# A Finite Axiomatization for Fork Algebras

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Abstract: Proper fork algebras are algebras of binary relations over a structured set. The underlying set has changed from a set of pairs to a set closed under an injective function. In this paper we present a representation theorem for their abstract counterpart, that entails that proper fork algebras — whose underlying set is closed under an injective function — constitute a finitely based variety.

Keywords: fork algebras, relation algebras, representability.

Resumo: As fork algebras próprias são álgebras de relações binárias sobre um conjunto estruturado. O conjunto base foi modificado a partir de um conjunto de pares para um conjunto fechado por uma função injetiva. Nesse artigo nós apresentamos um teorema de representação para as fork algebras abstractas pelo qual se prova que as fork algebras próprias – cujo conjunto base é fechado por uma função injetiva – constituem uma variedade finitamente baseada.

Palavras–chave: fork algebras, álgebras relacionais, representabilidade.

# 1 Introduction

Proper fork algebras (PFA's) arose in computer science with the work of Haeberer and Veloso as a relational framework for expressing non–deterministic programs and reasoning about them. They were extensively used for program specification, as well as for program development in relational calculi [1, 2, 4, 5, 6, 12, 13].

It is well known that relation algebras (RA's), the abstract counterpart of the algebras of binary relations [3, 23], have weak expressive power, since their logical counterpart — denoted by $\mathcal{L}^{\times}$ in [25] — is equipollent in means of expression and proof with a three variable fragment of first–order predicate logic (see [25] p.76–89 for a detailed proof of this). An example is shown in [25]: the formula

$$(\forall x, y, z)\,(\exists u)\,(\neg x = u \ \wedge \ \neg y = u \ \wedge \ \neg z = u)^{1}$$

from Tarski's elementary theory of binary relations [23] cannot be expressed by any formula from Tarski's calculus of binary relations[2]. Such lack of expressiveness had its impact in programming, for first–order specifications of programs are not likely to have meaningful translations into the calculus of relations. Thus, a change was required. If we examine the above formula and analyze the reasons why it was not expressible, we will realize that by losing individual variables we also lose information which should be kept. In order to preserve such information, we considered having relations on a structured universe, rather than on a plain set.

From 1990 —when we created the first class of fork algebras— until now, we defined several different classes of algebras of binary relations, with the aim of finding a framework suitable for programming needs. PFA's, as all these different versions were called, are algebras of binary relations on a structured set [7, 8, 27]. The underlying structured set changed from a true pairing structure —the objects were originally true set theoretical pairs in the first class defined in [26]— to a set closed under an injective function [10]. Whether or not PFA's are finitely axiomatizable is a question that immediately arises because frameworks with few simple rules are avidly sought. Furthermore, such finite presentation should be single sorted over relations, much as most functional calculi have variables ranging only over functions. This allows their properties to be succinctly expressed.

Mikulás, Sain, Simon, and Németi showed in [18, 21] that some of the other versions of PFA's [26, 9], on which the underlying domain was made of pairs, were not finitely axiomatizable. This was done by proving that the theory of natural numbers can be interpreted in the equational theory of PFA's, and thus leading to a non–recursively enumerable equational theory. In [10] we

[1] Notice that this formula has four variables ranging over individuals.

[2] See [23] for the presentation of the calculus of relations.

presented a class of proper fork algebras wich we proved is finitely axiomatizable over complete and atomic Boolean Algebras, but the axiomatization contained a non–equational axiom. Furthermore, because Boolean completeness cannot be expressed in first–order predicate logic, the class as a whole remained non axiomatizable.

In this paper we propose another version of proper fork algebras which is shown to be a finitely based variety. From our viewpoint, proper fork algebras originated as a framework to model the input/output semantics of specifications and programs. In such setting, the finitization result gives us a new tool: the abstract fork algebras which, with their lack of individual variables, are the relational counterpart of functional calculi.

In Section 2 we define a new version of PFA's, new because it differs from the previous ones [1, 2, 4, 5, 7, 8, 10, 12, 13, 26, 27] which are slight variations of the one originally introduced informally in [14]. This new version originated from remarks by Roger Maddux in [17], and also at the Dagstuhl RelMiCS encounter in January, 1994, later discussed during a meeting in Rio de Janeiro held in August, 1994. The obvious reason for introducing this class of fork algebras is to elude the previously mentioned Mikulás, Sain and Simon negative result [18]. The latter result was specially devised to show the non–finite axiomatizability of pairing algebras in general, and of the version of PFA's presented in [13] in particular. This negative result was also avoided in Németi [19, 20], and Saín–Németi [21] in such way that they did not change the original definition of PFA's, but instead used non–well–founded set theories. In Section 3, by using as a central result a theorem due to Tarski on the representability of quasi–projective relation algebras, we present the finite axiomatizability theorem for fork algebras.

# 2 Proper Fork Algebras

In order to define PFA's[3], we will first define the class of powerset $\star$PFA's by

DEFINITION 2.1 A powerset proper fork algebra (noted by $\star$PFA) is a two sorted structure with domains $\mathcal{P}(V)$ and $U$

$$<\mathcal{P}(V), U, \cup, \cap, {}^{\prime}, \emptyset, V, |, Id, {}^{\smile}, \underline{\nabla}, \star>$$

such that

1. $V$ is an equivalence relation with domain $U$,

[3] It is important to remark that PFA's are quasi–concrete structures since, as was pointed out by Andréka and Németi in a private communication, concrete structures must be fully characterized by their underlying domain, something that does not happen with proper fork algebras because of the (hidden) operation $\star$.

2. $|$, $Id$, $\breve{}$ and $'$ stand respectively for composition between binary relations, the diagonal relation on $U$, the converse of binary relations, and set complementation (with respect to $V$), thus making the reduct $<\mathcal{P}(V), \cup, \cap, ', \emptyset, V, |, Id, \breve{}>$ an algebra of binary relations,

3. $\star : U \times U \to U$ is an injective function when its domain is restricted to $V$,

4. whenever $xVy$ and $xVz$, also $xV \star (y, z)$,

5. $\mathrm{R}\underline{\nabla}\mathrm{S} = \{<x, \star(y,z)> : x\mathrm{R}y \text{ and } x\mathrm{S}z\}$.

Definition 2.2 The class of PFA's is defined as $\mathbf{S}\,\mathbf{Rd}\,[\star\mathsf{PFA}]$, where $\mathbf{Rd}$ takes reducts to the similarity type $<\cup, \cap, ', \emptyset, V, |, Id, \breve{}, \underline{\nabla}>$ and $\mathbf{S}$ takes subalgebras.

In Defs. 2.1 and 2.2, the function $\star$ performs the role of pairing, encoding pairs of objects into single objects. It is important to notice that there are $\star$ functions which are quite distinct from the function of set–theoretical pair formation, i.e., there are models in which $\star(x, y)$ is not the same as $(x, y)$.

Finally, we define the class of representable fork algebras, which is the candidate to be finitely axiomatized.

Definition 2.3 We define the class of representable fork algebras as RFA = $\mathbf{I}$ PFA[4].

# 3 Finite Axiomatizability of PFA's

In this section we will present a theorem ensuring the finite axiomatizability of PFA's. In order to prove the theorem we will define the class of abstract fork algebras (AFA's) as the models of a finite set of axioms $Ax$. After doing this, we will prove a representation theorem establishing that $\textsc{Mod}\,(Ax)$ = RFA. AFA's are an extension of RA's[5] by means of a new operator called fork — and denoted by $\nabla$— which, like the relation–algebraic operators, is axiomatized by equations whose variables range over relations.

Definition 3.1 An abstract fork algebra is an algebraic structure

$$<R, +, \bullet, ^{-}, \mathit{0}, \infty, ;, \mathit{1}, ^{\smile}, \nabla>$$

satisfying the following set of axioms

[4]Here $\mathbf{I}$ stands for the operator of taking the isomorphism class of a given class of algebras.
[5]For arithmetical properties of RA's refer to either of [3, 15, 23].

1. Axioms stating that $<R, +, \bullet, ^-, 0, \infty, ;, 1, ^\smile>$ is a relation algebra[6] where $<R, +, \bullet, ^-, 0, \infty>$ is the Boolean reduct, $<R, ;, 1>$ is the monoid reduct, and $^\smile$ stands for relational converse,
2. $r \nabla s = (r ; (1 \nabla \infty)) \bullet (s ; (\infty \nabla 1))$,
3. $(r \nabla s) ; (t \nabla q)^\smile = (r ; t^\smile) \bullet (s ; q^\smile)$,
4. $(1 \nabla \infty)^\smile \nabla (\infty \nabla 1)^\smile \preceq 1$[7].

In the next lemma we will prove some properties of the objects $(1 \nabla \infty)^\smile$ and $(\infty \nabla 1)^\smile$, which show that they form a pair of quasi–projections, as defined in [25] p.96.

Lemma 3.2 *The objects* $(1 \nabla \infty)^\smile$ *and* $(\infty \nabla 1)^\smile$ *are functional* [8]*. Moreover, the equation* $(1 \nabla \infty)^{\smile\smile} ; (\infty \nabla 1)^\smile = \infty$ *holds in every* AFA*.*

Proof Let us prove first the functionality of $(1 \nabla \infty)^\smile$.

| | | | |
|---|---|---|---|
| $(1 \nabla \infty)^{\smile\smile} ; (1 \nabla \infty)^\smile$ | $=$ | $(1 \nabla \infty) ; (1 \nabla \infty)^\smile$ | [$^\smile$ is an involution] |
| $(1 \nabla \infty) ; (1 \nabla \infty)^\smile$ | $=$ | $(1 ; 1^\smile) \bullet (\infty ; \infty^\smile)$ | [by Ax. 3 in Def. 3.1 ] |
| $(1 ; 1^\smile) \bullet (\infty ; \infty^\smile)$ | $=$ | $(1 ; 1) \bullet (\infty ; \infty)$ | [by properties of RA's] |
| $(1 ; 1) \bullet (\infty ; \infty)$ | $=$ | $1 \bullet \infty$ | [by properties of RA's] |
| $1 \bullet \infty$ | $=$ | $1$ | [$\infty$ is the Boolean unit] |

In a similar way we have the functionality of $(\infty \nabla 1)^\smile$.

| | | | |
|---|---|---|---|
| $(1 \nabla \infty)^{\smile\smile} ; (\infty \nabla 1)^\smile$ | $=$ | $(1 \nabla \infty) ; (\infty \nabla 1)^\smile$ | [$^\smile$ is an involution] |
| $(1 \nabla \infty) ; (\infty \nabla 1)^\smile$ | $=$ | $(1 ; \infty^\smile) \bullet (\infty ; 1^\smile)$ | [by Ax. 3 in Def. 3.1 ] |
| $(1 ; \infty^\smile) \bullet (\infty ; 1^\smile)$ | $=$ | $(1 ; \infty) \bullet (\infty ; 1)$ | [by properties of RA's] |
| $(1 ; \infty) \bullet (\infty ; 1)$ | $=$ | $\infty \bullet \infty$ | [$1$ is neuter regarding $;$] |
| $\infty \bullet \infty$ | $=$ | $\infty$ | [$\infty$ is the Boolean unit] |

■

We will call $(1 \nabla \infty)^\smile$ the first quasi–projection and $(\infty \nabla 1)^\smile$ the second quasi–projection of an AFA. We will denote the first quasi–projection by $\pi$ and the second by $\rho$. Relation algebras with projection elements have been widely studied. In [24] and [25] relation algebras having functional elements $A$ and $B$ satisfying $A^\smile ; B = \infty$ are called by Tarski quasi projective relation algebras (QRA's), and Maddux, in [16], calls structures $<R, A, B>$ ($R$ a relation algebra), pairing relation algebras (PRA's). From Lemma 3.2 we immediately obtain the following corollary.

[6] Equational axiomatic systems for relation algebras are given in [3, 22, 25].

[7] We use $\preceq$ for the partial ordering induced by the Boolean structure.

[8] A relation $r$ is said to be functional whenever it satisfies the inequation $r^\smile ; r \preceq 1$.

Corollary 3.3 *The relation algebra reduct of any* AFA *is a* QRA. *Moreover, if we call $\mathcal{R}$ the* RA *reduct of a given abstract fork algebra, then any structure* $<\mathcal{R}, \pi, \rho>$ *is a* PRA[9].

As the next step we will prove that the axioms characterizing AFA's are satisfied in any PFA, thus establishing that RFA $\subseteq$ AFA.

Theorem 3.4 RFA $\subseteq$ AFA.

Proof By the definition of PFA's the reduct of any PFA to the similarity type $<\cup, \cap, ', \emptyset, V, |, Id, ^{\smile}>$ is an algebra of relations, thus satisfying the axioms characterizing the RA reduct of AFA's.

In order to show that Ax. 2 in Def. 3.1 holds in every PFA, we will prove that for any relations R and S, $R \underline{\nabla} S = (R | (Id \underline{\nabla} V)) \cap (S | (V \underline{\nabla} Id))$.

We have $<x, y> \in R \underline{\nabla} S$ iff for some $v, w \in U$

$$<x, v> \in R, \; <x, w> \in S, \; y = \star(v, w) \text{ and } <v, w> \in V,$$

iff (since $<v, \star(v, w)> \in Id \underline{\nabla} V$, $<v, w> \in V$ and $<w, \star(v, w)> \in V \underline{\nabla} Id$ are equivalent), for some $v \in U$

$$<x, v> \in R \text{ and } <v, y> \in Id \underline{\nabla} V,$$

and for some $w \in U$

$$<x, w> \in S \text{ and } <w, y> \in V \underline{\nabla} Id$$

iff

$$<x, y> \in (R | (Id \underline{\nabla} V)) \cap (S | (V \underline{\nabla} Id)).$$

In order to prove that Ax. 3 from Def. 3.1 holds in every PFA, we will prove that for any relations R, S, T and Q

$$(R \underline{\nabla} S) | (T \underline{\nabla} Q)^{\smile} = (R | T^{\smile}) \cap (S | Q^{\smile}).$$

We have

$$<x, y> \in (R \underline{\nabla} S) | (T \underline{\nabla} Q)^{\smile}.$$

iff for some $z \in U$

$$<x, z> \in R \underline{\nabla} S \text{ and } <z, y> \in (T \underline{\nabla} Q)^{\smile},$$

[9] PRA's are presented in [16] p. 327.

iff for some $z$, $a$, $b$, $c$, $d \in U$

$$z = \star(a, b)\,,\ \ <x, a> \in \mathrm{R},\ \ <x, b> \in \mathrm{S},\ \ <a, b> \in V \tag{1}$$

and

$$z = \star(c, d)\,,\ \ <y, c> \in \mathrm{T},\ \ <y, d> \in \mathrm{Q},\ \ <c, d> \in V. \tag{2}$$

Since by hypothesis $\star$ is injective we have that 1 and 2 hold iff for some $a$, $b \in U$ with $<a, b> \in V$

$$<x, a> \in \mathrm{R}, <x, b> \in \mathrm{S}, <a, y> \in \mathrm{T}^{\smile} \text{ and } <b, y> \in \mathrm{Q}^{\smile}.$$

iff (since $\mathrm{R}^{\smile}|\mathrm{S} \subseteq V$)

$$<x, y> \in (\mathrm{R}|\mathrm{T}^{\smile}) \cap (\mathrm{S}|\mathrm{Q}^{\smile})\,.$$

In order to prove that Ax. 4 from Def. 3.1 holds in every PFA, we will prove that

$$(Id\underline{\nabla}V)^{\smile}\underline{\nabla}(V\underline{\nabla}Id)^{\smile} \subseteq Id.$$

Let us have

$$<x, y> \in (Id\underline{\nabla}V)^{\smile}\underline{\nabla}(V\underline{\nabla}Id)^{\smile}.$$

Then for some $v$, $w \in U$,

$$y = \star(v, w)\,,\ \ <v, x> \in Id\underline{\nabla}V,\ \ <w, x> \in V\underline{\nabla}Id \text{ and } <v, w> \in V.$$

So for some $v$, $w \in U$,

$$y = \star(v, w)\,,$$

for some $a$, $b \in U$

$$x = \star(a, b)\,,\ \ <v, a> \in Id,\ \ <v, b> \in V,\ \ <a, b> \in V$$

and for some $c$, $d \in U$

$$x = \star(c, d)\,,\ \ <w, c> \in V,\ \ <w, d> \in Id,\ \ <c, d> \in V.$$

Thus, since $\star : V \to U$ is injective and $Id$ is the diagonal, for some $v$, $w \in U$

$$x = \star(v, w) = y$$

as was to be proved. ■

Proposition 3.5 *Consider $A \in$ AFA with RA reduct $A'$. Given an RA homomorphism $\mathcal{H} : A' \to B$ into a proper relation algebra $B$, there exists an expansion of $B$ to $B^* \in$ PFA so that $\mathcal{H} : A \to B^*$ is an AFA homomorphism.*

Proof The algebra $B$ consists of relations over a set $U$, with unit element $V \subseteq U \times U$.

Since the quasi–projections $\pi,\ \rho \in A'$, we have $\mathcal{H}(\pi),\ \mathcal{H}(\rho) \in B$.

By Lemma 3.2, $A'$ satisfies

$$\pi^{\smile}\mathbf{;}\rho = \infty,\ \pi^{\smile}\mathbf{;}\pi \preceq \mathit{1} \text{ and } \rho^{\smile}\mathbf{;}\rho \preceq \mathit{1}.$$

So

$$\mathcal{H}(\pi)^{\sim}|\mathcal{H}(\rho) = \mathcal{H}(\infty) = V, \text{ and } \mathcal{H}(\pi) \text{ and } \mathcal{H}(\rho) \text{ are functional.}$$

We define the relation $F \subseteq (U \times U) \times U$ by

$$<<a,b>,c> \in F \quad \Longleftrightarrow \quad <c,a> \in \mathcal{H}(\pi) \text{ and } <c,b> \in \mathcal{H}(\rho).$$

1. $V \subseteq Dom(F)$.

   Given $<a,b> \in V \subseteq \mathcal{H}(\pi)^{\sim}|\mathcal{H}(\rho)$, there exists $c \in U$ such that $<a,c> \in \mathcal{H}(\pi)^{\sim}$ and $<c,b> \in \mathcal{H}(\rho)$, so $<<a,b>,c> \in F$.

2. $F$ is functional on $V$.

   Let $<<a,b>,x>,\ <<a,b>,y> \in F$. Then $<x,a> \in \mathcal{H}(\pi)$, $<x,b> \in \mathcal{H}(\rho)$ and $<y,a> \in \mathcal{H}(\pi)$, $<y,b> \in \mathcal{H}(\rho)$.

   So, $<x,y> \in (\mathcal{H}(\pi)|\mathcal{H}(\pi)^{\sim}) \cap (\mathcal{H}(\rho)|\mathcal{H}(\rho)^{\sim})$.

   But, by Axs. 2 and 4 in Def. 3.1, $A'$ satisfies $(\pi\mathbf{;}\pi^{\smile}) \bullet (\rho\mathbf{;}\rho^{\smile}) \preceq \mathit{1}$, thus

   $$(\mathcal{H}(\pi)|\mathcal{H}(\pi)^{\sim}) \cap (\mathcal{H}(\rho)|\mathcal{H}(\rho)^{\sim}) \subseteq \mathcal{H}(\mathit{1}) = Id.$$

   Hence $x = y$.

3. $F$ is injective on $V$.

   Consider $<a,b>, <c,d> \in V$ such that $<<a,b>,z>, <<c,d>,z> \in F$ for some $z \in U$.

   Then $<z,a>, <z,c> \in \mathcal{H}(\pi)$ and $<z,b>, <z,d> \in \mathcal{H}(\rho)$. Since $\mathcal{H}(\pi)$ and $\mathcal{H}(\rho)$ are functional, we have $a = c$ and $b = d$.

Hence, the restriction of $F$ to $V$ gives a well defined injective function $\star : V \to U$ such that

$$\star(a,b) = c \quad \Longleftrightarrow \quad <c,a> \in \mathcal{H}(\pi) \text{ and } <c,b> \in \mathcal{H}(\rho).$$

We expand $B$ to a PFA $B^*$ by

$$R\underline{\nabla}S = \{<x,\star(y,z)> : <x,y> \in R,\ <x,z> \in S,\ <y,z> \in V\}.$$

In order to show that $\underline{\nabla}$ is well defined, let us prove that

$$\mathcal{H}(R)\underline{\nabla}\mathcal{H}(S) = (\mathcal{H}(R)|\mathcal{H}(\pi)^{\sim}) \cap (\mathcal{H}(S)|\mathcal{H}(\rho)^{\sim}). \tag{3}$$

We have $<x,y> \in \mathcal{H}(R)\underline{\nabla}\mathcal{H}(S)$ iff for some $v, w \in U$

$$y = \star(v,w),\ <x,v> \in \mathcal{H}(R),\ <x,w> \in \mathcal{H}(S) \text{ and } <v,w> \in V,$$

iff for some $v, w \in U$ with $<v,w> \in V$

$$\begin{array}{ll} <x,v> \in \mathcal{H}(R), & <v,y> \in \mathcal{H}(\pi)^{\smallsmile} \quad \text{and} \\ <x,w> \in \mathcal{H}(S), & <w,y> \in \mathcal{H}(\rho)^{\smallsmile} \end{array}$$

iff (since $\mathcal{H}(R)^{\smallsmile}|\mathcal{H}(S) \subseteq V$)

$$<x,y> \in \mathcal{H}(R)\,|\mathcal{H}(\pi)^{\smallsmile} \text{ and } <x,y> \in \mathcal{H}(S)\,|\mathcal{H}(\rho)^{\smallsmile}.$$

We now wish to show that $\mathcal{H}$ is an AFA homomorphism from $A$ into $B^*$, i.e., that $\mathcal{H}$ preserves fork.

Consider $r, s \in A$. By Ax. 2 in Def. 3.1 $r\nabla s = (r;\pi^{\smallsmile}) \bullet (s;\rho^{\smallsmile})$.

So

$$\mathcal{H}(r\nabla s) = \mathcal{H}((r;\pi^{\smallsmile}) \bullet (s;\rho^{\smallsmile}))\,.$$

Since $\mathcal{H} : A' \to B$ is an RA homomorphism, we have

$$\mathcal{H}((r;\pi^{\smallsmile}) \bullet (s;\rho^{\smallsmile})) = (\mathcal{H}(R)\,|\mathcal{H}(\pi)^{\smallsmile}) \cap (\mathcal{H}(S)\,|\mathcal{H}(\rho)^{\smallsmile})$$

which by 3 equals $\mathcal{H}(r)\,\underline{\nabla}\mathcal{H}(s)$.

■

In the proof of Thm. 3.6 we give below, we use the result announced by Tarski in 1953 [24], on the representability of QRA's. This theorem is one of the central topics in Algebraic Logic, and different kinds of proofs for it can be found in Thm. 8.4(iii), p. 242 of Tarski–Givant [25], Maddux [16] and elsewhere.

Theorem 3.6 *Every* AFA *is isomorphic to some* PFA*.*

Proof Given $A \in$ AFA, its RA reduct $A'$ is a QRA by Cor. 3.3.

So, by [25] p. 242, we have an RA isomorphism $\mathcal{H} : A' \to B$ onto a proper relation algebra $B$.

By Prop.3.5, $B$ can be expanded to a PFA $B^*$, so that $\mathcal{H} : A \to B^*$ is an AFA isomorphism. ■

Theorem 3.7 AFA= RFA.

Proof By Thm. 3.6, AFA$\subseteq$ **I**[PFA] = RFA, and by Thm. 3.4, RFA$\subseteq$ AFA. ■

## 4 Conclusions

Classes of proper fork algebras (PFA's) as well as their abstract counterpart (AFA's) have been found useful for deriving programs from specifications [1, 4, 5, 6, 12, 13]. The representability was established for the complete and atomistic classes introduced in [10, 8]. In this paper we defined a different class of PFA's and AFA's by removing previously used assumptions. We have shown that this class has the following properties

- representability: every AFA is isomorphic to some PFA,
- finitely based variety: the closure of PFA's under isomorphism is axiomatizable by a finite number of equations,
- proper abstraction: every PFA is an AFA.

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