this article are: π and ρ have the same domain and $\pi;\breve{\pi}\cdot\rho;\breve{\rho}\subseteq 1'$.

24 varieties which are combibnations of these equations have been defined.

Some applications of the varieties in the literature (subsection 3.2) and in the specification of abstract data types (subsection 3.3) are exhibited.

The first problem studied in this article (section 4.1) is to answer the question: Are the relational reducts of the classes representable? In [Mad78, Mad89, TG87] the answer is affirmative when the "projection elements" are quasiprojections, in the other cases the answer is negative. In [Mad93] there is an example for the class of full products (that was studied in this article).

Following in the section 4.2 the representation problem for the defined classes is considered. There are two choices for the classes to represent: The classes defined, or subclasses of classes that were defined.

In the first case, it is necessary to represent classes containing models whose relational reducts aren't representable. For this reason, the decision to work in this article with the subclasses formed from the models with relational reduct representable was taken.

Another problem is to choose the proper models for each class. The idea is to weaken appropriately the definition of **PFA**s that represent **AFA** obtaining one set of proper algebras for each class. The proper algebras obtained are algebras of binary relations on a structured set. The structure of this set is formed by the set, two unary functions and a ternary relation. The properties that this structure must satisfy depend on the class considered.

In adition, in section 4.2 the finitization problem is studied for the defined classes of proper models.

Another problem is the comparison of classes that involve "projection elements". There are some examples in the literature. For example the comparison of direct products of heterogeneous relation algebras with fork algebras [BHSV93],groupoid representable fork algebras with set fork algebras [SN94] and **AFA** with Q-relation algebras [SN94].

In this paper, the problem is treated in two sections. Section 3.2 compares the classes defined in this article with some well-known classes in the literature. In section 5, a comparison between the classes defined in this article is made using the inclusion order. Some examples of comparison of the relational reduct of a class with the relational reduct of another are given.

Another goal is to characterize the finite models of the classes (section 6). This is possible when the projection elements are quasiprojections. The finite models whose relational reduct is a discrete relation algebra are characterized for each class. When a class satisfies that $\breve{\pi};\rho$ is an ideal, it isn't possible to characterize its finite models because it is necessary to have a characterization of all finite simple relation algebras.

In section 7, simple models in the classes are studied. Because the congruence lattice of a model of a class is equal to the congruence lattice of its relational reduct, a similar result to the characterization theorem of simple RA [CT51, JT52, Jón82, Mad91a] is given. Using the characterization of representable simple relation algebras [JT52], the simple models of the classes are described.

In section 8, is considered the problem of finding classes that use a fork like binary operation such that are equivalent with the classes studied. It is solved for the classes where the projection elements have the same domain.

Finally, some elementary properties for the classes are derived in section 8. The pioneering paper Tarski [Tar41] was the first major attempt at an axiomatic investigation of the theory of **RA**. A comprehensive account of the arithmetic properties of **RA** can be found in Chin, Tarski [CT51]. Here, a lot of them are used to prove identities valid in the classes. Properties of fork algebras have been studied in [BHVW92, FBHV93, DB93, FAN93, HBS93]. Some properties of full products and direct products are presented in [Mad93]. There are some laws of heterogeneous direct products in [Des89]. Finally, a few properties of Q-relation algebras and pairing algebras can be found in [TG87, AJN91]. Inspired in the above references, some elementary properties for the classes are studied. They are some arithmetical properties like distributive and semidistributive laws and some properties useful for program derivation like simplification, domain, codomain and monotonic laws.

2 Preliminaries

Definition 2.1 *A heterogeneous language is a pair $L = \langle S, F\rangle$ where S is a set of sort symbols and F is a set of function symbols. Each function symbol has an arity that is a pair formed with a finite sequence of elements of S and an element of S. A homogeneous language L is a set of function symbols. Each function symbol has an arity that is a natural number.*

Definition 2.2 *Let $\mathcal{A}$ be an algebra. The language of $\mathcal{A}$ is denoted by $L(\mathcal{A})$. If s is a sort symbol in $L(\mathcal{A})$ the corresponding set in $\mathcal{A}$ is denoted by $s^{\mathcal{A}}$. If f is an operation symbol in $L(\mathcal{A})$ the corresponding operation in $\mathcal{A}$ is denoted by $f^{\mathcal{A}}$.*

Definition 2.3 *Let $\mathbf{K}$ be a class of algebras and $\mathcal{A} \in \mathbf{K}$. Then $L(\mathbf{K})$ is the language of $\mathbf{K}$ and $L(\mathbf{K}) = L(\mathcal{A})$.*

Definition 2.4 *Let $\mathcal{A}$ be an algebra such that $L(\mathcal{A}) = \langle S, F\rangle$. Let $L = \langle S', F'\rangle$ be a language such that $L \subseteq L(\mathcal{A})$. Then the L-reduct of $\mathcal{A}$ is the algebra $\mathcal{A}\downarrow L$ such that:*

- *if $s \in S'$ then $s^{\mathcal{A}} = s^{\mathcal{A}\downarrow L}$*
- *$L(\mathcal{A}\downarrow L) = L$*
- *If $f \in F'$ then $f^{\mathcal{A}\downarrow L} = f^{\mathcal{A}}$*

The simplification of this definition to homogeneous algebras is straightforward.

Definition 2.5 *Let $\mathbf{K}$ be a class of algebras such that $L(\mathbf{K}) = \langle S, F\rangle$. Let $L = \langle S', F'\rangle$ such that $L \subseteq L(\mathbf{K})$ then the L-reduct of $\mathbf{K}$ is the class $\mathbf{K} \downarrow L$ such that $\mathbf{K} \downarrow L = \{\mathcal{A} \downarrow L : \mathcal{A} \in \mathbf{K}\}$*

Definition 2.6 *Let L be a language and $\mathbf{K}$ be a class of models such that $L(\mathbf{K}) \subseteq L$. Then the class of language L with $L(\mathbf{K})$-reducts in $\mathbf{K}$ is*

$$\mathbf{K}\uparrow L = \{\mathcal{A} : L(\mathcal{A}) = L \text{ and } \mathcal{A}\downarrow L(\mathbf{K}) \in \mathbf{K}\}$$

Proposition 2.7 *Let $\mathbf{V}$ be a variety and $\mathbf{K}$ a class of algebras such that $\mathbf{K} \subseteq \mathbf{V}\uparrow L$. Then $\mathbf{S}(\mathbf{K}) \subseteq \mathbf{V}\uparrow L$ and $\mathbf{I}(\mathbf{K}) \subseteq \mathbf{V}\uparrow L$.*

Definition 2.8 *The language of relation algebras LRA is the sequence of symbols $[+, \cdot, ^-, 0, 1, ;, ^\smile, 1']$. The sequence of their arities is $[2, 2, 1, 0, 0, 2, 1, 0]$.*

Definition 2.9 *Let LRAP be the language LRA concatenated with $[\pi, \rho]$. The new symbols have arity 0.*

Definition 2.10 *An algebra* $\mathcal{R} = \langle R, +, \cdot, ^-, 0, 1, ;, ^\smile, 1' \rangle$ *with language LRA is a relation algebra iff*

- $\langle R, +, \cdot, ^-, 0, 1 \rangle$ *is a Boolean algebra.*
- $\langle R, ;, 1' \rangle$ *is a monoid.*
- *for all r,s,t in R,* $r; s \subseteq t \Leftrightarrow \breve{r}; \overline{t} \subseteq \overline{s} \Leftrightarrow \overline{t}; \breve{s} \subseteq \overline{r}$

$0'$ *is used to abbreviate* $1'^-$.

The class of all the relation algebras is denoted by **RA**.

Definition 2.11 *Let t be a term in a language L,* $LRA \subseteq L$. *The following abbreviations are made:*

- *Eq(t) for* $t \approx \breve{t} \wedge t \approx t; t$ *(t is an equivalence element)*
- *Ie(t) for* $t \approx 1; t; 1$ *(t is an ideal element)*
- *Fn(t) for* $\breve{t}; t \subseteq 1'$ *(t is a functional element)*
- *Ct(t) for* $Fn(t) \wedge t \approx 1; t \wedge \neg(t \approx 0)$ *(t is a constant element)*

Definition 2.12 *Let t be a term in a language L,* $LRA \subseteq L$. *The following abbreviations are made:*

- t^d *for* $(t; \breve{t}) \cdot 1'$ *(domain of t)*
- t^i *for* $(\breve{t}; t) \cdot 1'$ *(codomain of t)*

If $r \subseteq A \times B$, *the same notation is used to define* r^d *and* r^i *as:*

$$r^d = \{\langle x, x\rangle : (\exists y) y \in B \wedge xry\}$$
$$r^i = \{\langle y, y\rangle : (\exists x) x \in A \wedge xry\}$$

Definition 2.13 *Let A be a set. The diagonal relation associated to A is defined by* $\triangle_A = \{\langle x, x\rangle : x \in A\}$.
Let r be a binary relation and A a set, such that $r \subseteq \triangle_A$. *The set associated to r is defined by* $Set(r) = \{x \in A : \langle x, x\rangle \in r\}$

Theorem 2.14 *The following properties hold in every relation algebra:*

1. $\breve{0} \approx 0$, $\breve{1} \approx 1$, $\breve{1'} \approx 1'$
2. $\breve{\breve{r}} \approx r$ *and* $\breve{\overline{r}} \approx \overline{\breve{r}}$
3. $(r \cdot s)^\smile \approx \breve{r} \cdot \breve{s}$, $(r + s)^\smile \approx \breve{r} + \breve{s}$ *and* $(r; s)^\smile \approx \breve{s}; \breve{r}$
4. $(r \subseteq s) \wedge (t \subseteq q) \rightarrow (r; t) \subseteq (s; q)$

5. $r \subseteq 1' \rightarrow \breve{r} \approx r,\ r \subseteq 1' \rightarrow r;r \approx r$

6. $r, s \subseteq 1' \rightarrow r;s \approx r \cdot s$

7. $\breve{r}^i \approx r^d$ *and* $\breve{r}^d \approx r^i$

8. $(r+s)^i \approx r^i + s^i$ *and* $(r+s)^d \approx r^d + s^d$

9. $(r;s)^d \approx (r;s^d)^d$ *and* $(r;s)^i \approx (r^i;s)^i$

10. $r \approx r^d;r$ *and* $r \approx r;r^i$

11. $(t;\breve{t}) \cdot 1' \approx (t;1) \cdot 1'$ *and* $(\breve{t};t) \cdot 1' \approx (1;t) \cdot 1'$

12. $r \subseteq 1' \rightarrow r^d \approx r^i \approx r$

13. $Ie(r) \leftrightarrow Eq(r) \wedge Eq(\overline{r})$ *and* $Ie(r) \rightarrow r \cdot (x;y) \approx (r \cdot x);(r \cdot y)$

14. $1;1 \approx 1$

15. $r \subseteq 1' \rightarrow r \approx (r;1;r) \cdot 1'$

16. $(r;1 \cdot s);t \approx (r;1) \cdot (s;t)$ *and* $(r \cdot (s;1)\breve{\ });t \approx r;(t \cdot (s;1))$

17. $1;r^i \approx 1;r$ *and* $\breve{r};1 \approx r^i;1$

18. $(x;y \approx 0) \wedge (x^i \approx y^d) \leftrightarrow (x \approx 0) \wedge (y \approx 0)$

PROOF 1 - 6 are proved in [CT51]. The axiomatization of relation algebras of this paper is proved equivalent to the axiomatization of [CT51] in [CT51]. 13 and 14 are also proved in [CT51]. 7, 10 and 12 are proved in [Mad82, 5.11]. 8 is obvious. 16 is proved in [CT51, Thm.5.2]. 11 is in [Mad91b, Thm.8, xxi and xxii]. 9 is in [FBHV93, 6.4]. Below 15, 17 and 18 are proved.

15. By 4, 14, 11 and 12 $(r;1;r) \cdot 1' \subseteq (r;1) \cdot 1' = r$
By 4 and 5 $r \subseteq r;1,\ r;r \subseteq r;1;r$ *and* $r \subseteq (r;1;r) \cdot 1'$.

17. By 1, 2, 3 and 16 $1;r^i = 1;((1;r) \cdot 1') = (((1;r) \cdot 1')\breve{\ };1)\breve{\ } = (((\breve{r};1) \cdot 1');1)\breve{\ }$
$= ((\breve{r};1) \cdot 1)\breve{\ } = (\breve{r};1)\breve{\ } = 1;r$.

By 1 and 3 and the last identity $\breve{r};1 = (1;r)\breve{\ } = (1;r^i)\breve{\ } = r^i;1$

18. Suppose that $x;y = 0$ and $x^i = y^d$ By 10 and 9
$0 = (x;y)^i = (x^i;y)^i = (y^d;y)^i = y^i$ then $y = 0$
By 10 and 9 $0 = (x;y)^d = (x;y^d)^d = (x;x^i)^d = x^d$ then $x = 0$

□

Definition 2.15 *An algebra* $\mathcal{R} = \langle R, +, \cdot, ^-, 0, 1, ;, ^{\smile}, 1' \rangle$ *with language LRA is called a proper relation algebra (or a relation set algebra) if its universe R is a nonempty family of binary relations between elements of a set U, and its fundamental operations respectively coincide with the appropriately restricted set-theoretical notions* $\cup, \cap, ^{\sim}, \emptyset, \cup R, |, ^{-1}, \triangle_U$*. Here* $\sim r$ *for any* $r \in R$ *is the complement of r relative to the relation* $\cup R$*. If R consists of all subrelations of* $U \times U$ *then* $\mathcal{R}$ *is referred to as the full relation algebra on U and it is denoted by* $\mathcal{F}r(U)$*. The class of all proper relation algebras is denoted by* **PRA***.*

Definition 2.16 $\mathcal{R} \in \mathbf{RA}$ *is representable if it is isomorphic with a proper relation algebra. The class of representable relation algebras is denoted by* **RRA***.*

Definition 2.17 *Let LAFA be the language LRA concatenated with the symbol* ∇ *of arity two.*

The following definitions of abstract fork algebras and proper fork algebras can be found in [Gyu95, FHV95].

Definition 2.18 *An algebra* $\mathcal{R} = \langle R, +, \cdot, ^-, 0, 1, ;, ^{\smile}, 1', \nabla \rangle$ *is an abstract fork algebra iff for all* $r, s, u, v \in R$

1. $\langle R, +, \cdot, ^-, 0, 1, ;, ^{\smile}, 1' \rangle$ *is a relation algebra*
2. $r \nabla s = (r; (1' \nabla 1)) \cdot (s; (1 \nabla 1'))$
3. $(r \nabla s); (u \nabla v)\breve{} = (r; \breve{u}) \cdot (s; \breve{v})$
4. $(1' \nabla 1)\breve{}\, \nabla\, (1 \nabla 1')\breve{} \subseteq 1'$

We denote by AFA the class of all abstract fork algebras.

Definition 2.19 *An algebra* $\langle \mathcal{R}, \nabla \rangle$ *is called a proper fork algebra if* $\mathcal{R}$ *is a proper relation algebra with base U and unit element V, and there is an injective function* $* : V \rightarrow U$ *such that* $r \nabla s = \{\langle x, *(y, z) \rangle \mid x, y, z \in U, xry, xsz\}$ *whenever r and s belong to the universe of* $\mathcal{R}$*.*

3 Construction of the classes

3.1 Definition of the classes

Definition 3.1 *Let r and s be terms in a language L and $LRA \subseteq L$. Then the formula $\breve{r}; s \approx \breve{r}; 1; s$ is abbreviated to Rn(r,s). ("rectangle")*

Definition 3.2 *To define the classes, the following formulas in the language $LRAP$ are considered:*

Q: $\breve{\pi}; \rho \approx 1$

R: $Rn(\pi, \rho)$

I: $Ie(\breve{\pi}; \rho)$

E: $Eq(\breve{\pi}; \rho)$

S: $Rn(\pi, \rho) \wedge 1; \pi \approx 1; \rho$

P: $(\pi; \breve{\pi}) \cdot (\rho; \breve{\rho}) \subseteq 1'$

D: $\pi; 1 \approx \rho; 1$

In the following, the names of the items are used for referring to the formulas.

Definition 3.3 *Let Sa be the following set of sets of axioms in LRAP:*
$\{\Sigma : \Sigma \subset \{Q, R, I, E, S, P, D\} \ \wedge \ at\, most\, one\, element\, of\, \{Q, R, I, E, S\}\, is\, in\, \Sigma\}$

Sa has 24 sets of axioms.

Definition 3.4 *For each $\Sigma \in Sa$ a class $\mathbf{K}_{\mathbf{\Sigma}}$ is defined such that:*
$\mathbf{K}_{\mathbf{\Sigma}} = \{\mathcal{R} : \mathcal{R}\!\downarrow LRA \in \mathbf{RA} \wedge \mathcal{R} \models \Sigma \cup \{Fn(\pi), Fn(\rho)\}\}$

3.2 Motivations

In this subsection, conections of $\mathbf{K}_{\mathbf{\Sigma}}$ classes with some well-known structures in the literature are studied.

$\mathbf{K}_{\{\mathbf{Q}\}}\!\downarrow LRA$ is the class of *Q-relation algebras* defined in [TG87, Def.8.4, p.242]. $\mathbf{K}_{\{\mathbf{R},\mathbf{P},\mathbf{D}\}}$ is the class of *full products* (See [Mad93]). $\mathbf{K}_{\{\mathbf{P},\mathbf{D}\}}$ is the class of *direct products* (presented in [Mad93]). $\mathbf{K}_{\{\mathbf{Q}\}} \cap \{\mathcal{R} : \mathcal{R} \models 0'; 0' \approx 1\}$ is the class of *pairing algebras* defined by Maddux in [Mad81, Def.17] (see also [AJN91], [Mad89] and [MSS92]).

Theorem 3.5 *The following statements hold in every relation algebra.*

1. $\breve{r}; 1; s \approx r^i; 1; s^i$
2. $r; 1 \approx s; 1 \leftrightarrow r^d \approx s^d$
3. $1; r \approx 1; s \leftrightarrow r^i \approx s^i$
4. $Eq(\breve{r}; s) \rightarrow 1; (r^d \cdot s^d); r \approx 1; (r^d \cdot s^d); s$
   $Eq(\breve{r}; s) \wedge r; 1 \approx s; 1 \rightarrow 1; r \approx 1; s$

PROOF

1. By 2.14.17 $\breve{r}; 1; s = r^i; 1; s = r^i; 1; s^i$.
2. $(\rightarrow)$ This implication is obvious.
   $(\leftarrow)$ By 2.14.16

$$\begin{aligned} r; 1 &= (r; 1) \cdot 1 = ((r; 1) \cdot 1'); 1 = ((\breve{r}; r) \cdot 1'); 1 \\ &= ((\breve{s}; s) \cdot 1'); 1 = ((s; 1) \cdot 1'); 1 = (s; 1) \cdot 1 = s; 1. \end{aligned}$$

3. By 2 and 2.14.[7,3] $1; r = 1; s \leftrightarrow \breve{r}; 1 = \breve{s}; 1 \leftrightarrow \breve{r}^d = \breve{s}^d \leftrightarrow r^i = s^i$.
4. By 2.14.[9,7,10,6] $(\breve{r}; s)^i = (\breve{r}^i; s)^i = (r^d; s)^i = ((r^d \cdot s^d); s)^i$.
   By 2.14.[9,7,10,6] $(\breve{s}; r)^i = (\breve{s}^i; r)^i = (s^d; r)^i = ((r^d \cdot s^d); r)^i$.
   But $\breve{r}; s$ is an equivalence element. Then $(\breve{r}; s)^i = (\breve{s}; r)^i$.
   Finally by 3. $1; (r^d \cdot s^d); r \approx 1; (r^d \cdot s^d); s$. If $r; 1 = s; 1$ then by 2. $r^d = s^d$ and $1; r = 1; s$ follows.

$\square$

There exist relation algebras that have r and s such that
$\neg(\neg(r; 1 \approx s; 1) \wedge Eq(\breve{r}; s) \rightarrow (1; r \approx 1; s))$ hold. (See 4.2.6.)

Now it is proved that the varieties of abstract fork algebras and $\mathbf{K}_{\{\mathbf{Q},\mathbf{P},\mathbf{D}\}}$ are equivalent.

Theorem 3.6 *(a) Let* $\mathcal{R} = \langle R, +, \cdot, ^-, 0, 1, ;, ^\smile, 1', \nabla \rangle \in AFA$ *then the algebra* $\langle R, +, \cdot, ^-, 0, 1, ;, ^\smile, 1', \pi, \rho \rangle$ *where* $\pi = (1'\nabla 1)^\smile$ *and* $\rho = (1\nabla 1')^\smile$ *is in* $\mathbf{K}_{\{\mathbf{Q},\mathbf{P},\mathbf{D}\}}$. *(b) Let* $\mathcal{R} = \langle R, +, \cdot, ^-, 0, 1, ;, ^\smile, 1', \pi, \rho \rangle \in \mathbf{K}_{\{\mathbf{Q},\mathbf{P},\mathbf{D}\}}$ *then the algebra* $\langle R, +, \cdot, ^-, 0, 1, ;, ^\smile, 1', \nabla \rangle$ *where for all* $r, s \in R$ $r\nabla s = (r; \breve{\pi}) \cdot (s; \breve{\rho})$ *is in* AFA.

PROOF

(a) By 2.18.3 and 2.14.[1,2]
$\breve{\pi}; \rho = (1'\nabla 1); (1\nabla 1')\breve{} = (1'; \breve{1}) \cdot (1; (\breve{1'})) = 1.$
By 2.18.3 and 2.14.14
$\breve{\pi}; \pi = (1'\nabla 1); (1'\nabla 1)\breve{} = 1' \cdot 1 = 1'.$
In the same way $\breve{\rho}; \rho = 1'$ (by 2.18.3 and 2.14.14).
By 2.18.[2,4]
$(\pi; \breve{\pi}) \cdot (\rho; \breve{\rho}) = ((1'\nabla 1); (1'\nabla 1)\breve{}) \cdot ((1\nabla 1'); (1\nabla 1')\breve{}) = (1'\nabla 1)\breve{}\nabla(1\nabla 1')\breve{} \subseteq 1'.$
By 2.18.2 and 2.14.[1,3]
$\pi = (1'\nabla 1)\breve{} = ((1'; (1'\nabla 1)) \cdot (1; (1\nabla 1')))\breve{} = (1'\nabla 1)\breve{} \cdot ((1\nabla 1')\breve{}; 1) = \pi \cdot (\rho; 1).$
In the same way $\rho = \rho \cdot (\pi; 1)$ (by 2.18.2 and 2.14.[1,3]).
By 2.14.16
$\pi; 1 = (\pi \cdot (\rho; 1)); 1 = (\pi; 1) \cdot (\rho; 1) = (\rho \cdot (\pi; 1)); 1 = \rho; 1$

(b) By **D** and 2.14.[1,3]
$(1'\nabla 1) = (1'; \breve{\pi}) \cdot (1; \breve{\rho}) = (\pi \cdot (\rho; 1))\breve{} = \breve{\pi}.$
In the same way $(1\nabla 1') = \breve{\rho}$ (by **D** and 2.14.[1,3]).
$r\nabla s = (r; \breve{\pi}) \cdot (s; \breve{\rho}) = (r; (1'\nabla 1)) \cdot (s; (1\nabla 1')).$
By **P** and 2.14.2
$(1'\nabla 1)\breve{}\nabla(1\nabla 1')\breve{} = ((1'\nabla 1)\breve{}; \breve{\pi}) \cdot ((1\nabla 1')\breve{}; \breve{\rho}) = (\pi; \breve{\pi}) \cdot (\rho; \breve{\rho}) \subseteq 1'.$

Because $\pi\ and\ \rho$ are quasiprojections by [TG87, 4.1(viii), p.97] and 2.14.3
$$(r; \breve{s}) \cdot (t; \breve{q}) = [(r; \breve{\pi}) \cdot (t; \breve{\rho})]; [(\pi; \breve{s}) \cdot (\rho; \breve{q})] = (r\nabla t); ((s; \breve{\pi}) \cdot (q; \breve{\rho}))\breve{}$$
$$= (r\nabla t); (s\nabla q)\breve{}$$

□

The following relational specification of direct products is presented in [Ber91, Section 4.2, p.9] (See also [SS93] and [BZ86]).

Spec PROD $\equiv$
param **types** m, n
rels None
laws None
target **types** prod
rels $\pi : prod \rightarrow m$
$\rho : prod \rightarrow n$
laws $\pi^T \pi = I$
$\rho^T \rho = I$
$\pi\pi^T \cap \rho\rho^T = I$
$\pi^T \rho = L$
end

This specification can be translated into a first order description of a class of algebras. In [BHSV93] this approach to products is compared with fork algebras.

Definition 3.7 *An algebra* $\langle \mathcal{A}, \pi, \rho, 1'_m, 1'_n, 1'_{prod} \rangle$ *where* $\mathcal{A} \in \mathbf{RA}$ *is a homogeneous direct product iff*

1. $1'_m, 1'_n, 1'_{prod} \subseteq 1'$
2. $\breve{\pi}; \pi = 1'_m$ *and* $\breve{\rho}; \rho = 1'_n$
3. $\pi \subseteq 1'_{prod}; 1; 1'_m$ *and* $\rho \subseteq 1'_{prod}; 1; 1'_n$
4. $(\pi; \breve{\pi}) \cdot 1'_{prod} = (\rho; \breve{\rho}) \cdot 1'_{prod}$
5. $(\pi; \breve{\pi}) \cdot (\rho; \breve{\rho}) = 1'_{prod}$
6. $\breve{\pi}; \rho = 1'_m; 1; 1'_n$

The class of homogeneous direct products is denoted by $\mathbf{HDP}$.

Finally, it is showed that the varieties $\mathbf{HDP}$ and $\mathbf{K}_{\{\mathbf{R},\mathbf{P},\mathbf{D}\}}$ are equivalent.

Theorem 3.8 *(a) Let* $\langle \mathcal{A}, \pi, \rho, 1'_m, 1'_n, 1'_{prod} \rangle \in \mathbf{HDP}$ *then the algebra* $\langle \mathcal{A}, \pi, \rho \rangle$ *is in* $\mathbf{K}_{\{\mathbf{R},\mathbf{P},\mathbf{D}\}}$ *-the class of full products-. (b) Let* $\langle \mathcal{A}, \pi, \rho \rangle \in \mathbf{K}_{\{\mathbf{R},\mathbf{P},\mathbf{D}\}}$ *then the algebra* $\langle \mathcal{A}, \pi, \rho, \pi^i, \rho^i, \pi^d \rangle \in \mathbf{HDP}$.

PROOF

(a) By 3.7.[1,5] $\mathbf{P}$ is true. By 3.7.3 and 2.14.[4,15]
$1' \cdot (\pi; \breve{\pi}) \subseteq 1' \cdot (1'_{prod}; 1; 1'_m; 1'_m; 1; 1'_{prod}) \subseteq 1' \cdot (1'_{prod}; 1; 1'_{prod}) = 1'_{prod}$.
In the same way $1' \cdot (\rho; \breve{\rho}) \subseteq 1'_{prod}$
By 3.7.4 $1' \cdot (\rho; \breve{\rho}) = 1'_{prod} \cdot (\rho; \breve{\rho}) = 1'_{prod} \cdot (\pi; \breve{\pi}) = 1' \cdot (\pi; \breve{\pi})$.
Finally by 3.5.2 $\mathbf{D}$ holds.
By 3.7.[2,6] $\breve{\pi}; \rho = 1'_m; 1; 1'_n = \pi^i; 1; \rho^i$
By 3.5.1 $\mathbf{R}$ holds.

(b) 3.7.1 Obviously holds.
3.7.2 holds because π *and* ρ are functionals, $\breve{\pi}; \pi = \pi^i$ and $\breve{\rho}; \rho = \rho^i$.
Now 3.7.3 is proved.
By 2.14.[10,4] $\pi \subseteq \pi^d; 1; \pi^i$
By $\mathbf{D}$ and 3.5.2 $\pi^d = \rho^d$
Finally by 2.14.[10,4] $\rho \subseteq \rho^d; 1; \rho^i = \pi^d; 1; \rho^i$
3.7.4 is proved by using $\mathbf{D}$ and 3.5.2
$(\pi; \breve{\pi}) \cdot \pi^d = (\pi; \breve{\pi}) \cdot 1' \cdot \pi^d = (\rho; \breve{\rho}) \cdot 1' \cdot \pi^d = (\rho; \breve{\rho}) \cdot \pi^d$.
3.7.5 is proved using $\mathbf{P}$.
$(\pi; \breve{\pi}) \cdot (\rho; \breve{\rho}) = (\pi; \breve{\pi}) \cdot (\rho; \breve{\rho}) \cdot 1' \subseteq \pi^d$.

□

3.3 Applications to the specification of abstract data types

The aim of this subsection is to show examples that motivate the selection of the properties used to build the classes $\mathbf{K}_{\mathbf{\Sigma}}$.

The $\mathbf{K}_{\mathbf{\Sigma}}$ classes can be used to specify selectors for abstract data types. The projection elements play the role of selectors.

Axioms of Full products ($\{\mathbf{R}, \mathbf{P}, \mathbf{D}\}$) can be used to specify selectors for stacks ([Des89]) and lists ([Gar96]). Here, stacks are specified.
$\{\mathbf{R}, \mathbf{P}, \mathbf{D}\}[\pi \rightarrow pop, \rho \rightarrow top]$
The $\rightarrow$ notation is used to say that π and ρ are renamed with the words on the right of them.
$empty; pop \approx 0$
$empty; top \approx 0$
$Ct(empty)$
$pop^i \approx 1_{stack}$
$(empty\breve{}; empty + (pop; pop\breve{} \cdot top; top\breve{})) \approx 1_{stack}$

In the above example the condition $\mathbf{D}$ can be deleted ([THE88]):
$\{\mathbf{R}, \mathbf{P}\}[\pi \rightarrow pop, \rho \rightarrow top]$
$empty; top \approx 0$
$empty; pop \approx empty$
$Ct(empty)$
$pop^i \approx 1_{stack}$
$(empty\breve{}; empty + (pop; pop\breve{} \cdot top; top\breve{})) \approx 1_{stack}$
It is easy to show that $\neg(top^d \approx pop^d)$.

The $\mathbf{S}$ condition can be used to specify binary trees with internal nodes without information ([Des89]).
$\{\mathbf{S}, \mathbf{P}, \mathbf{D}\}[\pi \rightarrow left, \rho \rightarrow right]$
$make; make\breve{} \approx 1_N$
$make; left \approx 0$
$make; right \approx 0$
$left^i \approx 1_T$
$make\breve{}; make + ((left; left\breve{}) \cdot (right; rigth\breve{})) \approx 1_T$
$1_T \subseteq 1$

The $\mathbf{E}$ statement can be used to specify a dictionary of synonyms.
$\{\mathbf{E}, \mathbf{P}, \mathbf{D}\}[\pi \rightarrow word, \rho \rightarrow synonym]$
$isSynonymOf \approx synonym\breve{}; word$

The **I** condition can be used in the above example.
$\{\mathbf{I}, \mathbf{P}, \mathbf{D}\}[\pi \rightarrow word, \rho \rightarrow synonym]$
$isSynonymOf \approx synonym\breve{}; word$

But this implies that the complement of $word\breve{}; synonym$ is an equivalence relation. Suppose for example that
$1 \approx 1_{word} + 1_{person}$
$word^i \approx synonym^i \approx 1_{word}$
$0 \approx 1_{word} \cdot 1_{person}$
Due to the **I** condition, the models that satisfy the above equations don't have relations with domain contained in 1_{word} and codomain contained in 1_{person}.

In the specification of a dictionary none of the conditions **Q, R, S, E** and **I** are needed.
$\{\mathbf{P}, \mathbf{D}\}[\pi \rightarrow word, \rho \rightarrow definition]$
$dictionary \approx word\breve{}; definition.$

In the specification of an abstract data type that needs more than two selectors, more than one $\mathbf{K}_{\mathbf{\Sigma}}$ class is needed. For example binary trees.
$\{\mathbf{R}, \mathbf{D}\}[\pi \rightarrow left, \rho \rightarrow root]$
$\{\mathbf{R}, \mathbf{D}\}[\pi \rightarrow right, \rho \rightarrow root]$
$\{\mathbf{S}, \mathbf{D}\}[\pi \rightarrow left, \rho \rightarrow right]$
$root\breve{}; root \approx 1_N$
$left^i \approx 1_T$
$(left; left\breve{}) \cdot (right; rigth\breve{}) \cdot root; root\breve{} \approx 1_T$

4 Representation results

4.1 Representation of the relational reducts of the classes

The question to answer is if it is true that $\mathbf{K}_{\mathbf{\Sigma}} \downarrow LRA \subseteq \mathbf{RRA}$.
In [Mad78, Cor.8, p.168] it is proved that the answer is affirmative when $\mathbf{Q} \in \Sigma$ (there is another proof in [TG87, p.243]).
The answer is negative if $\{\mathbf{P}, \mathbf{R}, \mathbf{D}\} \subseteq \Sigma$. In [Mad93, Thr.5, p.7] there is a counterexample.

Theorem 4.1 $\langle \mathcal{R}, \pi, \rho \rangle \in \mathbf{K}_{\mathbf{\Sigma} \cup \{\mathbf{I}\}}$ *iff there exists* $\langle \mathcal{R}_1, \pi_1, \rho_1 \rangle \in \mathbf{K}_{\mathbf{\Sigma} \cup \{\mathbf{Q}\}}$ *and* $\langle \mathcal{R}_2, \pi_2, \rho_2 \rangle \in \mathbf{K}_{\mathbf{\Sigma} \cup \{\mathbf{I}\}}$ *where* $\breve{\pi}_2; \rho_2 = 0^{\mathcal{R}_2}$ *such that* $\langle \mathcal{R}, \pi, \rho \rangle$ *is isomorphic to* $\langle \mathcal{R}_1, \pi_1, \rho_1 \rangle \times \langle \mathcal{R}_2, \pi_2, \rho_2 \rangle$.

PROOF

($\Rightarrow$) To simplify the notation names of LRA are used for the operations of $\mathcal{R}$.
For $r \in \mathcal{R}$ define $B_r = \{s \in \mathcal{R} : s \subseteq r\}$.
Let r be an ideal. In [JT52, Thr.4.9] is defined the algebra $\mathcal{R}(r)$ by $\langle B_r, +, \cdot, ^{-^r}, 0, r, \breve{}, ;, 1' \cdot r\rangle$ where $s^{-^r} = s^- \cdot r$ is a relational algebra and the function $\Phi_r : \mathcal{R} \rightarrow \mathcal{R}(r)$ where $\Phi_r(s) = s \cdot r$ is an epimorphism.
Let $\Phi : \mathcal{R} \rightarrow \mathcal{R}(\breve{\pi};\rho) \times \mathcal{R}((\breve{\pi};\rho)^-)$ where $\Phi(s) = \langle \Phi_{\breve{\pi};\rho}(s), \Phi_{((\breve{\pi};\rho)^-)}(s)\rangle$.
By [JT52, Thr.4.12 and Thr.4.13] Φ is an isomorphism.
It is obvious that Φ is an isomorphism from $\langle \mathcal{R}, \pi, \rho\rangle$ to $\langle \mathcal{R}(\breve{\pi};\rho), \pi \cdot (\breve{\pi};\rho), \rho \cdot (\breve{\pi};\rho)\rangle \times \langle \mathcal{R}(\breve{\pi};\rho)^-, \pi \cdot (\breve{\pi};\rho)^-, \rho \cdot (\breve{\pi};\rho)^-\rangle$
and $\langle \mathcal{R}(\breve{\pi};\rho)^-, \pi \cdot (\breve{\pi};\rho)^-, \rho \cdot (\breve{\pi};\rho)^-\rangle \in \mathbf{K}_{\mathbf{\Sigma}\cup\{\mathbf{I}\}}$.

By 2.14.[3,13] $\pi \cdot (\breve{\pi};\rho)$ and $\rho \cdot (\breve{\pi};\rho)$ are quasiprojections.

$$
\begin{aligned}
(\pi \cdot (\breve{\pi};\rho))\breve{};(\rho \cdot (\breve{\pi};\rho)) &= (\breve{\pi} \cdot (\breve{\pi};\rho)\breve{});(\rho \cdot (\breve{\pi};\rho)) \\
&= (\breve{\pi} \cdot (\breve{\pi};\rho));(\rho \cdot (\breve{\pi};\rho)) \\
&= (\breve{\pi};\rho) \cdot (\breve{\pi};\rho) = (\breve{\pi};\rho)
\end{aligned}
$$

Finally by 2.14.[2,3,13]

$$(\pi \cdot (\breve{\pi};\rho)^-)\breve{});(\rho \cdot (\breve{\pi};\rho)^-) = (\breve{\pi} \cdot (\breve{\pi};\rho)^{\breve{}-});(\rho \cdot (\breve{\pi};\rho)^-) = (\breve{\pi};\rho) \cdot (\breve{\pi};\rho)^- = 0.$$

($\Leftarrow$) It is obvious that $\langle \mathcal{R}_1, \pi_1, \rho_1\rangle \in \mathbf{K}_{\mathbf{\Sigma}\cup\{\mathbf{I}\}}$. Then $\langle \mathcal{R}, \pi, \rho\rangle \in \mathbf{K}_{\mathbf{\Sigma}\cup\{\mathbf{I}\}}$.

□

Corollary 4.2 $\langle \mathcal{R}, \pi, \rho\rangle \in \mathbf{K}_{\mathbf{\Sigma}\cup\{\mathbf{I},\mathbf{D}\}}$ *iff there exist* $\langle \mathcal{R}_1, \pi_1, \rho_1\rangle \in \mathbf{K}_{\mathbf{\Sigma}\cup\{\mathbf{Q},\mathbf{D}\}}$ *and* $\langle \mathcal{R}_2, 0^{\mathcal{R}_2}, 0^{\mathcal{R}_2}\rangle \in \mathbf{RA}\!\uparrow LRAP$ *such that* $\langle \mathcal{R}, \pi, \rho\rangle \cong \langle \mathcal{R}_1, \pi_1, \rho_1\rangle \times \langle \mathcal{R}_2, 0^{\mathcal{R}_2}, 0^{\mathcal{R}_2}\rangle$.

PROOF

($\Rightarrow$) Using the last theorem, there exists $\langle \mathcal{R}_1, \pi_1, \rho_1\rangle \in \mathbf{K}_{\mathbf{\Sigma}\cup\{\mathbf{Q}\}}$ and $\langle \mathcal{R}_2, \pi_2, \rho_2\rangle \in \mathbf{K}_{\mathbf{\Sigma}\cup\{\mathbf{I}\}}$ where $\breve{\pi}_2;\rho_2 = 0_2^{\mathcal{R}}$ such that $\langle \mathcal{R}, \pi, \rho\rangle \cong \langle \mathcal{R}_1, \pi_1, \rho_1\rangle \times \langle \mathcal{R}_2, \pi_2, \rho_2\rangle$.
Then $\pi_2;1 = \rho_2;1$. By 3.5.2 $\pi_2^d = \rho_2^d$. Finally by 2.14.18 $\pi_2 = \rho_2 = 0$.

($\Leftarrow$) The two algebras of the product satisfy $\mathbf{D}$.

□

In 5.2.5, it is constructed a model $\langle \mathcal{R}, \pi, \rho\rangle \in \mathbf{K}_{\{\mathbf{I}\}}$ such that there exists $\langle \mathcal{R}_1, \pi_1, \rho_1\rangle$ in $\mathbf{K}_{\{\mathbf{Q}\}}$ and $\langle \mathcal{R}_2, \pi_2, \rho_2\rangle$ in $\mathbf{K}_{\{\mathbf{I}\}}$ where $\breve{\pi}_2;\rho_2 = 0^{\mathcal{R}_2}$ such that $\langle \mathcal{R}, \pi, \rho\rangle$ is isomorphic to $\langle \mathcal{R}_1, \pi_1, \rho_1\rangle \times \langle \mathcal{R}_2, \pi_2, \rho_2\rangle$ and $\pi_2 \neq 0^{\mathcal{R}_2}$ and $\rho_2 \neq 0^{\mathcal{R}_2}$

Theorem 4.3 *There exists an algebra* $\langle \mathcal{R}, \pi, \rho\rangle \in \mathbf{K}_{\mathbf{\Sigma}\cup\{\mathbf{I}\}}$ *such that* $\mathcal{R}$ *is not in* **RRA**.

PROOF Let $\langle\mathcal{R},\pi,\rho\rangle \in \mathbf{K}_{\mathbf{\Sigma}\cup\{\mathbf{Q},\mathbf{D}\}}$ and $\mathcal{S} \in \mathbf{RA} \sim \mathbf{RRA}$. By the last corollary $\langle\mathcal{R},\pi,\rho\rangle\times\langle\mathcal{S},0^{\mathcal{S}},0^{\mathcal{S}}\rangle \in \mathbf{K}_{\mathbf{\Sigma}\cup\{\mathbf{I},\mathbf{D}\}}$. If $\mathcal{R}\times\mathcal{S} \in \mathbf{RRA}$ because $\mathbf{RRA}$ is a variety $\mathcal{S} \in \mathbf{RRA}$. This is a contradiction.
□
From the finite example of the last theorem, an infinite example can be constructed.
If $\langle\mathcal{R},\pi,\rho\rangle \in \mathbf{K}_{\mathbf{\Sigma}\cup\{\mathbf{I},\mathbf{D}\}}$ and $\mathcal{R} \in \mathbf{RA} \sim \mathbf{RRA}$ then $\langle\mathcal{R},\pi,\rho\rangle^{\omega} \in \mathbf{K}_{\mathbf{\Sigma}\cup\{\mathbf{I},\mathbf{D}\}}$
If $\mathcal{R}^{\omega} = \langle\mathcal{R},\pi,\rho\rangle^{\omega} \downarrow LRA \in \mathbf{RRA}$ then $\mathcal{R} \in \mathbf{RRA}$ because $\mathbf{RRA}$ is a variety. This is a contradiction.

Theorem 4.4 *If $\langle\mathcal{R},\pi,\rho\rangle \in \mathbf{K}_{\mathbf{\Sigma}\cup\{\mathbf{I},\mathbf{D}\}}$ then* $\mathbf{S}$ *holds.*

PROOF By 4.2 there exist $\langle\mathcal{R}_1,\pi_1,\rho_1\rangle$ in $\mathbf{K}_{\mathbf{\Sigma}\cup\{\mathbf{Q},\mathbf{D}\}}$ and $\langle\mathcal{R}_2,0^{\mathcal{R}_2},0^{\mathcal{R}_2}\rangle$ in $\mathbf{RA}\uparrow LRAP$ such that $\langle\mathcal{R},\pi,\rho\rangle \cong \langle\mathcal{R}_1,\pi_1,\rho_1\rangle\times\langle\mathcal{R}_2,0^{\mathcal{R}_2},0^{\mathcal{R}_2}\rangle$. $Rn(\pi,\rho)$ holds in $\langle\mathcal{R}_1,\pi_1,\rho_1\rangle$ because by 2.14.4 $\breve{\pi}_1;\rho_1 \subseteq \breve{\pi}_1;1;\rho_1 \subseteq 1 = \breve{\pi}_1;\rho_1$. $Rn(\pi,\rho)$ trivially holds in $\langle\mathcal{R}_2,0^{\mathcal{R}_2},0^{\mathcal{R}_2}\rangle$.
By 3.5.4 and 2.14.13 $1;\pi = 1;\rho$.
□
The theorem 4.3 says that the answer to the question of this section is negative if $\mathbf{E} \in \Sigma$ or $\mathbf{S} \in \Sigma$ or $\mathbf{R} \in \Sigma$. If $\langle\mathcal{R},\pi,\rho\rangle \in \mathbf{K}_{\mathbf{\Sigma}\cup\{\mathbf{I},\mathbf{D}\}}$ then by 4.4 and 2.14.13 $\langle\mathcal{R},\pi,\rho\rangle \in \mathbf{K}_{\mathbf{\Sigma}\cup\{\mathbf{E}\}}, \mathbf{K}_{\mathbf{\Sigma}\cup\{\mathbf{S}\}}, \mathbf{K}_{\mathbf{\Sigma}\cup\{\mathbf{R}\}}$.

4.2 Representation theorems for the classes

In the literature, there exist representation theorems for the variety of fork algebras (cf. e.g. [FHV95] and [Gyu95]). In 3.6 it is proved that the variety of fork algebras is equivalent with the class $\mathbf{K}_{\{\mathbf{Q},\mathbf{P},\mathbf{D}\}}$. By [Mad78] it is obvious that $\mathbf{K}_{\{\mathbf{Q},\mathbf{P},\mathbf{D}\}} \subseteq \mathbf{K}_{\mathbf{\Sigma}} \cap (\mathbf{RRA} \upharpoonright LRAP)$ for each Σ.

Because the classes $\mathbf{K}_{\mathbf{\Sigma}}$ where $\mathbf{Q}$ is not in Σ don't have their relational reduct representable, the aim of this section is to prove that the classes $\mathbf{K}_{\mathbf{\Sigma}} \cap (\mathbf{RRA} \upharpoonright LRAP)$ are representable.

From the definition of **PFA** in [FHV95], the set $\mathbf{Pr}(\mathbf{K}_{\{\mathbf{Q},\mathbf{P},\mathbf{D}\}})$ of the proper elements of $\mathbf{K}_{\{\mathbf{Q},\mathbf{P},\mathbf{D}\}}$ is built.

The definition of $\mathbf{Pr}(\mathbf{K}_{\{\mathbf{Q},\mathbf{P},\mathbf{D}\}})$ is generalized to obtain the set $\mathbf{Pr}(\mathbf{K}_{\mathbf{\Sigma}})$ of proper elements of $\mathbf{K}_{\mathbf{\Sigma}}$ satisfying $\mathbf{Pr}(\mathbf{K}_{\{\mathbf{Q},\mathbf{P},\mathbf{D}\}}) \subseteq \mathbf{Pr}(\mathbf{K}_{\mathbf{\Sigma}})$.

In this section, the representation theorem in [FHV95] is generalized and it is proved that $\mathbf{K}_{\mathbf{\Sigma}} \cap (\mathbf{RRA} \upharpoonright LRAP)$ is representable (by $\mathbf{Pr}(\mathbf{K}_{\mathbf{\Sigma}})$).

It is reasonable to use theorem 3.6.(a) to define $\mathbf{Pr}(\mathbf{K}_{\{\mathbf{Q},\mathbf{P},\mathbf{D}\}})$ as the set of the corresponding elements of **PFA**.

Definition 4.5 *Let V be an equivalence relation on U and $* \subseteq V \times U$. The following relations on U are defined:*

$$\widehat{\pi} = \{\langle x, y\rangle \in V : (\exists v)\; v \in U \wedge \langle y, v\rangle * x\}$$
$$\widehat{\rho} = \{\langle x, y\rangle \in V : (\exists v)\; v \in U \wedge \langle v, y\rangle * x\}.$$

It is easy to prove using the definition in [FHV95] of **PFA** that:

$\mathbf{Pr}(\mathbf{K}_{\{\mathbf{Q},\mathbf{P},\mathbf{D}\}}) = \{\langle \mathcal{R}, \pi, \rho\rangle : \mathcal{R} \in \mathbf{PRA}$ with base set U and unit element V and there is an injective function $* \subseteq V \times U$ such that $\pi = \widehat{\pi}$ and $\rho = \widehat{\rho}\}$

The algebras of $\mathbf{Pr}(\mathbf{K}_{\{\mathbf{Q},\mathbf{P},\mathbf{D}\}})$ have elements that are binary relations in a structured set U. The function $*$ performs the role of encoding pairs of objects into single objects. For each element of $\mathbf{Pr}(\mathbf{K}_{\{\mathbf{Q},\mathbf{P},\mathbf{D}\}})$ there is a structure $\langle U, V, *\rangle$ (where V is an equivalence relation on U and $* \subseteq V \times U$ is an injective function) that is used to define the projection elements. The definition of these structures will be extended to build the classes $\mathbf{Pr}(\mathbf{K}_{\mathbf{\Sigma}})$. The name of pairing structures will be used for them. This name is used in [SN94] to define different candidate classes for proper fork algebras.

For pairing structures of the form $\langle U, V, *\rangle$ where V is an equivalence relation on U and $* \subseteq V \times U$ is a functional element, it is easy to prove that $\widehat{\pi}$ and $\widehat{\rho}$ satisfy **P**. For the reason that there exist members of $\mathbf{K_Q}$ where **P** doesn't hold, it is necessary the weaker requirement that $*$ is a relation in the definition of pairing structures for the classes $\mathbf{K_\Sigma}$.

For pairing structures of the form $\langle U, V, *\rangle$ where V is an equivalence relation on U and $* \subseteq V \times U$, it is easy to see that $\widehat{\pi}^d = \widehat{\rho}^d = *^i$. For the reason that there exist members of $\mathbf{K_Q}$ where the projection elements have different domain, it is necessary to consider more components in the definition of pairing structures. It is reasonable to consider two functional elements r_1 and r_2 contained in V, such that $r_1^d \cap r_2^d = \emptyset$, $r_1^d \cup r_2^d \subseteq \triangle_{U \sim *^i}$ and the projection elements of the proper models are $\pi = \widehat{\pi} \cup r_1$ and $\rho = \widehat{\rho} \cup r_2$.

Proposition 4.6 *Let V be an equivalence relation on U and $* \subseteq V \times U$ then $*$ is injective if and only if $\widehat{\pi}$ and $\widehat{\rho}$ are functions.*

PROOF
($\Rightarrow$) Let $z\widehat{\pi}u$ and $z\widehat{\rho}x$. Then there exists v, $y \in U$ such that $\langle u, v\rangle * z$ and $\langle x, y\rangle * z$. But $*$ is injective. Then $u = x$ and $v = y$. The same proof is used to see that $\widehat{\rho}$ is a function.
($\Leftarrow$) Let $\langle u, v\rangle * z$ and $\langle x, y\rangle * z$ then $z\widehat{\pi}u$ and $z\widehat{\pi}x$. But $\widehat{\pi}$ is function then $u = x$. Using that $\widehat{\rho}$ is a function $v = y$.
$\square$
The last proposition says that, in a pairing structure, $*$ must be injective.

Definition 4.7 *Let LPAR be the language:*
Sort symbols $= S_1, S_2$
Function symbols $= r_1, r_2 : \langle\langle S_1\rangle, S_1\rangle$
Relation symbols $= * : \langle S_2, S_1\rangle$.

Definition 4.8 *A LPAR-structure $\langle U, V, r_1, r_2, *\rangle$ is a pairing structure iff V is an equivalence relation on U, $r_1, r_2 \subseteq V$, r_1 and r_2 are functions, $r_1^d \cap r_2^d = \emptyset$ and $(r_1 \cup r_2)^d \subseteq \triangle_{U \sim *^i}$ and $*$ is injective.*

The definition of a pairing structure was made to capture the properties that are necessary for all the classes of $\mathbf{Pr}(\mathbf{K_\Sigma})$. But they aren't enough to capture an especific class $\mathbf{Pr}(\mathbf{K_\Sigma})$. For this reason, it is introduced the name $\mathbf{K_\Sigma}$-pairing structure in the definition of $\mathbf{Pr}(\mathbf{K_\Sigma})$. They will be pairing structures with additional properties depending on Σ.

Definition 4.9 *Let* $\mathbf{K_\Sigma}$ *be one of the classes defined in LRAP. The class* $\mathbf{Pr(K_\Sigma)}$ *of proper algebras in* $\mathbf{K_\Sigma}$ *is defined by:*

$\mathbf{Pr(K_\Sigma)} = \{ \langle \mathcal{R}, \pi, \rho \rangle : \mathcal{R} \in \mathbf{PRA}$ *with base set* U *and unit element* V *and there exist a pairing LPAR-structure* $\langle U, V, r_1, r_2, * \rangle$ *such that* $\pi = \widehat{\pi} \cup r_1$ *and* $\rho = \widehat{\rho} \cup r_2$ *and* $\langle U, V, r_1, r_2, * \rangle$ *is a* $\mathbf{K_\Sigma}$*-pairing structure* $\}$

The following result presents some sufficient and necessary conditions that pairing structures must satisfy to capture properties such as **P, D, E, I and Q** and will be useful in the definition of $\mathbf{K_\Sigma}$-pairing structures.

Theorem 4.10 *Let* $\langle U, V, r_1, r_2, * \rangle$ *be a pairing structure. Let* $\pi = \widehat{\pi} \cup r_1$ *and* $\rho = \widehat{\rho} \cup r_2$*. Then the following statements hold.*

1. $\widehat{\pi} \,|\, V = \widehat{\rho} \,|\, V$.
2. $\widehat{\pi}^{-1} \,|\, \widehat{\rho} = \pi^{-1} \,|\, \rho$.
3. $(\pi \,|\, \pi^{-1}) \cap (\rho \,|\, \rho^{-1}) = (\widehat{\pi} \,|\, \widehat{\pi}^{-1}) \cap (\widehat{\rho} \,|\, \widehat{\rho}^{-1})$
4. *If* $\pi^{-1} \,|\, \rho = \pi^{-1} \,|\, V \,|\, \rho$ *then* $\widehat{\pi}^{-1} \,|\, \widehat{\rho} = \widehat{\pi}^{-1} \,|\, V \,|\, \widehat{\rho}$
5. $*$ *is a function iff* $(\widehat{\pi} \,|\, \widehat{\pi}^{-1}) \cap (\widehat{\rho} \,|\, \widehat{\rho}^{-1}) \subseteq \triangle_U$.
6. *If* $E \subseteq V$ *then* $*^d = \triangle_E$ *iff* $\pi^{-1} \,|\, \rho = E$.
7. *If* $A,\ B \subseteq U$ *and* $*^d = \triangle_{V \cap (A \times B)}$ *then*
   (a) $\widehat{\pi}^i \subseteq \triangle_A$ *and* $\widehat{\rho}^i \subseteq \triangle_B$.
   (b) *If* $A = B$ *then* $\widehat{\pi}^i = \triangle_A = \widehat{\rho}^i$
8. *There exists* $A, B \subseteq U$ *such that* $*^d = \triangle_{V \cap (A \times B)}$ *iff* $\pi^{-1} \,|\, \rho = \widehat{\pi}^{-1} \,|\, V \,|\, \widehat{\rho}$
9. *There exists* $A \subseteq U$ *such that* $V = (V \cap (A \times A)) \cup (V \cap ((U \sim A) \times (U \sim A)))$ *and* $*^d = \triangle_{V \cap (A \times A)}$ *iff* $\pi^{-1} \,|\, \rho = V \,|\, \pi^{-1} \,|\, \rho \,|\, V$.

PROOF

1. Let $x(\widehat{\pi} \mid V)y$. There exists $w,\ v \in U$ such that $\langle w, v \rangle * x$ and wVy. Then $x\widehat{\rho}v$, wVy and wVv. But V is an equivalence relation on U. Then $\widehat{\pi} \,|\, V \subseteq \widehat{\rho} \,|\, V$. The proof of the other inclusion is similar.
2. $\pi^{-1} \,|\, \rho = (\widehat{\pi}^{-1} \,|\, \widehat{\rho}) \cup (\widehat{\pi}^{-1} \,|\, r_2) \cup (r_1^{-1} \,|\, \widehat{\rho}) \cup (r_1^{-1} \,|\, r_2)$. The last summand is $\emptyset$ because $r_1^d \cap r_2^d = \emptyset$. The second and third summands are $\emptyset$ because $r_1^d,\ r_2^d \subseteq U \sim *^i$.

3. Using distributive laws

$$(\pi\,|\,\pi^{-1}) \cap (\rho\,|\,\rho^{-1}) = [(\widehat{\pi}\,|\,\widehat{\pi}^{-1}) \cup (\widehat{\pi}\,|\,r_1^{-1}) \cup (r_1\,|\,\widehat{\pi}^{-1}) \cup (r_1\,|\,r_1^{-1})] \cap$$
$$[(\widehat{\rho}\,|\,\widehat{\rho}^{-1}) \cup (\widehat{\rho}\,|\,r_2^{-1}) \cup (r_2\,|\,\widehat{\rho}^{-1}) \cup (r_2\,|\,r_2^{-1})]$$

Appying again distributive laws 16 terms are obtained. One is $(\widehat{\pi}\,|\,\widehat{\pi}^{-1}) \cap (\widehat{\rho}\,|\,\widehat{\rho}^{-1})$. The other terms are $\emptyset$ by 1. and because $r_1^d \cap r_2^d = r_1^d \cap \widehat{\pi}^d = \widehat{\rho}^d \cap r_2^d = \emptyset$.

4. By 2. $\pi^{-1}\,|\,\rho = \widehat{\pi}^{-1}\,|\,\widehat{\rho} \subseteq \widehat{\pi}^{-1}\,|\,V\,|\,\widehat{\rho}$ and by hypothesis $\widehat{\pi}^{-1}\,|\,V\,|\,\widehat{\rho} \subseteq \pi^{-1}\,|\,\rho$ then $\widehat{\pi}^{-1}\,|\,V\,|\,\widehat{\rho} = \widehat{\pi}^{-1}\,|\,\widehat{\rho} = \pi^{-1}\,|\,V\,|\,\rho$.

5. ($\Rightarrow$) Let $x((\widehat{\pi}\,|\,\widehat{\pi}^{-1}) \cap (\widehat{\rho}\,|\,\widehat{\rho}^{-1}))y$. There exists $u,\ v \in U$ such that $x\widehat{\pi}u$, $y\widehat{\pi}u$, $x\widehat{\rho}v$ and $y\widehat{\rho}v$. There exists $w,\ w',\ z,\ z' \in U$ such that $\langle u, w\rangle * x$, $\langle z, v\rangle * x$, $\langle u, w'\rangle * y$ and $\langle z', v\rangle * y$. But $*$ injective implies $u = z = z'$ and $v = w = w'$. Then $\langle u, v\rangle * x$ and $\langle u, v\rangle * y$. But $*$ is a function, then $x = y$.

   ($\Leftarrow$) Let $\langle u, v\rangle * x$ and $\langle u, v\rangle * y$. Then $x\widehat{\pi}u$, $x\widehat{\rho}v$, $y\widehat{\pi}u$ and $y\widehat{\rho}v$. But $(\widehat{\pi}\,|\,\widehat{\pi}^{-1}) \cap (\widehat{\rho}\,|\,\widehat{\rho}^{-1}) \subseteq \triangle_U$. Then $x = y$.

6. ($\Rightarrow$) If $x(\pi^{-1}\,|\,\rho)y$ then by 2. there exists $z \in V$ such that $z\widehat{\pi}x$ and $z\widehat{\rho}y$. There exist $u, v \in U$ such that $\langle x, u\rangle * z$ and $\langle v, y\rangle * z$. But $x = v$ and $u = y$ because $*$ is injective. Then $\langle x, y\rangle * z$, $\langle x, y\rangle \in E$ and $\pi^{-1}\,|\,\rho \subseteq E$ holds. The other inclusion is trivial.

   ($\Leftarrow$) If xEy then by 2. there exists $z \in U$ such that $z\widehat{\pi}x$ and $z\widehat{\rho}y$. There exist $u, v \in U$ such that $\langle x, u\rangle * z$ and $\langle v, y\rangle * z$. But $x = v$ and $u = y$ because $*$ is injective. Then $\langle x, y\rangle * z$ and $\triangle_E \subseteq *^d$ holds. The other inclusion is trivial.

7. (a) If $x\widehat{\pi}^i x$ then there exists $y, v \in U$ such that $\langle x, v\rangle * y$. Then $x \in A$. Similarly $\widehat{\rho}^i \subseteq \triangle_B$.

   (b) Let $x \in A, \triangle_A \subseteq V$ then there exists $w \in U$ such that $\langle x, x\rangle * w$. Then $x\widehat{\pi}^i x$ and $x\widehat{\rho}^i x$. Using (a) the result follows.

8. ($\Rightarrow$) By 3.5.1, 7.(a) and 6. $\widehat{\pi}^{-1}\,|\,V\,|\,\widehat{\rho} = \widehat{\pi}^i\,|\,V\,|\,\widehat{\rho}^i \subseteq \triangle_A\,|\,V\,|\,\triangle_B = V \cap (A \times B) = \widehat{\pi}^{-1}\,|\,\widehat{\rho}$. The implication follows using 2.

   ($\Leftarrow$) By 3.5.1 $\widehat{\pi}^{-1}\,|\,V\,|\,\widehat{\rho} = \widehat{\pi}^i\,|\,V\,|\,\widehat{\rho}^i = V \cap (Set(\widehat{\pi}^i) \times Set(\widehat{\rho}^i))$. By 6. $*^d = \triangle_{V \cap (Set(\widehat{\pi}^i) \times Set(\widehat{\rho}^i))}$.

9. First it is proved that if $E \subseteq V$ then $E = V\,|\,E\,|\,V$ iff $\exists A\ A \subseteq U$ such that $E = V \cap A \times A$ and $V = V \cap ((A \times A) \cup (U \sim A) \times (U \sim A))$.

   ($\Rightarrow$) Let $A = Set(E \cap \triangle_U)$. It is easy to prove that $E \subseteq (A \times A)$ and $V \sim E \subseteq (U \sim A) \times (U \sim A)$. Then $V = V \cap ((A \times A) \cup (U \sim A) \times (U \sim A))$. $E \subseteq V \cap A \times A$. Let $x(V \cap A \times A)y$. If $x(V \sim E)y$ then $x((U \sim A) \times (U \sim A))y$. This is a contradiction. Then xEy.

($\Leftarrow$) E and $V \sim E$ are equivalence relations. Then by 2.14.13 $E = V \,|\, E \,|\, V$.
Now the main result is proved.
$V \,|\, \pi^{-1} \,|\, \rho \,|\, V = \pi^{-1} \,|\, \rho$ iff there exists $A \subseteq U$ such that
$\pi^{-1} \,|\, \rho = V \cap (A \times A)$ and $V = (V \cap (A \times A)) \cup (V \cap ((U \sim A) \times (U \sim A)))$.
Iff by 6. $*^d = \triangle_{V \cap (A \times A)}$ and $V = (V \cap (A \times A)) \cup (V \cap ((U \sim A) \times (U \sim A)))$.

□

The last theorem gives only some necessary conditions that pairing structures must satisfy when $\mathbf{R} \in \Sigma$. Bellow, a stronger necessary and sufficient condition is proved. For this purpose the well known concept of saturated sets is considered.

Definition 4.11 *Let V be an equivalence relation on U and $r \subseteq \triangle_U$ then the saturated of r is defined by $\triangle_U \cap (V \mid r \mid V)$. sat(r) is used to abbreviate the saturated of r.*

It is easy to see that the saturated of r is $\bigcup\{[y] : \triangle_{[y]} \cap r \neq \emptyset \,\wedge\, y \in U\}$. Other trivial identities are:

1. $sat(r) \cup sat(s) = sat(r \cup s)$.
2. $(t \,|\, V \,|\, u)^d = t \cap sat(u)$.
3. $(t \,|\, V \,|\, u)^i = u \cap sat(t)$.

Theorem 4.12 *Let $\langle U, V, r_1, r_2, *\rangle$ be a pairing structure. Let $\pi = \widehat{\pi} \cup r_1$ and $\rho = \widehat{\rho} \cup r_2$. Then $\pi^{-1} \,|\, \rho = \pi^{-1} \,|\, V \,|\, \rho$ iff $\widehat{\pi}^{-1} \,|\, \widehat{\rho} = \widehat{\pi}^{-1} \,|\, V \,|\, \widehat{\rho}$, $r_1^i \cap sat(\rho^i) \subseteq \widehat{\pi}^i$ and $r_2^i \cap sat(\pi^i) \subseteq \widehat{\rho}^i$.*

PROOF
It is proved that the theorem is a consequence of the following fact:
If $r, s, t, u \subseteq \triangle_U$ then $r \,|\, V \,|\, s \supseteq t \,|\, V \,|\, u$ iff $t \cap sat(u) \subseteq r$ and $u \cap sat(t) \subseteq s$.
This result is intuitive by items (2) and (3).
($\Rightarrow$) If $x(t \cap sat(u))x$ then $(\exists z)(z \in U) \wedge xtx \wedge zuz \wedge xVz$ this implies $x(t \,|\, V \,|\, u)z$.
By hypothesis $x(r \,|\, V \,|\, s)z$ and xrx. The proof of the other inequality is similar.
($\Leftarrow$) If $x(t \,|\, V \,|\, u)y$ then $x(t \cap sat(u))x$ and $y(u \cap sat(t))y$. By hypothesis xrx and ysy.
Second, it is proved the result:
If $r, s, t, u \subseteq \triangle_U$ then $r \mid V \mid s \supseteq (t \mid V \mid s) \cup (t \mid V \mid u) \cup (r \mid V \mid u)$ iff $t \cap sat(u \cup s) \subseteq r$ and $u \cap sat(t \cup r) \subseteq s$
$r \,|\, V \,|\, s \supseteq r \,|\, V \,|\, u$ iff $u \cap sat(r) \subseteq s$ and $r \,|\, V \,|\, s \supseteq t \,|\, V \,|\, s$ iff $t \cap sat(s) \subseteq r$. Then $r \,|\, V \,|\, s \supseteq (t \,|\, V \,|\, s) \cup (t \,|\, V \,|\, u) \cup (r \,|\, V \,|\, u)$ iff $u \cap sat(r) \subseteq s$ and $t \cap sat(s) \subseteq r$ and $t \cap sat(u) \subseteq r$ and $u \cap sat(t) \subseteq s$ iff $t \cap sat(u \cup s) \subseteq r$ and $u \cap sat(t \cup r) \subseteq s$.
Now the main result is proved.
($\Rightarrow$) By 4.10.4 $(r_1^{-1} \,|\, V \,|\, \widehat{\rho}) \cup (\widehat{\pi}^{-1} \,|\, V \,|\, r_2) \cup (r_1^{-1} \,|\, V \,|\, r_2) \subseteq (\widehat{\pi}^{-1} \,|\, V \,|\, \widehat{\rho})$. By 3.5.1 $(r_1^i \,|\, V \,|\, \widehat{\rho}^i) \cup (\widehat{\pi}^i \,|\, V \,|\, r_2^i) \cup (r_1^i \,|\, V \,|\, r_2^i) \subseteq (\widehat{\pi}^i \,|\, V \,|\, \widehat{\rho}^i)$. Then $r_1^i \cap sat(\rho^i) \subseteq \widehat{\pi}^i$ and $r_2^i \cap sat(\pi^i) \subseteq \widehat{\rho}^i$.

($\Leftarrow$) $(r_1^i \,|\, V \,|\, \widehat{\rho}^{\,i}) \cup (\widehat{\pi}^{\,i} \,|\, V \,|\, r_2^i) \cup (r_1^i \,|\, V \,|\, r_2^i) \subseteq (\widehat{\pi}^i \,|\, V \,|\, \widehat{\rho}^i)$ and
$\widehat{\pi}^{-1} \,|\, \widehat{\rho} = \widehat{\pi}^{-1} \,|\, V \,|\, \widehat{\rho}$ by hypothesis. Then by 3.5.1
$r_1^{-1} \,|\, V \,|\, \widehat{\rho} \cup \widehat{\pi}^{-1} \,|\, V \,|\, r_2 \cup r_1^{-1} \,|\, V \,|\, r_2 \subseteq \widehat{\pi}^{-1} \,|\, V \,|\, \widehat{\rho}$. Using 4.10.2.
$\pi^{-1} \,|\, \rho = \pi^{-1} \,|\, V \,|\, \rho$.
$\square$

The last two results give the necessary information for the definition of $\mathbf{K}_{\mathbf{\Sigma}}$-pairing structure and the proof of the representation theorem.

Definition 4.13 *$\langle U, V, r_1, r_2, *\rangle$ is $\mathbf{K}_{\mathbf{\Sigma}}$-pairing structure iff $\langle U, V, r_1, r_2, *\rangle$ is a pairing structure and*

1. $*^d = \triangle_V$ *if* $\mathbf{Q} \in \Sigma$
2. $(\exists C)(\exists D)$ $(C, D \subseteq U) \wedge (*^d = \triangle_{V \cap (C \times D)}) \wedge (r_1^i \cap sat(\rho^i) \subseteq \widehat{\pi}^{\,i}) \wedge$
   $(r_2^i \cap sat(\pi^i) \subseteq \widehat{\rho}^{\,i})$ *if* $\mathbf{R} \in \Sigma$
3. $(\exists A)$ $(A \subseteq U) \wedge (V = (V \cap (A \times A)) \cup (V \cap ((U \sim A) \times (U \sim A)))) \wedge$
   $(*^d = \triangle_{V \cap (A \times A)})$ *if* $\mathbf{I} \in \Sigma$
4. $(\exists W)(*^d = \triangle_W) \wedge (W = W^{-1}) \wedge (W = W \mid W)$ *if* $\mathbf{E} \in \Sigma$
5. $(\exists C)(C \subseteq U) \wedge (*^d = \triangle_{V \cap (C \times C)}) \wedge ((r_1 + r_2)^{\,i} \subseteq \widehat{\pi}^{\,i})$ *if* $\mathbf{S} \in \Sigma$
6. $(\exists W)(*^d = \triangle_W)$ *if* **Q, R, I, E** *and* **S** *aren't in* Σ
7. $*$ *is a function if* $\mathbf{P} \in \Sigma$
8. $r_1 = r_2 = \emptyset$ *if* $\mathbf{D} \in \Sigma$

In the next step, it will be showed that the axioms characterizing $\mathbf{K}_{\mathbf{\Sigma}}$ are satisfied in any element of $\mathbf{Pr}(\mathbf{K}_{\mathbf{\Sigma}})$.

Theorem 4.14 *Let $\mathbf{K}_{\mathbf{\Sigma}}$ be one of the classes defined in LRAP. Then*
$\mathbf{I}(\mathbf{Pr}(\mathbf{K}_{\mathbf{\Sigma}})) \subseteq \mathbf{K}_{\mathbf{\Sigma}}$.

PROOF Definition 4.13 and theorem 4.10 will be applied several times.
π and ρ are functions then by 4.6 $*$ is injective.
If $\mathbf{Q} \in \Sigma$ then $*^d = \triangle_V$. By 4.10.6 $\pi^{-1} \,|\, \rho = V$.

If $\mathbf{R} \in \Sigma$ then $(\exists C)(\exists D)$ $(C, D \subseteq U) \wedge (*^d = \triangle_{V \cap (C \times D)}) \wedge (r_1^i \cap sat(\rho^i) \subseteq \widehat{\pi}^{\,i}) \wedge$
$(r_2^i \cap sat(\pi^i) \subseteq \widehat{\rho}^{\,i})$

By 4.10.8 and 4.10.2 $\widehat{\pi}^{-1} \,|\, \widehat{\rho} = \widehat{\pi}^{-1} \,|\, V \,|\, \widehat{\rho}$. Using 4.12 $\pi^{-1} \,|\, \rho = \pi^{-1} \,|\, V \,|\, \rho$.

If $\mathbf{I} \in \Sigma$ then $(\exists A)(A \subseteq U) \wedge (V = (V \cap (A \times A)) \cup (V \cap ((U \sim A) \times (U \sim A))))$.
By 4.10.9 $\pi^{-1} \,|\, \rho = V \,|\, \pi^{-1} \,|\, \rho \,|\, V$.

If $\mathbf{E} \in \Sigma$ then $(\exists W)(*^d = \triangle_W) \wedge (W = W^{-1}) \wedge (W = W \mid W)$. By 4.10.6 $\pi^{-1} \,|\, \rho = W$, then $\pi^{-1} \,|\, \rho$ is an equivalence relation.

If $\mathbf{S} \in \Sigma$ then $(\exists C)(C \subseteq U) \wedge (*^d = \triangle_{V \cap (C \times C)}) \wedge ((r_1 + r_2)^i \subseteq \widehat{\pi}^i)$.
By 4.10.7.(b) $\widehat{\pi}^i = \triangle_C = \widehat{\rho}^i$. Then $\pi^i = \widehat{\pi}^i + r_1^i = \widehat{\pi}^i = \widehat{\pi}^i + r_2^i = \widehat{\rho}^i + r_2^i = \rho^i$.
$r_1^i \cap sat(\rho^i) = r_1^i \cap sat(\pi^i) \subseteq r_1^i \subseteq \widehat{\pi}^i$
$r_2^i \cap sat(\pi^i) = r_2^i \cap sat(\rho^i) \subseteq r_2^i \subseteq \widehat{\pi}^i = \widehat{\rho}^i$
By 4.10.[8,2] $\pi^{-1} \,|\, \rho = \widehat{\pi}^{-1} \,|\, \widehat{\rho} = \widehat{\pi}^{-1} \,|\, V \,|\, \widehat{\rho}$.
Using 4.12 $\pi^{-1} \,|\, \rho = \pi^{-1} \,|\, V \,|\, \rho$.

If **Q, R, I, E** and **S** aren't in Σ then $(\exists W)(*^d = \triangle_W)$. By 4.10.6 $\pi^{-1} \,|\, \rho = W$.

If $\mathbf{P} \in \Sigma$ then $*$ is a function. Then by 4.10.[5,4] $(\pi \,|\, \pi^{-1}) \cap (\rho \,|\, \rho^{-1}) \subseteq \triangle_U$.

If $\mathbf{D} \in \Sigma$ then $r_1 = r_2 = \emptyset$. By 4.10.1 $\pi \,|\, V = \rho \,|\, V$.

□

Before proving the representation theorem, it is showed that for each model of $\mathbf{K_\Sigma} \cap (\mathbf{PRA} \upharpoonright LRAP)$ there is a pairing structure that represents its projection elements.

Theorem 4.15 *Let $\langle \mathcal{R}, \pi_1, \rho_1 \rangle$ where $\mathcal{R} \in PRA$ and π_1, ρ_1 are functionals. Let V be the top element of $\mathcal{R}$ and $\triangle_U$ the diagonal element of $\mathcal{R}$. Then there exist a pairing LPAR-structure $\langle U, V, r_1, r_2, * \rangle$ such that $\pi_1 = \widehat{\pi} \cup r_1$ and $\rho_1 = \widehat{\rho} \cup r_2$.*

PROOF Let $* \subseteq V \times U$ such that $\langle a, b \rangle * c$ iff $c\pi_1 a$ and $c\rho_1 b$.
Let $r_1 = (\triangle_{U \sim *^i}) \,|\, \pi_1$ and $r_2 = (\triangle_{U \sim *^i}) \,|\, \rho_1$.
Take the LPAR-structure $\langle U, V, r_1, r_2, * \rangle$.
Let $x(r_1^d \cap r_2^d)x$ then $x(\pi_1^d \cap \rho_1^d)x$.There exists $y, z \in U$ such that $x\pi_1 y$ and $x\rho_1 z$. Then $\langle y, z \rangle * x$ and $x \in *^i$. This is a contradiction. Then $r_1^d \cap r_2^d = \emptyset$.
The other conditions for the admisibility of $\langle U, V, r_1, r_2, * \rangle$ are obvious.
Let $\langle a, b \rangle * e$ and $\langle c, d \rangle * e$. By definition of $*$ $e\pi_1 a$, $e\rho_1 b$, $e\pi_1 c \, and \, e\rho_1 d$. But π_1 and ρ_1 are functions then $a = c$ and $b = d$.
Now the equality $\pi_1 = \widehat{\pi} \cup r_1$ is proved.
$(\supseteq)$ Let $x\widehat{\pi}y$. There exists $v \in U$ such that $\langle y, v \rangle * x$, then $x\pi_1 y$.
$(\subseteq)$ Let $x\pi_1 y$. If $x \in U \sim *^i$ then $xr_1 y$. If $x \in *^i$ then there exists $u, v \in U$ such that $\langle u, v \rangle * x$. Then $x\pi_1 u$. But π_1 is functional. Then $u = y$ and $x\widehat{\pi}y$.
The proof of the equality $\rho_1 = \widehat{\rho} \cup r_2$ is similar.
□

Theorem 4.16 *Let $\mathbf{K_\Sigma}$ be one of the classes defined in LRAP. Then*

$$\begin{aligned} \mathbf{I}(\mathbf{Pr}(\mathbf{K_\Sigma})) &= \{\mathcal{A} \in \mathbf{K_\Sigma} : \mathcal{A} \downarrow LRA \in \mathbf{RRA}\} \\ &= \mathbf{K_\Sigma} \cap (\mathbf{RRA} \upharpoonright LRAP) \end{aligned}$$

PROOF $\{\mathcal{A} \in \mathbf{K}_{\boldsymbol{\Sigma}} : \mathcal{A}\downarrow LRA \in \mathbf{RRA}\} = \mathbf{K}_{\boldsymbol{\Sigma}} \cap (\mathbf{RRA}\uparrow LRAP)$ by 2.6. The equality of this set with $\mathbf{I}(\mathbf{Pr}(\mathbf{K}_{\boldsymbol{\Sigma}}))$ is proved bellow.

($\subseteq$) By definition 4.9, $\mathbf{Pr}(\mathbf{K}_{\boldsymbol{\Sigma}}) \subseteq (\mathbf{RRA}\uparrow LRAP)$. By 2.7 and because $\mathbf{RRA}$ is a variety $\mathbf{I}(\mathbf{Pr}(\mathbf{K}_{\boldsymbol{\Sigma}})) \subseteq (\mathbf{RRA}\uparrow LRAP)$. Using theorem 4.14 the inclusion follows.

($\supseteq$) Let $\mathcal{A} = \langle \mathcal{R}, \pi_1, \rho_1\rangle \in \mathbf{K}_{\boldsymbol{\Sigma}} \cap (\mathbf{RRA}\uparrow LRAP)$.
Suppose that $\mathcal{A}\downarrow LRA \in \mathbf{PRA}$.
π_1 *and* ρ_1 are functionals because $\mathcal{A} \in \mathbf{K}_{\boldsymbol{\Sigma}}$.
Let V be the top element of $\mathcal{R}$ and $\triangle_U$ the diagonal element of $\mathcal{R}$. Then by 4.15 there exist a pairing LPAR-structure $\langle U, V, r_1, r_2, *\rangle$ such that $*$ is injective, $\pi_1 = \widehat{\pi} \cup r_1$ and $\rho_1 = \widehat{\rho} \cup r_2$.
It will be proved that $\mathcal{A} \in \mathbf{Pr}(\mathbf{K}_{\boldsymbol{\Sigma}})$ by showing that $\langle U, V, r_1, r_2, *\rangle$ is a $\mathbf{K}_{\boldsymbol{\Sigma}}$-pairing structure.

If $\mathbf{Q} \in \Sigma$ then by 4.10.6 $*^d = V$.

If $\mathbf{R} \in \Sigma$ then by 4.12 $\widehat{\pi}^{-1}|\widehat{\rho} = \widehat{\pi}^{-1}|V|\widehat{\rho}$, $r_1^i \cap sat(\rho^i) \subseteq \widehat{\pi}^i$ and $r_2^i \cap sat(\pi^i) \subseteq \widehat{\rho}^i$.
By 4.10.[8,2] $(\exists A)(\exists B)(A, B \subseteq U) \wedge (*^d = \triangle_{V\cap(A\times B)})$.

If $\mathbf{I} \in \Sigma$ by 4.10.9 $V = (V\cap(A\times A))\cup(V\cap((U\sim A)\times(U\sim A)))$ and $*^d = \triangle_{V\cap(A\times A)}$.

If $\mathbf{E} \in \Sigma$ then by 4.10.6 $*^d = \triangle_{\pi^{-1}|\rho}$.

If $\mathbf{S} \in \Sigma$ then by 4.12 $\widehat{\pi}^{-1}|\widehat{\rho} = \widehat{\pi}^{-1}|V|\widehat{\rho}$, $r_1^i \cap sat(\rho^i) \subseteq \widehat{\pi}^i$ and
$r_2^i \cap sat(\pi^i) \subseteq \widehat{\rho}^i$. Using $\pi^i = \rho^i$: $r_1^i \cap sat(\pi^i) \subseteq \widehat{\pi}^i$ then $r_1^i \subseteq \widehat{\pi}^i$.
Similarly $r_2^i \subseteq \widehat{\rho}^i$.
$\widehat{\pi}^i = \widehat{\pi}^i + r_1^i = \pi^i = \rho^i = \widehat{\rho}^i + r_2^i = \widehat{\rho}^i$. Then $(r_1 + r_2)^i \subseteq \widehat{\pi}^i$.
By 4.10.2 and 3.5.1
$\pi^i|\rho^i = \widehat{\pi}^{-1}|V|\widehat{\rho} = \widehat{\pi}^i|V|\widehat{\rho}^i = \widehat{\pi}^i|V|\widehat{\pi}^i = V \cap (Set(\widehat{\pi}^i)\times Set(\widehat{\pi}^i))$.
By 4.10.6 $*^d = \triangle_{V\cap(Set(\widehat{\pi}^i)\times Set(\widehat{\pi}^i))}$.

If $\mathbf{Q}$, $\mathbf{R}$, $\mathbf{I}$, $\mathbf{E}$ and $\mathbf{S}$ aren't in Σ then by 4.10.6 $*^d = \triangle_{\pi^{-1}|\rho}$.

If $\mathbf{P} \in \Sigma$ then by 4.10.5 $*$ is a function.

If $\mathbf{D} \in \Sigma$ then $\pi|V = \rho|V$. By 4.10.1 $\widehat{\pi}|V = \widehat{\rho}|V$. Using that $\langle U, V, r_1, r_2, *\rangle$ is a pairing structure, 3.5.2 and properties of Boolean algebra, it is easy to prove that $r_1 = r_2 = \emptyset$.

Suppose that $\mathcal{A}\downarrow LRA \in \mathbf{PRA}$ is not true. Then $\mathcal{R}$ is isomorphic to an algebra $\mathcal{S}$ where $\mathcal{S} \in \mathbf{PRA}$. Let be $h : \mathcal{R} \to \mathcal{S}$ an isomorphism and define

$\pi_2 = h(\pi_1)$ *and* $\rho_2 = h(\rho_1)$. But h *preserves* Σ. Then $\langle \mathcal{S}, \pi_2, \rho_2 \rangle \in \mathbf{K}_{\mathbf{\Sigma}}$. Then $\langle \mathcal{S}, \pi_2, \rho_2 \rangle \in \mathbf{Pr}(\mathbf{K}_{\mathbf{\Sigma}})$. This implies that $\mathcal{A} \in \mathbf{I}(\mathbf{Pr}(\mathbf{K}_{\mathbf{\Sigma}}))$.
$\square$

In [Gar96] the representation theorem is used to build the proper models of a theory of lists (that is an extension of the class $\mathbf{K}_{\{\mathbf{R},\mathbf{P},\mathbf{D}\}}$) and in the proof of a representation theorem for lists.

The same can be done with each of the abstract data types specified in section 3.3. The proper models of them are extensions with operations of a subclass of the respective $\mathbf{Pr}(\mathbf{K}_{\mathbf{\Sigma}})$ classes.

4.3 The finitization problem

Here it is proved that when $\mathbf{Q}$ is not in Σ the class $\mathbf{Pr}(\mathbf{K}_{\mathbf{\Sigma}})$ is not finitely axiomatizable.

In [Gar96] a theory of lists based on the class $\mathbf{K}_{\{\mathbf{R},\mathbf{P},\mathbf{D}\}}$ is built. This theory have models with relational reduct representable. This is because the theory is also an extension of **AFA** and **AFA** have its relational reduct representable. This is a good idea to solve the finitization problem for the proper models of the specifications of abstract data types. In addition, the expressive power of **AFA** is available. In section 9 it is proved that **AFA** have properties that are easier to apply during derivations because the respective properties in the classes $\mathbf{K}_{\mathbf{\Sigma}}$ have more complicate applicability conditions.

Theorem 4.17 *If* $\mathbf{Q}$ *is not in* Σ *then* $\mathbf{K}_{\mathbf{\Sigma}} \cap (\mathbf{RRA} \restriction LRAP)$ *is nonfinitely axiomatizable.*

PROOF Suppose that $\mathbf{K}_{\mathbf{\Sigma}} \cap (\mathbf{RRA} \restriction LRAP)$ where $\mathbf{Q}$ is not in Σ is finitely axiomatizable with axiom set Γ. Then the class
$\mathbf{K} = \{\langle \mathcal{R}, \pi, \rho \rangle \in \mathbf{K}_{\mathbf{\Sigma}} \cap (\mathbf{RRA} \restriction LRAP) : \pi = \rho = 0\}$
is finitely axiomatizable with axiom set $\Gamma \cup \{\pi \approx 0, \rho \approx 0\}$. But $\mathbf{K} = \{\langle \mathcal{R}, 0^{\mathcal{R}}, 0^{\mathcal{R}} \rangle : \mathcal{R} \in \mathbf{RRA}\}$. $\Gamma \cup \{\pi \approx 0, \rho \approx 0\}$ is equivalent to $\Gamma^* \cup \{\pi \approx 0, \rho \approx 0\}$. Such that Γ^* is obtained from Γ by replacing π and ρ by 0 and simplifying. Γ^* is in LRA.
$\mathcal{R} \models \Gamma^* \Leftrightarrow \langle \mathcal{R}, 0^{\mathcal{R}}, 0^{\mathcal{R}} \rangle \models \Gamma \cup \{\pi \approx 0, \rho \approx 0\} \Leftrightarrow \langle \mathcal{R}, 0^{\mathcal{R}}, 0^{\mathcal{R}} \rangle \in \mathbf{K} \Leftrightarrow \mathcal{R} \in \mathbf{RRA}$
Then **RRA** is finitely axiomatizable. By [D.64, Thm.2] this is a contradiction.
$\square$
By [Mad78] if $\mathbf{Q} \in \Sigma$ then $\mathbf{K}_{\mathbf{\Sigma}} \cap (\mathbf{RRA} \restriction LRAP)$ is finitely axiomatizable because it is equal to $\mathbf{K}_{\mathbf{\Sigma}}$.

5 Comparison of the classes

Theorem 5.1 *Let $\langle \mathcal{R}, \pi, \rho\rangle \in \mathbf{RA} \upharpoonright LRAP$ such that π and ρ are functionals. Then the following statements hold in $\langle \mathcal{R}, \pi, \rho\rangle$*

1. $\mathbf{Q} \to \mathbf{R} \wedge \mathbf{S} \wedge \mathbf{E} \wedge \mathbf{I}$
2. $\mathbf{I} \wedge \mathbf{D} \to \mathbf{R} \wedge \mathbf{S}$
   $\mathbf{I} \to \mathbf{E}$
3. $\mathbf{S} \to \mathbf{R} \wedge \mathbf{E}$

PROOF

1. Let $\breve{\pi};\rho = 1$. By 2.14.[1,14] $Eq(\breve{\pi};\rho)$ holds.
   By 2.14.14 and 2.11 $Ie(\breve{\pi};\rho)$ holds.
   By 2.14.4 $\breve{\pi};\rho \subseteq \breve{\pi};1;\rho$. Then $1 = \breve{\pi};1;\rho$ and $Rn(\pi,\rho)$ holds.
   By 2.14.4 $\breve{\pi};\rho \subseteq \breve{\pi};1$. Then by By 2.14.3 $1;\pi = 1$.
   By 2.14.4 $\breve{\pi};\rho \subseteq 1;\rho$. Then $1;\rho = 1$ and $\mathbf{S}$ holds.
2. By 2.14.13 $\mathbf{E}$ holds and by 4.4 $\mathbf{R}$ and $\mathbf{S}$ hold.
3. Obviously $\mathbf{R}$ holds.

   By 2.14.[2,3] and 3.5.[1,3] $\breve{\rho};\pi = (\breve{\pi};\rho)^{\smile} = (\pi^i;1;\rho^i)^{\smile} = \rho^i;1;\pi^i$
   $= \pi^i;1;\rho^i = \breve{\pi};\rho$

   Now the identity $1;r = (1;r);(1;r)$ is proved.
   By 2.14.[16,4] $r;1 = (r;1)\cdot(1;r;1) = ((r;1)\cdot 1);(r;1) = (r;1);(r;1)$.
   Finally by 2.14.3 $1;r = (1;r);(1;r)$

   By 2.14.6 and 3.5.[1,3] $(\breve{\pi};\rho);(\breve{\pi};\rho) = (\pi^i;1;\rho^i);(\pi^i;1;\rho^i) = (\pi^i;1;\rho^i);(\rho^i;1;\rho^i)$
   $= (\pi^i;1;\rho^i;1;\rho^i) = (\pi^i;1;\rho^i) = \breve{\rho};\pi$

□

Definition 5.2 *Let V be an equivalence relation on a set U and $* \subseteq V \times U$. Let $\mathcal{F}r(V,*)$ be the following model in LRAP.*
$\mathcal{F}r(V,*) = \langle \mathcal{P}(V), \widehat{\pi}, \widehat{\rho}\rangle$. $\mathcal{P}(V)$ is the realtion algebra of all subrelations of V.
$\widehat{\pi} = \{\langle x,y\rangle \in V : (\exists v) v \in U \wedge \langle y,v\rangle * x\}$, $\widehat{\rho} = \{\langle x,y\rangle \in V : (\exists v) v \in U \wedge \langle v,y\rangle * x\}$

Theorem 5.3 *Each of the following statements holds for some algebra $\langle \mathcal{R}, \pi, \rho \rangle$ in* $\mathbf{RA} \upharpoonright LRAP$ *where π and ρ are functionals.*

1. $\neg(\mathbf{I} \rightarrow \mathbf{Q})$
2. $\neg(\mathbf{R} \rightarrow \mathbf{I}) \wedge \neg(\mathbf{R} \rightarrow \mathbf{E})$
3. $\neg(\mathbf{S} \rightarrow \mathbf{I})$
4. $\neg(\mathbf{E} \rightarrow \mathbf{R})$
5. $\neg((\mathbf{I} \wedge \neg\mathbf{D}) \rightarrow \mathbf{R})$
6. $\neg((\mathbf{I} \wedge \neg\mathbf{D}) \rightarrow (1;\pi = 1;\rho))$

PROOF

1. Let $A \neq \emptyset$, $A \subseteq U$ and an injective function $* : A \times A \rightarrow U$.
   Let $V = (A \times A) \cup (U \sim A) \times (U \sim A)$. By 4.10.9 $Ie(\widehat{\pi}^{-1} \mid \widehat{\rho})$ holds in $\mathcal{F}r(V, *)$.
   If $\widehat{\pi}^{-1} \mid \widehat{\rho} = V$ then by 4.10.6 $*^d = V$ and $V = A \times A$. This is a contradiction.

2. Let $A \neq B$, $A, B \subseteq U$ and an injective function $* : A \times B \rightarrow U$.
   Let $V = U \times U$. By 4.10.8 $Rn(\widehat{\pi}, \widehat{\rho})$ holds in $\mathcal{F}r(V, *)$.
   By 4.10.6 $\widehat{\pi}^{-1} \mid \widehat{\rho} = A \times B$. Then $\mathbf{E}$ doesn't hold in $\mathcal{F}r(V, *)$. By 2.14.13 $\mathbf{I}$ doesn't hold in $\mathcal{F}r(V, *)$.

3. Let $A \neq \emptyset$, $A \subseteq U$ and an injective function $* : A \times A \rightarrow U$.
   Let $V = (U \times U)$. By 4.10.8 $Rn(\widehat{\pi}, \widehat{\rho})$ holds in $\mathcal{F}r(V, *)$.
   By 4.10.[13,12] $\widehat{\pi}^i = \widehat{\rho}^i = \triangle_A$. By 3.5.3 $V \mid \widehat{\pi} = V \mid \widehat{\rho}$ and $\mathbf{S}$ holds in $\mathcal{F}r(V, *)$.
   But $A \times A \neq (A \times A) \cup ((U \sim A) \times (U \sim A))$ because $A \neq U$. Then by 4.10.9 $\mathbf{I}$ doesn't hold in $\mathcal{F}r(V, *)$.

4. Let $A \subseteq U$ and A have at least two elements. Let $* : \triangle_A \rightarrow U$ be an injective function. Let $V = (U \times U)$. By 4.10.6 $\widehat{\pi}^{-1} \mid \widehat{\rho} = \triangle_A$. $\triangle_A$ is an equivalence relation. Then $\mathbf{E}$ holds in $\mathcal{F}r(V, *)$. If $\mathbf{R}$ holds in $\mathcal{F}r(V, *)$ by 4.10.8 there exist $C, D \subseteq U$ such that $*^d = \triangle_{C \times D}$.
   By 4.10.6 $\widehat{\pi}^{-1} \mid \widehat{\rho} = (C \times D)$. Then $(C \times D) = \triangle_A$. This is a contradiction.

5. Let $A \neq \emptyset$, $A \subseteq U$ and $U \sim A$ has at least two elements. Let $* : A \times A \rightarrow U$ be an injective function such that $(U \sim *(A \times A)) \cap (U \sim A)$ has at least two elements. Let $V = (A \times A) \cup ((U \sim A) \times (U \sim A))$. Let $a, b \in U \sim A$, $a \neq b$ and $c, d \in (U \sim *(A \times A)) \cap (U \sim A)$. Let $r_1 = \{\langle c, a \rangle\}$, $r_2 = \{\langle d, b \rangle\}$, $\pi = \widehat{\pi} \cup r_1$ and $\rho = \widehat{\rho} \cup r_2$. It is easy to see that r_1, $r_2 \subseteq V$, r_1 and r_2 are functions, $r_1 \cap r_2 = \emptyset$ and $r_1^{\,d} \cup r_2^{\,d} \subseteq \triangle_{U \sim \{y : (\exists x) x \in V \wedge x * y\}}$.
   Now $\langle \mathcal{P}(V), \pi, \rho \rangle$ is used as the example. By 4.10.6 $(\pi^{-1} \mid \rho) = A \times A$.
   By 2.14.4 $r_1^{-1} \mid V \mid r_2 \subseteq \pi^{-1} \mid V \mid \rho$

6. Let $\langle\mathcal{P}(V),\pi,\rho\rangle$ be the example in 5. By 4.10.9 $Ie(\pi^{-1} \mid \rho)$ holds in $\langle\mathcal{P}(V),\pi,\rho\rangle$. $r_1 \mid V \subseteq \pi \mid V$ and $r_1 \mid V \cap \rho \mid V = \emptyset$. Then $\neg\mathbf{D}$ holds in $\langle\mathcal{P}(V),\pi,\rho\rangle$. By 4.10.7 $\widehat{\pi}^i = \widehat{\rho}^i = \triangle_A$. Then by 2.14.8 $\pi^i = \triangle_A \cup \{\langle a,a\rangle\}$ and $\rho^i = \triangle_A \cup \{\langle b,b\rangle\}$. Then $\pi^i \neq \rho^i$ and by 3.5.3 $V \mid \pi = V \mid \rho$ doesn't hold on $\langle\mathcal{P}(V),\pi,\rho\rangle$. But $\langle a,b\rangle \in r_1^{-1} \mid V \mid r_2 \cap (U \sim A)\times(U \sim A)$. Then $\mathbf{R}$ doesn't hold in $\langle\mathcal{P}(V),\pi,\rho\rangle$.

□

In the literature, there are at least three basic criteria to compare varieties: By proving the equivalence of the classes, by using the inclusion order and by the use of class operators. They can be combined to compare models. For example: an interesting problem is to compare Q-relation algebras with the relational reduct of fork algebras using the inclusion order. The following two results are examples of comparisons between the relational reduct of a class with the relational reduct of another.

Theorem 5.4 *There exists an algebra $\langle\mathcal{R},\pi,\rho\rangle \in \mathbf{K}_{\boldsymbol{\Sigma}}$ such that $\mathbf{R} \in \Sigma$ and for all $\widehat{\pi}$, $\widehat{\rho}$ in the universe of $\mathcal{R}$, $\langle\mathcal{R},\widehat{\pi},\widehat{\rho}\rangle$ is not in $\mathbf{K}_{(\boldsymbol{\Sigma}\sim\{\mathbf{R}\})\cup\{\mathbf{Q}\}}$.*

PROOF Let $U = \{a,b,c\}$, $A = \{a,b\}$ and $B = \{c\}$. Consider an injective function $* : A\times B \to U$. By 4.10.8 $\mathbf{R}$ holds in $\mathcal{F}r(U\times U,*)$. The relational reduct of $\mathcal{F}r(U\times U,*)$ is $\mathcal{F}r(U)$. Suppose that $\mathbf{Q}$ holds in $\langle\mathcal{F}r(U),\pi,\rho\rangle$ for some $\pi,\rho \subseteq U\times U$. Then $\mathcal{F}r(U)$ is a finite quasiprojective relation algebra. By [TG87, p.244] U must be a singleton. This is a contradiction.
□

Theorem 5.5 *Let $\langle\mathcal{R},\pi,\rho\rangle \in \mathbf{K}_{\boldsymbol{\Sigma}}$ and $\mathbf{S}$ is not in Σ. Then there exists $\widehat{\pi},\widehat{\rho} \in \mathcal{R}$ such that $\langle\mathcal{R},\widehat{\pi},\widehat{\rho}\rangle \in \mathbf{K}_{\boldsymbol{\Sigma}\cup\{\mathbf{D}\}}$*

PROOF Let $\widehat{\pi} = (\pi^d \cdot \rho^d);\pi$ and $\widehat{\rho} = (\pi^d \cdot \rho^d);\rho$
By 2.14.6 $\widehat{\pi}^d = ((\pi^d \cdot \rho^d);\pi^d)^d = \pi^d \cdot \rho^d = ((\pi^d \cdot \rho^d);\rho^d)^d = \widehat{\rho}^d$ and $\mathbf{D}$ holds in $\langle\mathcal{R},\widehat{\pi},\widehat{\rho}\rangle$.
If $\mathbf{P} \in \Sigma$ then by 2.14.4 $\widehat{\pi};\widehat{\pi}^{\smile} \cdot \widehat{\rho};\widehat{\rho}^{\smile} \subseteq \pi;\breve{\pi} \cdot \rho;\breve{\rho} \subseteq 1'$. Then $\mathbf{P}$ holds in $\langle\mathcal{R},\widehat{\pi},\widehat{\rho}\rangle$.

$$\begin{aligned}\text{By 2.14.[6,9]}\quad \widehat{\pi}^{\smile};\widehat{\rho} &= ((\pi^d \cdot \rho^d);\pi)^{\smile};((\pi^d \cdot \rho^d);\rho) = \breve{\pi};(\pi^d \cdot \rho^d);\rho \\ &= \breve{\pi};\pi^d;\rho^d;\rho = \breve{\pi};\rho.\end{aligned}$$

If $\mathbf{Q} \in \Sigma$ then by the last identity proved $\mathbf{Q}$ holds in $\langle\mathcal{R},\widehat{\pi},\widehat{\rho}\rangle$
If $\mathcal{R} \in \Sigma$ then by 2.14.4 $\widehat{\pi}^{\smile};1;\widehat{\rho} = \breve{\pi};(\pi^d \cdot \rho^d);1;(\pi^d \cdot \rho^d);\rho \subseteq \breve{\pi};1;\rho = \breve{\pi};\rho = \widehat{\pi}^{\smile};\widehat{\rho}$ and $\widehat{\pi}^{\smile};\widehat{\rho} \subseteq \widehat{\pi}^{\smile};1;\widehat{\rho}$, then R holds in $\langle\mathcal{R},\widehat{\pi},\widehat{\rho}\rangle$.
If $\mathbf{I} \in \Sigma$ then $1;\widehat{\pi}^{\smile};\widehat{\rho};1 = 1;\breve{\pi};\rho;1 = \breve{\pi};\rho = \widehat{\pi}^{\smile};\widehat{\rho}$. Then $\mathbf{I}$ holds in $\langle\mathcal{R},\widehat{\pi},\widehat{\rho}\rangle$
□

6 Finite models

Theorem 6.1 *Let U be an infinite set and $\langle \mathcal{R}, \pi, \rho \rangle$ be a subalgebra of $\langle \mathcal{F}r(U), \pi, \rho \rangle$ if $\langle \mathcal{R}, \pi, \rho \rangle \in \mathbf{K}_{\mathbf{\Sigma}}$ with $\mathbf{Q} \in \Sigma$ then $\langle \mathcal{R}, \pi, \rho \rangle$ is an infinite model.*

PROOF Let $*, r_1, r_2$ be defined as in 4.15. In 4.15 $\pi = \widehat{\pi} \cup r_1$ and $\rho = \widehat{\rho} \cup r_2$, $*$ is injective and $r_1^d \cap r_2^d = \emptyset$.
Let the following recursive definition
$\pi^0 = \triangle_U$
$\pi^{n+1} = \pi \,|\, \pi^n$.
It will be proved that if $i \neq j$ then $\pi^i \neq \pi^j$.
Let $z_1, z_2, ..., z_{j+2} \in U$ such that $z_n \neq z_m$ for all $n \neq m$, $1 \leq n, m \leq j+2$.
By 4.7.5 $*$ is total. Then there exist $x_1, x_2, ..., x_{j+1} \in U$ such that
$\langle z_1, z_2 \rangle * x_1$
$\langle x_m, z_{m+2} \rangle * x_{m+1}$ for all $1 \leq m \leq j$.
Clearly $x_{j+1} \pi^j x_1$ and $x_{j+1} \pi^i x_{j-i+1}$. If $x_1 = x_{j-i+1}$ then $z_2 = z_{j+2}$ because $*$ is injective. This is a contradiction. Then $x_1 \neq x_{j-i+1}$. But π^i and π^j are functionals. Then it is not true that $x_{j+1} \pi^i x_1$. This implies that $\pi^i \neq \pi^j$.
$\Box$

Definition 6.2 **2** *is used to denote the relation algebra $\mathcal{F}r(\{\emptyset\})$*

V(2) is the variety of the discrete relation algebras (i.e. the class of all the relation algebras $\mathcal{R}$ where $;^{\mathcal{R}} = \cdot^{\mathcal{R}}$, $^{\smile\mathcal{R}} = id$ and $1'^{\mathcal{R}} = 1^{\mathcal{R}}$).

Theorem 6.3 *If $\langle \mathcal{R}, \pi, \rho \rangle \in \mathbf{K}_{\mathbf{\Sigma}}$ with $\mathbf{Q} \in \Sigma$ and $\langle \mathcal{R}, \pi, \rho \rangle$ is finite then $\langle \mathcal{R}, \pi, \rho \rangle \cong \langle 2, 1^2, 1^2 \rangle^n$ for some positive number n.*

PROOF Suppose that $\mathcal{R}$ is not trivial. By [Mad78, Cor.8, p.168] $\mathcal{R} \in \mathbf{RRA}$. Because $\mathcal{R}$ is finite, there is an isomorphism $\alpha : \mathcal{R} \to \prod_{i=1}^n \mathcal{R}_i$ where all the $\mathcal{R}_i$ are directly idecomposable, are not trivial and are in **PRA**. By [JT52, Thm.4.10, p.132 and Thm.4.13, p.134] $\mathcal{R}_i \in \mathbf{Simple}(\mathbf{PRA})$ for all $1 \leq i \leq n$. By [JT52, Thm.4.28, p.143] for all $1 \leq i \leq n$, $\mathcal{R}_i \subseteq \mathcal{F}r(U_i)$ for some set U_i.
$\langle \mathcal{R}_i, \pi_i(\alpha(\pi)), \pi_i(\alpha(\rho)) \rangle \in \mathbf{K}_{\mathbf{\Sigma}}$ because α is an isomorphism. Then $\mathcal{F}r(U_i)$ is a Q-relation algebra because $\mathbf{Q} \in \Sigma$.
If U_i is finite, by [TG87, p.244] U_i is a one element set. Then $\mathcal{F}r(U_i) \cong 2$. $\mathcal{R}_i \cong 2$ because $\mathcal{R}_i$ is not trivial.
If U_i is infinite, by the last theorem $\langle \mathcal{R}_i, \pi_i(\alpha(\pi)), \pi_i(\alpha(\rho)) \rangle$ is an infinite model. This is a contradiction.
$\Box$
The last theorem fails in some algebras where **R** holds and don't satisfy **E**.

Theorem 6.4 *The following statements hold.*

1. *$\langle 2^m, \pi, \rho\rangle \in \mathbf{K}_{\mathbf{\Sigma}}$ for some Σ such that $\mathbf{Q} \in \Sigma$ iff $\pi = \rho = 1^{2^m}$.*
2. *$\langle 2^m, \pi, \rho\rangle \in \mathbf{K}_{\mathbf{\Sigma}}$ for some Σ such that $\mathbf{D} \in \Sigma$ or $\mathbf{S} \in \Sigma$ iff $\pi = \rho$.*
3. *$\langle 2^m, \pi, \rho\rangle \in \mathbf{K}_{\mathbf{\Sigma}}$ and $\pi \neq \rho$ then $\Sigma \cap \{\mathbf{Q}, \mathbf{D}, \mathbf{S}\} = \emptyset$.*
4. *Let $\langle 2^m, \pi, \rho\rangle$ such that $\pi \neq \rho$ then for every Σ such that $\Sigma \cap \{\mathbf{Q}, \mathbf{D}, \mathbf{S}\} = \emptyset$ the algebra $\langle 2^m, \pi, \rho\rangle$ is in $\mathbf{K}_{\mathbf{\Sigma}}$.*

PROOF

1. $\langle 2^m, \pi, \rho\rangle \in \mathbf{K}_{\mathbf{\Sigma}}$ for some Σ such that $\mathbf{Q} \in \Sigma$ iff $1^{2^m} = \pi \cdot \rho$ iff $1^{2^m} = \pi = \rho$.
2. $\langle 2^m, \pi, \rho\rangle \in \mathbf{K}_{\mathbf{\Sigma}}$ for some Σ such that $\mathbf{D} \in \Sigma$ *or* $\mathbf{S} \in \Sigma$ iff $\pi \cdot 1^{2^m} = \rho \cdot 1^{2^m}$ iff $\pi = \rho$.
3. Obvious by 1. and 2.
4. **P,E,R,I** hold in $\langle 2^m, \pi, \rho\rangle$.

□

Theorem 6.5 *If $\langle \mathcal{R}, \pi, \rho\rangle \in \mathbf{K}_{\mathbf{\Sigma}}$ with $\{\mathbf{I}, \mathbf{D}\} \subseteq \Sigma$ then $\langle \mathcal{R}, \pi, \rho\rangle$ is finite iff $\langle \mathcal{R}, \pi, \rho\rangle \cong \langle 2^m, 1^{2^m}, 1^{2^m}\rangle \times \langle \mathcal{R}_1, 0^{\mathcal{R}_1}, 0^{\mathcal{R}_1}\rangle$ for some positive number m and finite relation algebra $\mathcal{R}_1$.*

PROOF
By 4.2 there exist $\langle \mathcal{R}_1, \pi_1, \rho_1\rangle \in \mathbf{K}_{\mathbf{\Sigma} \cup \{\mathbf{Q}, \mathbf{D}\}}$ and $\langle \mathcal{R}_2, 0^{\mathcal{R}_2}, 0^{\mathcal{R}_2}\rangle$ is in $\mathbf{RA} \uparrow LRAP$ such that $\langle \mathcal{R}, \pi, \rho\rangle \cong \langle \mathcal{R}_1, \pi_1, \rho_1\rangle \times \langle \mathcal{R}_2, 0^{\mathcal{R}_2}, 0^{\mathcal{R}_2}\rangle$. But $\langle \mathcal{R}, \pi, \rho\rangle$ is finite then the factors are finite. By 6.3 $\langle \mathcal{R}_1, \pi_1, \rho_1\rangle \cong \langle 2, 1^2, 1^2\rangle^m \cong \langle 2^m, 1^{2^m}, 1^{2^m}\rangle$ for some positive integer m.
□
The last theorem says that theorem 6.3 doesn't hold when **E** or **S** belong to Σ.

7 Simple models

Definition 7.1 *Let* $\mathbf{K}$ *be a class of algebras.* $\mathbf{Simple(K)}$ *denotes the class of simple elements of* $\mathbf{K}$*,* $\mathbf{Si(K)}$ *denotes the class of subdirectly irreducible members of* $\mathbf{K}$ *and* $\mathbf{Di(K)}$ *denotes the set of directly idecomposable elements of* $\mathbf{K}$*.*

Theorem 7.2 *Let* $\mathbf{K_\Sigma}$ *be one of the classes defined in LRAP. Then*
$\mathbf{Simple}(\mathbf{K_\Sigma}) = (\mathbf{Simple}(\mathbf{RA}){\restriction} LRAP) \cap \mathbf{K_\Sigma} = \mathbf{Si}(\mathbf{K_\Sigma}) = \mathbf{Di}(\mathbf{K_\Sigma})$

PROOF Let $\langle \mathcal{R}, \pi, \rho \rangle \in \mathbf{K_\Sigma}$. Then $\mathbf{Con}(\langle \mathcal{R}, \pi, \rho \rangle) = \mathbf{Con}(\mathcal{R})$. It follows that:
$\langle \mathcal{R}, \pi, \rho \rangle$ is simple iff $\mathcal{R}$ is simple.
$\langle \mathcal{R}, \pi, \rho \rangle$ is subdirectly irreducible iff $\mathcal{R}$ is subdirectly irreducible.
$\langle \mathcal{R}, \pi, \rho \rangle$ is directly idecomposable iff $\mathcal{R}$ is directly idecomposable.
Then, it is easy to check that:
$\mathbf{Simple}(\mathbf{K_\Sigma}) = (\mathbf{Simple}(\mathbf{RA}){\restriction} LRAP) \cap \mathbf{K_\Sigma}$
$\mathbf{Si}(\mathbf{K_\Sigma}) = (\mathbf{Si}(\mathbf{RA}){\restriction} LRAP) \cap \mathbf{K_\Sigma}$
$\mathbf{Di}(\mathbf{K_\Sigma}) = (\mathbf{Di}(\mathbf{RA}){\restriction} LRAP) \cap \mathbf{K_\Sigma}$
The result follows from the fact that $\mathbf{Simple}(\mathbf{RA}) = \mathbf{Di}(\mathbf{RA}) = \mathbf{Si}(\mathbf{RA})$ proved in [JT52, Thm.4.10, p.132 and Thm.4.13, p.134] (The same result is cited in [Jón82] and [Mad91a]).
□
The fact that $\mathbf{Simple(RRA)} = \mathbf{I}(\mathbf{S}(\{\mathcal{F}\mathbf{r}(\mathbf{U}) : \mathbf{U\ set}\}))$ is used in the following:

Theorem 7.3 *Let* $\langle \mathcal{R}, \pi, \rho \rangle \in \mathbf{K_\Sigma}$*. Then*

1. *If* $\mathbf{Q} \in \Sigma$ *then the class* $\mathbf{Simple}(\mathbf{K_\Sigma})$ *is equal to the set*
   $\mathbf{I}(\mathbf{S}(\{\langle \mathcal{F}r(U), \pi, \rho \rangle \in \mathbf{K_\Sigma} : U \textit{ is an infinite set}\})) \cup \mathbf{I}(\mathbf{S}(\{\langle \mathbf{2}, 1^2, 1^2 \rangle\}))$
2. *If* $\mathbf{I} \in \Sigma$ *then* $\mathbf{Simple}(\mathbf{K_\Sigma}) = \mathbf{Simple}(\mathbf{K}_{\mathbf{\Sigma}\sim\{\mathbf{I}\}\cup\{\mathbf{Q}\}}) \cup$
   $\{\langle \mathcal{R}, \pi, \rho \rangle \in \mathbf{K_\Sigma} : \mathcal{R} \in \mathbf{Simple}(\mathbf{RA}) \textit{ and } \breve{\pi}; \rho = 0^{\mathcal{R}}\}$
3. *If* $\mathbf{Q}, \mathbf{I}$ *aren't in* Σ *then the class* $\mathbf{Simple}(\mathbf{K_\Sigma} \cap (\mathbf{RRA}{\restriction}\mathbf{LRAP}))$ *is equal to* $\mathbf{I}(\mathbf{S}(\{\langle \mathcal{F}r(U), \pi, \rho \rangle \in \mathbf{K_\Sigma} : U \textit{ is a set}\}))$

PROOF

1. Let $\langle \mathcal{R}, \pi, \rho \rangle \in \mathbf{Simple}(\mathbf{K_\Sigma})$. If $\mathbf{Q} \in \Sigma$ then by [Mad78, Cor.8, p.168] $\mathbf{K_\Sigma} \downarrow LRA \subseteq \mathbf{RRA}$. Then $\mathcal{R} \in \mathbf{Simple}(\mathbf{RRA})$.
   By [JT52, Thm.4.28, p.143] $\mathbf{Simple}(\mathbf{RRA}) = \mathbf{I}(\mathbf{S}(\{\mathcal{F}\mathbf{r}(\mathbf{U}) : \mathbf{U\ set}\}))$. Then there exists a set U and an embedding $\alpha : \mathcal{R} \to \mathcal{F}r(U)$. Then $\langle \mathcal{F}r(U), \alpha(\pi), \alpha(\rho) \rangle$ is in $\mathbf{K_\Sigma}$. By [TG87, p.244] U has only one element or U is an infinite set.
   If U has only one element then $\mathcal{F}r(U) \cong 2$. It follows that $\mathcal{R}$ is trivial or $\mathcal{R} \cong 2$. By 6.4.1 $\langle \mathcal{R}, \pi, \rho \rangle \in \mathbf{I}(\mathbf{S}(\{\langle \mathbf{2}, 1^2, 1^2 \rangle\}))$.
   If U is an infinite set then $\alpha : \langle \mathcal{R}, \pi, \rho \rangle \to \langle \mathcal{F}r(U), \alpha(\pi), \alpha(\rho) \rangle$ is an embedding.
   Using the characterization of $\mathbf{Simple(RRA)}$ the other inclusion is trivial.

2. Let $\langle \mathcal{R}, \pi, \rho \rangle \in \mathbf{Simple}(\mathbf{K}_{\mathbf{\Sigma}})$. Then, by the last theorem $\mathbf{R}$ is simple. By [JT52, Thm.4.10, p.132] the only ideals of R are $0^{\mathcal{R}}$ *and* $1^{\mathcal{R}}$. But $\breve{\pi}; \rho$ is an ideal. Then $\breve{\pi}; \rho = 0^{\mathcal{R}}$ *or* $\breve{\pi}; \rho = 1^{\mathcal{R}}$. Using the last theorem and 4.1.1 the other inclusion follows.

3. Let $\langle \mathcal{R}, \pi, \rho \rangle \in \mathbf{Simple}(\mathbf{K}_{\mathbf{\Sigma}} \cap (\mathbf{RRA} \upharpoonright \mathbf{LRAP}))$. By the last theorem $\mathcal{R} \in \mathbf{Simple}(\mathbf{RRA})$. Then there exists a set U and an embedding $\alpha : \mathcal{R} \to \mathcal{F}r(U)$. Then $\alpha : \langle \mathcal{R}, \pi, \rho \rangle \to \langle \mathcal{F}r(U), \alpha(\pi), \alpha(\rho) \rangle$ is an embedding. Using the characterization of **Simple(RRA)** the other inclusion is trivial.

$\square$

If it is added to the hypothesis of 7.3.2 that $\mathbf{D} \in \Sigma$; by corollary 4.2:
$\mathbf{Simple}(\mathbf{K}_{\mathbf{\Sigma}}) = \mathbf{Simple}(\mathbf{K}_{\mathbf{\Sigma} \sim \{\mathbf{I}\} \cup \{\mathbf{Q}\}}) \cup \{\langle \mathcal{R}, 0^{\mathcal{R}}, 0^{\mathcal{R}} \rangle : \mathcal{R} \in \mathbf{Simple}(\mathbf{RA})\}$.

8 Equivalent varieties with a binary operation like ∇

In section 3.2 it was proved that $\mathbf{K}_{\{\mathbf{Q},\mathbf{P},\mathbf{D}\}}$ is equivalent to the variety **AFA**. The question of this section is: is it possible to build varieties in the language LAFA that are equivalent to $\mathbf{K}_{\mathbf{\Sigma}}$?. Here an affirmative answer is given for the classes $\mathbf{K}_{\mathbf{\Sigma}} \cap (\mathbf{RRA} \upharpoonright LRAP)$ with $\mathbf{D} \in \Sigma$.

The first problem is: how to define ∇ in a class $\mathbf{K}_{\mathbf{\Sigma}}$?.In section 3.2, for $\mathbf{K}_{\{\mathbf{Q},\mathbf{P},\mathbf{D}\}}$ it was defined as in the following definition.

Definition 8.1 *Let r, s be terms in LRAP. The following abbreviations are made:*

1. $(r;\breve{\pi}) \cdot (s;\breve{\rho})$ *is abbreviated to* $r\nabla s$ *(fork)*
2. $(\pi;r)\nabla(\rho;s)$ *is abbreviated to* $r \otimes s$ *(cross)*
3. $(1'\nabla 1')$ *is abbreviated to* 2. *(make a copy)*

The last two abbreviations will are used only in the next section.

The following theorem gives an insight on how the properties of ∇ may be defined.

Theorem 8.2 *(a) Let $\mathcal{R} \in \mathbf{RRA}$. Then the following statements hold in $\langle \mathcal{R}, \pi, \rho \rangle$.*

1. $(\breve{v};x) \cdot (\breve{w};y) \subseteq \breve{\pi};\rho \rightarrow ((v;\breve{w}) \cdot (x;\breve{y})) \approx (v\nabla x);(w\nabla y)\breve{}$
2. $Fn(\pi) \wedge Fn(\rho) \wedge \mathbf{Q} \rightarrow ((v;\breve{w}) \cdot (x;\breve{y})) \approx (v\nabla x);(w\nabla y)\breve{}$
3. $\mathbf{R} \rightarrow [[(v^i \subseteq \pi^i \wedge x^i \subseteq \rho^i) \vee (w^i \subseteq \pi^i \wedge y^i \subseteq \rho^i)] \rightarrow (v;\breve{w}) \cdot (x;\breve{y}) \approx (v\nabla x);(w\nabla y)\breve{}]$
4. $\mathbf{S} \rightarrow [[((v+x)^i \subseteq (\pi+\rho)^i) \vee ((w+y)^i \subseteq (\pi+\rho)^i)] \rightarrow ((v;\breve{w}) \cdot (x;\breve{y})) \approx (v\nabla x);(w\nabla y)\breve{}]$
5. *completar caso E*

(b) Let $\langle \mathcal{R}, \pi, \rho \rangle$ where $\mathcal{R} \in \mathbf{RA}$ then the following statement holds
$Fn(\pi) \wedge Fn(\rho) \wedge \mathbf{I} \wedge \mathbf{D} \rightarrow (v;\breve{w}) \cdot (x;\breve{y}) \cdot (\breve{\pi};\rho) \approx (v\nabla x);(w\nabla y)\breve{}$

PROOF

(a) 1. This result is cited in [Mad93].

2. A proof of this fact is in [TG87, p.97]. But it is an easy consequence of item 1.

3. Suppose that $v^i \subseteq \pi^i \wedge x^i \subseteq \rho^i$. Then $\breve{v}; x = v^i; \breve{v}; x; x^i \subseteq \pi^i; \breve{v}; x; \rho^i$ and $\pi^i; \breve{v}; x; \rho^i \subseteq \pi^i; 1; \rho^i = \breve{\pi}; 1; \rho = \breve{\pi}; \rho$. Similarly $\breve{w}; y \subseteq \breve{\pi}; \rho$ when $w^i \subseteq \pi^i \wedge y^i \subseteq \rho^i$. Finally the result follows using 1.

4. The proof is similar to item 3.

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(b) By corollary 4.2 there exist $\langle \mathcal{R}_1, \pi_1, \rho_1 \rangle$ in $\mathbf{K}_{\mathbf{\Sigma} \cup \{\mathbf{Q}, \mathbf{D}\}}$ and $\langle \mathcal{R}_2, 0^{\mathcal{R}_2}, 0^{\mathcal{R}_2} \rangle$ in $\mathbf{RA} \upharpoonright LRAP$ such that $\langle \mathcal{R}, \pi, \rho \rangle \cong \langle \mathcal{R}_1, \pi_1, \rho_1 \rangle \times \langle \mathcal{R}_2, 0^{\mathcal{R}_2}, 0^{\mathcal{R}_2} \rangle$. By item 1. of part (a), the desired property trivially holds in $\langle \mathcal{R}_1, \pi_1, \rho_1 \rangle$. By definition of ∇ and elementary properties of 0 the result holds in $\langle \mathcal{R}_2, 0^{\mathcal{R}_2}, 0^{\mathcal{R}_2} \rangle$.

□

Note that the properties of part (a) can be derived using the first property,that is valid only in classes with representable relational reduct. In [Mad93] There is an example of a member of $\mathbf{K}_{\{\mathbf{R}, \mathbf{P}, \mathbf{D}\}}$ with relational reduct not representable in which the first formula of (a) fails.

Before defining the formulas of ∇ for the equivalence candidates, it is necesary to express π and ρ in the language LAFA (using ∇). The following result solves the problem.

Proposition 8.3 $\mathbf{K}_{\mathbf{\Sigma}} \models \{(1'\nabla 1)\breve{} \approx \pi, (1 \nabla 1')\breve{} \approx \rho\}$ *iff* $\mathbf{D} \in \Sigma$

PROOF

$(\rightarrow)$ By definition of ∇ and 2.14.[1,2, 3,16]

$$\begin{aligned}\pi; 1 &= (1'\nabla 1)\breve{}; 1 = ((1'; \breve{\pi}) \cdot (1; \breve{\rho}))\breve{}; 1 = (\pi \cdot (\rho; 1)); 1 = (\pi; 1) \cdot (\rho; 1) \\ &= ((\pi; 1) \cdot \rho); 1 = ((1; \breve{\pi}) \cdot (1'; \breve{\rho}))\breve{}; 1 = (1 \nabla 1')\breve{}; 1 = \rho; 1\end{aligned}$$

Then $\mathbf{D} \in \Sigma$.

$(\leftarrow)$ By definition of ∇, $\mathbf{D}$ and 2.14.[1,2,3]
$(1'\nabla 1)\breve{} = ((1'; \breve{\pi}) \cdot (1; \breve{\rho}))\breve{} = (\pi \cdot (\rho; 1)) = (\pi \cdot (\pi; 1)) = \pi$
The other identity is proved in the same way.

□

Now all the information needed to define the properties of the equivalence candidates in the language LAFA is available.

Definition 8.4 *To define the classes, the following formulas in the language LAFA are considered:*

DQ: $((v;\breve{w})\cdot(x;\breve{y}))\approx(v\nabla x);(w\nabla y)\breve{}$

DR: $[(v^i\subseteq((1'\nabla 1)\breve{})^i\wedge x^i\subseteq((1\nabla 1')\breve{})^i)\vee(w^i\subseteq((1'\nabla 1)\breve{})^i\wedge y^i\subseteq((1\nabla 1')\breve{})^i)]$
$\rightarrow(v;\breve{w})\cdot(x;\breve{y})\approx(v\nabla x);(w\nabla y)\breve{}$

DI: $(v;\breve{w})\cdot(x;\breve{y})\cdot((1'\nabla 1);((1\nabla 1')\breve{}))\approx(v\nabla x);(w\nabla y)\breve{}$

DE: *completar*

DS: $[((v+x)^i\subseteq(((1'\nabla 1)+(1\nabla 1'))\breve{})^i)\vee((w+y)^i\subseteq(((1'\nabla 1)+(1\nabla 1'))\breve{})^i)]$
$\rightarrow((v;\breve{w})\cdot(x;\breve{y}))\approx(v\nabla x);(w\nabla y)\breve{}$

DN: $Fn((1'\nabla 1)\breve{})\wedge Fn((1\nabla 1')\breve{})\wedge[(\breve{v};x)\cdot(\breve{w};y)\subseteq(1'\nabla 1);((1\nabla 1')\breve{})\rightarrow(v;\breve{w})\cdot(x;\breve{y})\approx(v\nabla x);(w\nabla y)\breve{}]$

DP: $(1'\nabla 1)\breve{}\,\nabla\,(1\nabla 1')\breve{}\subseteq 1'$

D∇: $(r\nabla s)\approx(r;(1'\nabla 1))\cdot(s;(1\nabla 1'))$

In the following, the names of the items are used for referring to the formulas.

Definition 8.5 *Let $S\nabla a$ be the following set of sets of axioms in LAFA:*

$S\nabla a=\{\Omega:\Omega\subset\{DQ,DR,DI,DE,DS,DN,DP\}\ \wedge$
$\qquad\qquad exactly\ one\ element\ of\ \{DQ,DR,DI,DE,DS,DN\}\ is\ in\ \Omega\}$

$S\nabla a$ has 12 sets of axioms. Each set is used to define a class in LAFA.

Definition 8.6 *For each $\Omega\in S\nabla a$ a class $\mathbf{AFA_\Omega}$ is defined such that:*
$\mathbf{AFA_\Omega}=\{\mathcal{R}:\mathcal{R}\downarrow LRA\in\mathbf{RA}\wedge\mathcal{R}\models\Omega\cup\{D\nabla\}\}$

Note that $\mathbf{AFA}=\mathbf{AFA_{\{DQ,DP\}}}$.

Theorem 8.7 *(a) Let $\mathcal{A}=\langle\mathcal{R},\pi,\rho\rangle\in\mathbf{K_\Sigma}\cap(\mathbf{RRA}\uparrow LRAP)$ with $\mathbf{D}\in\Sigma$. Then the algebra $\langle\mathcal{R},\nabla\rangle$ where for all $r,s\in\mathcal{R}$, $(r\nabla s)=(r;\breve{\pi})\cdot(s;\breve{\rho})$ is in $\mathbf{AFA_\Omega}$ such that*

- $\mathbf{D}l\in\Omega$ *iff* $l\in\Sigma-\{\mathbf{D}\}$
- $\mathbf{DN}\in\Omega$ *iff* $\{\mathbf{Q},\mathbf{R},\mathbf{S},\mathbf{E}\}\cap\Sigma=\emptyset$

(b) Let $\mathcal{A}=\langle\mathcal{R},\pi,\rho\rangle\in\mathbf{K_{\{I,D\}}}$. Then the algebra $\langle\mathcal{R},\nabla\rangle$ where for all $r,s\in\mathcal{R}$, $(r\nabla s)=(r;\breve{\pi})\cdot(s;\breve{\rho})$ is in $\mathbf{AFA_\Omega}$ such that $\mathbf{D}l\in\Omega$ iff $l\in\Sigma-\{\mathbf{D}\}$.

PROOF If $\mathbf{P}\in\Sigma$, by the identities for π and ρ of 8.3
$(\pi;\breve{\pi})\cdot(\rho;\breve{\rho})=(\pi\nabla\rho)=((1'\nabla 1)\breve{}\nabla(1\nabla 1')\breve{})\subseteq 1'$
Then $\mathbf{DP}\in\Omega$.

(a) Because $\mathbf{D} \in \Sigma$, by 8.2 and 8.3, if $l \in \Sigma \cap \{\mathbf{Q}, \mathbf{R}, \mathbf{S}, \mathbf{E}\}$ then $\mathbf{D}l$ holds in $\langle \mathcal{R}, \nabla \rangle$. If $\{\mathbf{Q}, \mathbf{R}, \mathbf{S}, \mathbf{E}\} \cap \Sigma = \emptyset$, because $\mathcal{R} \in \mathbf{RRA}$, by theorem 8.2 $\mathbf{DN}$ holds in $\langle \mathcal{R}, \nabla \rangle$.

(b) Because $\mathbf{D} \in \Sigma$, by 8.2 and 8.3, $\mathbf{DI}$ holds in $\langle \mathcal{R}, \nabla \rangle$.

□

Theorem 8.8 *Let* $\mathcal{A} = \langle \mathcal{R}, \nabla \rangle \in \mathbf{AFA}_{\mathbf{\Omega}}$. *Then the algebra* $\langle \mathcal{R}, \pi, \rho \rangle$ *where* $\pi = (1'\nabla 1)^{\smile}$ *and* $\rho = (1\nabla 1')^{\smile}$ *is in* $\mathbf{K}_{\mathbf{\Sigma}}$ *where* $\mathbf{D} \in \Sigma$ *and* $l \in \Sigma$ *iff* $\mathbf{D}l \in \Omega$.

PROOF By fact 8.3 $\mathbf{D} \in \Sigma$.

If $\mathbf{DQ} \in \Omega$ The proof that π and ρ are functionals and that $\mathbf{Q} \in \Sigma$ is in 3.6.

Let $\mathbf{DR} \in \Omega$. By $\mathbf{D}\nabla$, and 2.14.[7,10]

$$\begin{aligned}(1'\nabla 1) &= (1';(1'\nabla 1)) \cdot (1;(1\nabla 1')) = (1';(1'\nabla 1)^d;(1'\nabla 1)) \cdot (1;(1\nabla 1')^d;(1\nabla 1')) \\ &= (1';(1'\nabla 1)^d \nabla (1;(1\nabla 1')^d) = (1';(\breve{\pi})^d \nabla (1;(\breve{\rho})^d) = (\pi^i \nabla (1;\rho^i))\end{aligned}$$

In the same way can be proved that $(1\nabla 1') = ((1;\pi^i)\nabla \rho^i)$.
By $\mathbf{DR}$ and 2.14.5

$$\begin{aligned}\breve{\pi};\pi &= (1'\nabla 1);(1'\nabla 1)^{\smile} = (\pi^i \nabla (1;\rho^i));(\pi^i \nabla (1;\rho^i))^{\smile} \\ &= (\pi^i;\pi^i) \cdot (1;\rho^i;\rho^i;1) = \pi^i \cdot (1;\rho^i;1) \subseteq 1'\end{aligned}$$

In the same way it is proved that ρ is functional.

By $\mathbf{DR}$, 2.14.[1,3, 5,16] and 3.5.1

$$\begin{aligned}\breve{\pi};\rho &= (1'\nabla 1);(1\nabla 1')^{\smile} = (\pi^i \nabla (1;\rho^i));((1;\pi^i)\nabla \rho^i)^{\smile} = (\pi^i;\pi^i;1) \cdot (1;\rho^i;\rho^i) \\ &= (\pi^i;1) \cdot (1;\rho^i) = ((\pi^i;1) \cdot (1;\rho^i));1' = ((\pi^i;1) \cdot (\rho^i;1)^{\smile});1' \\ &= (\pi^i;1);(1' \cdot (\rho^i;1)) = \pi^i;1;\rho^i = \breve{\pi};1;\rho\end{aligned}$$

If $\mathbf{DS} \in \Omega$, the proof that π and ρ are functional and that $R \in \Sigma$ is similar to the proof when $\mathbf{DR} \in \Omega$.
By $\mathbf{DS}$ and 2.14.5

$$\begin{aligned}(\pi+\rho)^i &= (\pi+\rho)^i \cdot (1;(\pi+\rho)^i;1) = ((\pi+\rho)^i \nabla 1;(\pi+\rho)^i);((\pi+\rho)^i \nabla 1;(\pi+\rho)^i)^{\smile} \\ &\subseteq (1'\nabla 1);(1'\nabla 1)^{\smile} = \breve{\pi};\pi = \pi^i\end{aligned}$$

Similarly it can be proved that $(\pi+\rho)^i \subseteq \rho^i$. Then $\pi^i = \rho^i$. By 2.14.17 $1;\pi = 1;\rho$ and $\mathbf{S} \in \Sigma$.

If $\mathbf{DI} \in \Omega$ then by 2.14.[1,14]
$\breve{\pi};\pi = (1'\nabla 1);(1'\nabla 1)^{\smile} = 1';1' \cdot 1;1 \cdot ((1'\nabla 1);(1\nabla 1')^{\smile}) \subseteq 1'$
To see that ρ is functional, the proof is similar.
By $\mathbf{D}$, 2.14.[1,2,3, 14], $\mathbf{DI}$ and $\mathbf{D}\nabla$.

$$1;(\breve{\pi};\rho);1 = ((1;\breve{\pi})\cdot(1;\breve{\rho}));((\rho;1)\cdot(\pi;1)) = (1\nabla 1);(1\nabla 1)\breve{}$$
$$= 1\cdot((1'\nabla 1);(1\nabla 1')\breve{}) = \breve{\pi};\rho$$

If $\mathbf{DE}\in\Omega$ completar

If $\mathbf{DP}\in\Omega$ By $\mathbf{DP}$ and $\mathbf{D}\nabla$
$(1'\nabla 1)\breve{}\,\nabla\,(1\nabla 1')\breve{} = (\pi\nabla\rho) = (\pi;(1'\nabla 1))\cdot(\rho;(1\nabla 1')) = (\pi;\breve{\pi})\cdot(\rho;\breve{\rho}) \subseteq 1'$
Then $\mathbf{P}\in\Sigma$.
$\Box$
Now the theorem of equivalence can be formulated.

Theorem 8.9 *(a) Each class* $\mathbf{K}_{\boldsymbol{\Sigma}} \cap (\mathbf{RRA}\upharpoonright LRAP)$ *where* $\mathbf{D}\in\Sigma$ *and* $\mathbf{I}$ *is not in* Σ*, is equivalent to the class* $\mathbf{AFA}_{\boldsymbol{\Omega}} \cap (\mathbf{RRA}\upharpoonright LRAP)$ *where* $\mathbf{D}l\in\Omega$ *if* $l\in\Sigma-\{D,I\}$ *and* $\mathbf{DN}\in\Omega$ *if* $\Sigma\cap\{R,S,E,Q\}=\emptyset$.
(b) $\mathbf{K}_{\{\mathbf{I},\mathbf{D}\}}$ *is equivalent to* $\mathbf{AFA}_{\{\mathbf{DI}\}}$ *and* $\mathbf{K}_{\{\mathbf{I},\mathbf{D},\mathbf{P}\}}$ *is equivalent to* $\mathbf{AFA}_{\{\mathbf{DI},\mathbf{DP}\}}$.

9 Elementary properties

Some elementary properties for the classes are studied in this section.

Given a class of algebras, two kinds of operations can be considered in the study of its properties: basic operations (for the interpretation of the language) and non basic operations (defined in terms of basic operations, for instance in **RA**: domain and codomain [Mad82, Mad91a, DB93, FBHV93], symmetric quotients [BSZ86, BSZ89], residuals [JT52, JT93]).
Here, the non basic operations are:

- domain and codomain (definition 2.12).
- fork, cross and make a copy (definition 8.1).

Depending on how they are used, properties can be classificated as basic (axioms and rules relating the different operations. i.e.: Dedekind rule in **RA**) and non basic (properties derived from the basic laws).

The basic properties relating the different operations used here are the formulas listed in theorem 8.2. There are two reasons to work in this manner:

- The existence of equivalent classes to the classes $\mathbf{K}_{\boldsymbol{\Sigma}} \cap (\mathbf{RRA} \upharpoonright LRAP)$ where $\mathbf{D} \in \Sigma$ using a binary fork like operation (Theorem 8.9).
- The existence of papers that define in some classes with projection elements (i.e. heterogeneous direct products, Q-relation algebras), fork like operations and have some properties of them.

There is not a classification of the properties of a class of models. But certain classes of properties were intensively studied in each well-known class. They are arithmetical properties: commutative, associative, distributive and modular laws. In [CT51] Chin and Tarski talk about them.
Properties for specification and derivation of programs: for instance, simplification, domain and codomain laws, rules relating the diferent operations and monotonic properties. In [Par90], Partsch made an interesting classification of them.

Here, the arithmetical properties are distributive and semidistributive laws and properties useful for program derivation are simplifications, domain and codomain properties and monotonic laws.

The pioneering paper Tarski [Tar41] was the first major attempt at an axiomatic investigation of the theory of **RA**. A comprehensive account of the arithmetic properties of **RA** can be found in Chin, Tarski [CT51]. Properties of fork algebras were specially investigated in [FBHV93, FHV95, DB93, BHVW92, FAN93, HBS93]. Some properties of full products and direct products are presented in [Mad93]. There are some laws of heterogeneous direct products in

[Des89]. Finally, a few properties of Q-relation algebras and pairing algebras can be found in [TG87, AJN91]. Inspired in the above references, some elementary properties for the representable classes are studied.

Theorem 9.1 *Let* $\mathcal{R} \in \mathbf{RA}$. *Then the following statements hold in* $\langle \mathcal{R}, \pi, \rho \rangle$.

1. $(r \otimes s)\breve{} \approx (\breve{r} \otimes \breve{s})$
2. $r;(s \nabla t) \subseteq (r;s)\nabla(r;t)$
   $Fn(r) \rightarrow r;(s\nabla t) \approx (r;s)\nabla(r;t)$
3. $(r \subseteq s) \wedge (t \subseteq q) \rightarrow (r \nabla t) \subseteq (s \nabla q)$ *and* $(r \subseteq s) \wedge (t \subseteq q) \rightarrow (r \otimes t) \subseteq (s \otimes q)$
4. $(r\nabla 0) \approx (0 \nabla r) \approx (r \otimes 0) \approx (0 \otimes r) \approx 0$
5. $Fn(\pi) \wedge Fn(\rho) \rightarrow (r\nabla s) \cdot (t \nabla q) \approx (r \cdot t)\nabla(s \cdot q)$ *and*
   $Fn(\pi) \wedge Fn(\rho) \rightarrow (r \otimes s) \cdot (t \otimes q) \approx (r \cdot t) \otimes (s \cdot q)$
6. $r\nabla(s+t) \approx (r\nabla s) + (r \nabla t)$ *and* $r \otimes (s+t) \approx (r \otimes s) + (r \otimes t)$
   $(r+s)\nabla t \approx (r \nabla t) + (s \nabla t)$ *and* $(r+s) \otimes t \approx (r \otimes t) + (s \otimes t)$
7. $Fn(\pi) \wedge Fn(\rho) \rightarrow (r \cdot t);\breve{\pi} \approx (r;\breve{\pi}) \cdot (t;\breve{\pi})$ *and*
   $Fn(\pi) \wedge Fn(\rho) \rightarrow (r \cdot t);\breve{\rho} \approx (r;\breve{\rho}) \cdot (t;\breve{\rho})$
8. $Fn(\pi) \wedge Fn(\rho) \rightarrow \pi;(r \cdot t) \approx (\pi;r) \cdot (\pi;t)$ *and*
   $Fn(\pi) \wedge Fn(\rho) \rightarrow \rho;(r \cdot t) \approx (\rho;r) \cdot (\rho;t)$
9. $Fn(\pi) \wedge Fn(\rho) \rightarrow (1' \otimes 1') \cdot 1' = (1' \otimes 1')^d$
10. $Fn(\pi) \wedge Fn(\rho) \wedge \pi;1 \subseteq \rho;1 \rightarrow \pi^d \approx (1' \otimes 1')^d$
    $Fn(\pi) \wedge Fn(\rho) \wedge (\pi;1 \subseteq \rho;1) \wedge ((1' \otimes 1') \subseteq 1') \rightarrow \pi^d \approx (1' \otimes 1')$
    $Fn(\pi) \wedge Fn(\rho) \wedge \rho;1 \subseteq \pi;1 \rightarrow \rho^d \approx (1' \otimes 1')^d$
    $Fn(\pi) \wedge Fn(\rho) \wedge (\rho;1 \subseteq \pi;1) \wedge ((1' \otimes 1') \subseteq 1') \rightarrow \rho^d \approx (1' \otimes 1')$
11. $(\breve{\pi};\rho \approx 1) \wedge (\pi;1 \subseteq \rho;1) \rightarrow (\pi^i \approx 1')$
    $(\breve{\pi};\rho \approx 1) \wedge (\rho;1 \subseteq \pi;1) \rightarrow (\rho^i \approx 1')$
12. $Fn(\pi) \wedge Fn(\rho) \wedge r \subseteq (1\nabla 1) \rightarrow r;(1' \otimes 1')^d \approx r$
    $Fn(\pi) \wedge Fn(\rho) \wedge r \subseteq (1\nabla 1) \wedge ((1' \otimes 1') \subseteq 1') \rightarrow r;(1' \otimes 1') \approx r$
13. $Fn(\pi) \wedge Fn(\rho) \wedge r \subseteq (1 \otimes 1) \rightarrow (1' \otimes 1')^d;r \approx r$
    $Fn(\pi) \wedge Fn(\rho) \wedge r \subseteq (1 \otimes 1) \wedge ((1' \otimes 1') \subseteq 1') \rightarrow (1' \otimes 1');r \approx r$

PROOF
The result of item 9 is proved bellow. By definition of domain and item 1.
$(1' \otimes 1')^d = ((1' \otimes 1');(1' \otimes 1')\breve{}) \cdot 1' = ((1' \otimes 1');(1' \otimes 1')) \cdot 1'$
By definition of $\otimes$ and monotonicity properties.
$(1' \otimes 1')^d = (((\pi;\breve{\pi}) \cdot (\rho;\breve{\rho}));((\pi;\breve{\pi}) \cdot (\rho;\breve{\rho}))) \cdot 1' \subseteq (\pi;(\breve{\pi};\pi);\breve{\pi})) \cdot (\rho;(\breve{\rho};\rho);\breve{\rho})) \cdot 1'$
By functionality of π and ρ and definition of $\otimes$.
$(1' \otimes 1')^d \subseteq (\pi;\breve{\pi}) \cdot (\rho;\breve{\rho}) \cdot 1' = (1' \otimes 1') \cdot 1'$
The inclusion $(1' \otimes 1') \cdot 1' \subseteq (1' \otimes 1')^d$ is obvious.

To make a proof of item 10 similar to the proof in [FBHV93, DB93], the representability of $\mathcal{R}$ is needed. But bellow is showed that this condition is not necesary.

By **D** and definition of ∇.
$\pi^d = \pi;\breve{\pi} \cdot 1' = ((1'\nabla 1)\breve{};(1'\nabla 1)) \cdot 1' = ((\pi \cdot (\rho;1));(\breve{\pi} \cdot (1;\breve{\rho}))) \cdot 1'$
By monotonicity properties.
$\pi^d \subseteq (\pi;\breve{\pi}) \cdot (\rho;1;\breve{\rho}) \cdot 1' \subseteq (\pi;\breve{\pi}) \cdot (\rho;1) \cdot 1'$
By domain and $\otimes$ definition.
$\pi^d \subseteq (\pi;\breve{\pi}) \cdot 1' \cdot \rho^d = (\pi;\breve{\pi}) \cdot (\rho;\breve{\rho}) \cdot 1' = (1' \otimes 1') \cdot 1'$
By domain and $\otimes$ definition.
$(1' \otimes 1') \cdot 1' = (\pi;\breve{\pi}) \cdot (\rho;\breve{\rho}) \cdot 1' \subseteq (\pi;\breve{\pi}) \cdot 1' = \pi^d$
Then the first property of item 10 holds. The third is similar. By 2.14.12 the other two statements follow.

Item 12 is proved bellow.

By definition of ∇ and 2.14.[3,1,7,16].
$(1\nabla 1)^i = ((1;\breve{\pi}) \cdot (1;\breve{\rho}))^i = ((\pi;1) \cdot (\rho;1))^d = (((\pi;1) \cdot \rho);1)^d$
By 2.14.10.
$(1\nabla 1)^i = ((\pi;1) \cdot \rho)^d \subseteq (\pi;1)^d \cdot \rho^d = \pi^d \cdot \rho^d$
By domain definition, definition of $\otimes$ and item 9
$(1\nabla 1)^i \subseteq (\pi;\breve{\pi}) \cdot (\rho;\breve{\rho}) \cdot 1' = (1' \otimes 1') \cdot 1' = (1' \otimes 1')^d$
By hipothesis $r^i \subseteq (1' \otimes 1')^d$ and by using 2.14.[10,6].
$r;(1' \otimes 1')^d = r;r^i;(1' \otimes 1')^d = r;(r^i \cdot (1' \otimes 1')^d) = r;r^i = r$

The proof of the other results of this theorem can be found in [FBHV93, DB93]. $\square$

The properties of the following theorem hold for classes with representable relational reduct. The properties considering the **Q** condition can be found in [FBHV93, DB93]. The correspondent properties considering the **S** condition are omitted.

Theorem 9.2 *Let $\langle \mathcal{R}, \pi, \rho \rangle$ where $\mathcal{R} \in$ **RRA** where π and ρ are functionals. Then the following statements hold in $\langle \mathcal{R}, \pi, \rho \rangle$.*

1. $(\breve{r}; s) \subseteq (\breve{\pi}; \rho) \rightarrow (r \nabla s)\breve{} \approx (\breve{r} \nabla \breve{s}); \breve{2}$
   $Rn(\pi, \rho) \wedge (r^i \subseteq \pi^i) \wedge (s^i \subseteq \pi^i) \rightarrow (r \nabla s)\breve{} \approx (\breve{r} \nabla \breve{s}); \breve{2}$

2. $((\breve{r}; s) \subseteq (\breve{\pi}; \rho)) \wedge ((\pi; 1) \subseteq (\rho; 1)) \wedge (r^i \subseteq s^i) \rightarrow ((r \nabla s); \pi) \approx r$

   $Rn(\pi, \rho) \wedge (r^i \subseteq \pi^i) \wedge (s^i \subseteq \rho^i) \wedge ((\pi; 1) \subseteq (\rho; 1)) \wedge (r^i \subseteq s^i)$
   $\rightarrow ((r \nabla s); \pi) \approx r$

   $((\breve{r}; s) \subseteq (\breve{\pi}; \rho)) \wedge ((\rho; 1) \subseteq (\pi; 1)) \wedge (s^i \subseteq r^i) \rightarrow ((r \nabla s); \rho) \approx s$

   $Rn(\pi, \rho) \wedge (s^i \subseteq \rho^i) \wedge (r^i \subseteq \pi^i) \wedge ((\rho; 1) \subseteq (\pi; 1)) \wedge (s^i \subseteq r^i)$
   $\rightarrow ((r \nabla s); \rho) \approx s$

3. $((\breve{r}; s) \subseteq (\breve{\pi}; \rho)) \wedge ((\pi; 1) \subseteq (\rho; 1)) \rightarrow (r \otimes s); \pi \approx (1' \otimes s^d); \pi; r$
   $Rn(\pi, \rho) \wedge (r^i \subseteq \pi^i) \wedge (s^i \subseteq \rho^i) \wedge ((\pi; 1) \subseteq (\rho; 1)) \rightarrow (r \otimes s); \pi \approx (1' \otimes s^d); \pi; r$
   $((\breve{r}; s) \subseteq (\breve{\pi}; \rho)) \wedge ((\rho; 1) \subseteq (\pi; 1)) \rightarrow (r \otimes s); \rho \approx (r^d \otimes 1'); \rho; s$
   $Rn(\pi, \rho) \wedge (r^i \subseteq \pi^i) \wedge (s^i \subseteq \rho^i) \wedge ((\pi; 1) \subseteq (\rho; 1)) \rightarrow (r \otimes s); \rho \approx (r^d \otimes 1'); \rho; s$

4. $((\breve{r}; \breve{\pi}; \rho; s) \subseteq (\breve{\pi}; \rho)) \rightarrow (r \otimes s)^d \approx (r^d \otimes s^d)^d$
   $((\breve{r}; \breve{\pi}; \rho; s) \subseteq (\breve{\pi}; \rho)) \wedge (1' \otimes 1') \subseteq 1' \rightarrow (r \otimes s)^d \approx (r^d \otimes s^d)$
   $Rn(\pi, \rho) \wedge ((\pi; r)^i \subseteq \pi^i) \wedge ((\rho; s)^i \subseteq \rho^i) \rightarrow (r \otimes s)^d \approx (r^d \otimes s^d)^d$
   $Rn(\pi, \rho) \wedge (r^i \subseteq \pi^i) \wedge (s^i \subseteq \rho^i) \rightarrow (r \otimes s)^d \approx (r^d \otimes s^d)^d$

   $Rn(\pi, \rho) \wedge ((\pi; r)^i \subseteq \pi^i) \wedge ((\rho; s)^i \subseteq \rho^i) \wedge ((1' \otimes 1') \subseteq 1')$
   $\rightarrow (r \otimes s)^d \approx (r^d \otimes s^d)$

5. $(r; \breve{s}) \subseteq (\breve{\pi}; \rho) \rightarrow (r \otimes s)^i \approx (r^i \otimes s^i)^d$
   $(r; \breve{s}) \subseteq (\breve{\pi}; \rho) \wedge (1' \otimes 1') \subseteq 1' \rightarrow (r \otimes s)^i \approx (r^i \otimes s^i)$
   $Rn(\pi, \rho) \wedge (\breve{r}^i \subseteq \pi^i) \wedge (\breve{s}^i \subseteq \rho^i) \rightarrow (r \otimes s)^i \approx (r^i \otimes s^i)^d$
   $Rn(\pi, \rho) \wedge (\breve{r}^i \subseteq \pi^i) \wedge (\breve{s}^i \subseteq \rho^i) \wedge ((1' \otimes 1') \subseteq 1') \rightarrow (r \otimes s)^i \approx (r^i \otimes s^i)^d$

6. $(r; \breve{s}) \subseteq (\breve{\pi}; \rho) \rightarrow (r \nabla s)^d \approx r^d \cdot s^d$
   $Rn(\pi, \rho) \wedge (r^i \subseteq \pi^i) \wedge (s^i \subseteq \rho^i) \rightarrow (r \nabla s)^d \approx r^d \cdot s^d$

7. $((\pi; 1) \subseteq (\rho; 1)) \wedge ((\breve{r}; s) \subseteq (\breve{\pi}; \rho)) \wedge ((\breve{r}; s) \subseteq (\breve{\rho}; \pi)) \rightarrow (r \nabla s)^i \subseteq (r^i \otimes s^i)^d$
   $((\pi; 1) \subseteq (\rho; 1)) \wedge ((\breve{r}; s) \subseteq (\breve{\pi}; \rho)) \wedge Eq(\breve{\pi}; \rho) \rightarrow (r \nabla s)^i \subseteq (r^i \otimes s^i)^d$

   $((\pi; 1) \subseteq (\rho; 1)) \wedge ((\breve{r}; s) \subseteq (\breve{\pi}; \rho)) \wedge ((\breve{r}; s) \subseteq (\breve{\rho}; \pi)) \wedge ((1' \otimes 1') \subseteq 1')$
   $\rightarrow (r \nabla s)^i \subseteq (r^i \otimes s^i)$

   $((\pi; 1) \subseteq (\rho; 1)) \wedge ((\breve{r}; s) \subseteq (\breve{\pi}; \rho)) \wedge Eq(\breve{\pi}; \rho) \wedge ((1' \otimes 1') \subseteq 1')$
   $\rightarrow (r \nabla s)^i \subseteq (r^i \otimes s^i)$

$$((\pi;1) \subseteq (\rho;1)) \wedge Rn(\pi,\rho) \wedge (r^i \subseteq \pi^i) \wedge (s^i \subseteq \rho^i) \rightarrow (r\nabla s)^i \subseteq (r^i \otimes s^i)^d$$

$$((\pi;1) \subseteq (\rho;1)) \wedge Rn(\pi,\rho) \wedge (r^i \subseteq \pi^i) \wedge (s^i \subseteq \rho^i) \wedge ((1' \otimes 1') \subseteq 1') \rightarrow (r\nabla s)^i \subseteq (r^i \otimes s^i)$$

8. $((\pi;1) \subseteq (\rho;1)) \wedge ((1' \otimes 1') \subseteq 1') \wedge ((\breve{p};q) \cdot (r;(\breve{\pi};\rho);\breve{s})) \subseteq (\breve{\pi};\rho) \rightarrow (p\nabla q);(r \otimes s) \approx (p;r)\nabla(q;s)$

$$((\pi;1) \subseteq (\rho;1)) \wedge ((1' \otimes 1') \subseteq 1') \wedge Rn(\pi,\rho) \wedge [((p^i \subseteq \pi^i) \wedge (q^i \subseteq \rho^i)) \vee (((r;\breve{\pi})^d \subseteq \pi^i) \wedge ((\rho;\breve{s})^i \subseteq \rho^i))] \rightarrow (p\nabla q);(r \otimes s) \approx (p;r)\nabla(q;s)$$

9. $((\pi;1) \subseteq (\rho;1)) \wedge ((1' \otimes 1') \subseteq 1') \wedge ((\breve{p};(\breve{\pi};\rho);q) \cdot (r;(\breve{\pi};\rho);\breve{s})) \subseteq (\breve{\pi};\rho) \rightarrow (p \otimes q);(r \otimes s) \approx (p;r) \otimes (q;s)$

$$((\pi;1) \subseteq (\rho;1)) \wedge ((1' \otimes 1') \subseteq 1') \wedge Rn(\pi,\rho) \wedge [(((\pi;p)^i \subseteq \pi^i) \wedge ((\rho;q)^i \subseteq \rho^i)) \vee (((r;\breve{\pi})^d \subseteq \pi^i) \wedge ((\rho;\breve{s})^i \subseteq \rho^i))] \rightarrow (p \otimes q);(r \otimes s) \approx (p;r) \otimes (q;s)$$

PROOF For each of these results only one basic property can be considered. Each result can be easily proved taking the correspondent proofs in [FBHV93, DB93] and verifying that for each application of the second axiom of **AFA** the correspondent basic property is applicable. These verification steps are easy because the needed applicability conditions are included in the hypothesis of the result. □

Theorem 9.3 *Let $\langle \mathcal{R},\pi,\rho\rangle$ where $\mathcal{R} \in$ **RA** where π and ρ are functionals and $\{$**I, D**$\}$ holds in $\langle \mathcal{R},\pi,\rho\rangle$. Then the following statements hold in $\langle \mathcal{R},\pi,\rho\rangle$.*

1. $(r\nabla s);(t \otimes q) \approx ((r;t)\nabla(s;q)) \cdot (\breve{\pi};\rho)$
   $(r\nabla s);(t \otimes q) \approx ((r;t)\nabla(s;q));(1' \otimes 1')$
   $((1' \otimes 1') \subseteq 1') \rightarrow (r\nabla s);(t \otimes q) \approx ((r;t)\nabla(s;q))$
2. $(r \otimes s);(t \otimes q) \approx ((r;t) \otimes (s;q)) \cdot (\breve{\pi};\rho)$
   $(r \otimes s);(t \otimes q) \approx ((r;t)\nabla(s;q));(1' \otimes 1')$
   $((1' \otimes 1') \subseteq 1') \rightarrow (r \otimes s);(t \otimes q) \approx ((r;t)\nabla(s;q))$
3. $(\breve{\pi};\rho) \cdot (r\nabla s)\breve{} \approx (\breve{r} \otimes \breve{s});\breve{2}$
   $(1' \otimes 1');(r\nabla s)\breve{} \approx (\breve{r} \otimes \breve{s});\breve{2}$
4. $((\pi;1) \subseteq (\rho;1)) \wedge (r^d \subseteq s^d) \rightarrow (r\nabla s);\pi \approx r \cdot (\breve{\pi};\rho)$
   $((\rho;1) \subseteq (\pi;1)) \wedge (s^d \subseteq r^d) \rightarrow (r\nabla s);\rho \approx s \cdot (\breve{\pi};\rho)$
5. $((\pi;1) \subseteq (\rho;1)) \rightarrow (r \otimes s);\pi \approx (1' \otimes s^d);\pi;r$
   $((\rho;1) \subseteq (\pi;1)) \rightarrow (r \otimes s);\rho \approx (r^d \otimes 1');\rho;s$

6. $((\pi;1) \approx (\rho;1)) \rightarrow \pi^i \approx \rho^i \approx (\breve{\pi};\rho) \cdot 1'$

7. $(r\nabla s)^d \approx (r^d \cdot s^d) \cdot (\breve{\pi};\rho)$

8. $(r \otimes s)^d = (r^d \otimes s^d)^d$
$((1' \otimes 1') \subseteq 1') \rightarrow (r \otimes s)^d = (r^d \otimes s^d)$

9. $(r \otimes s)^i = (r^i \otimes s^i)^d$
$((1' \otimes 1') \subseteq 1') \rightarrow (r \otimes s)^i = (r^i \otimes s^i)$

10. $(r\nabla s)^i \subseteq (r^i \otimes s^i)^i$

PROOF Only the items 1, 5 and 10 are proved. The others are very easy.

1 By 9.1.1, 2.14.2, ∇ definition and 8.2.

$$\begin{aligned}(r\nabla s);(t \otimes q) &= (r\nabla s);(\breve{t} \otimes \breve{q})^{\smile} = (r\nabla s);(\pi;\breve{t}\nabla\rho;\breve{q})^{\smile}\\ &= (r;(\pi;\breve{t})^{\smile}) \cdot (s;(\rho;\breve{q})^{\smile}) \cdot (\breve{\rho};\pi)\end{aligned}$$

By 2.14.[3,2] and definition of ∇.
$(r\nabla s);(t \otimes q) = (r;t;\breve{\pi}) \cdot (s;q;\breve{\rho}) \cdot (\breve{\rho};\pi) = (r;t\nabla s;q) \cdot (\breve{\rho};\pi)$

By 8.2, definition of $\otimes$ and 9.1.1.

$$\begin{aligned}(r\nabla s);(t \otimes q) &= (r;t\nabla s;q);(\pi\nabla\rho)^{\smile} = (r;t\nabla s;q);(1' \otimes 1')^{\smile}\\ &= (r;t\nabla s;q);(1' \otimes 1')\end{aligned}$$

5 By hypothesis, 2.14.[3,2] and 9.1.1.
$(r \otimes s);\pi = (r \otimes s);(1'\nabla 1)^{\smile} = ((1'\nabla 1);(r \otimes s)^{\smile})^{\smile} = ((1'\nabla 1);(\breve{r} \otimes \breve{s}))^{\smile}$
By item 1, 2.14.[3,2,1] 9.1.1 and definition of ∇.
$(r\otimes s);\pi = ((\breve{r}\nabla 1;\breve{s});(1'\otimes 1'))^{\smile} = (1'\otimes 1');(\breve{r}\nabla 1;\breve{s})^{\smile} = (1'\otimes 1');(\breve{r};\breve{\pi}\cdot 1;\breve{s};\breve{\rho})^{\smile}$
By 2.14.[3,2,16, 17].

$$\begin{aligned}(r \otimes s);\pi &= (1' \otimes 1');((\pi;r) \cdot (\rho;s;1)) = (1' \otimes 1');((\pi;r) \cdot (\rho;s^i;1))\\ &= (1' \otimes 1');(\pi \cdot (\rho;s^i;1));r\end{aligned}$$

By 2.14.[3,2,1], item 1 and hypothesis.

$$\begin{aligned}(r \otimes s);\pi &= (1' \otimes 1')^{\smile};(\breve{\pi} \cdot (1;s^{d\smile};\breve{\rho}))^{\smile};r = ((1'\nabla(1;s^d))^{\smile};(1' \otimes 1'))^{\smile};r\\ &= ((1'\nabla 1);(1' \otimes s^d))^{\smile};r = (1' \otimes s^d);\pi;r\end{aligned}$$

10 By 9.1.[12,1] and 2.14.[3, 2,1].
$((r\nabla s)^{\smile})^d = (((r\nabla s);(1\otimes 1))^{\smile})^d = (((r\nabla s);(1\otimes 1)^{\smile})^{\smile})^d = ((1\otimes 1);(r\nabla s)^{\smile})^d$
By item 3 and 2.14.9.
$((r\nabla s)^{\smile})^d = ((\breve{r} \otimes \breve{s});\breve{2})^d = ((\breve{r} \otimes \breve{s});\breve{2}^d)^d$

By 2.14.[3,2] and by definition of 2, ∇ and $\otimes$.
$\breve{2}^d = ((1'\nabla 1')\breve{})^d = (\pi \cdot \rho)^d = ((\pi \cdot \rho);(\pi \cdot \rho)\breve{}) \cdot 1' \subseteq (\pi \cdot \rho);(\pi \cdot \rho)\breve{}$
$\breve{2}^d \subseteq (\pi;\breve{\pi}) \cdot (\rho;\breve{\rho}) = (1' \otimes 1')$
Using the last inequality.
$((r\nabla s)\breve{})^d \subseteq ((\breve{r} \otimes \breve{s});(1' \otimes 1'))^d$

By 2.14.[7,9], 9.1.13 and item 8.
$(r^i \otimes s^i)^d = (\breve{r}^d \otimes \breve{s}^d)^d = (\breve{r} \otimes \breve{s})^d = ((\breve{r} \otimes \breve{s});(1' \otimes 1')^d)^d = ((\breve{r} \otimes \breve{s});(1' \otimes 1'))^d$

$\square$

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