A new Cut-free Sequent Calculus for S5

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Abstract: The definition of a cut-free sequent calculus for S5 is a well-known problem. Traditional cut-elimination strategies do not seem to work for S5 and the main reason for this fact is that sequent calculus systems for S5 do not seem to be closed under "permutative" reductions. On the other hand, this problem is already solved for at least one Natural Deduction formalization of S5, where strong structural restrictions on the form of the branches of derivations guarantee a normalization result [2]. The aim of the present work is to define a new sequent calculus, (SC_5) for S5 where these strong global structural restrictions are simulated.

Keywords: Sequent Calculus, Cut-elimination, Modal Logic, $S5$.

Resumo: A definição de um cálculo de sequentes para a Lógica Modal S5, é um problema bem conhecido. As estratégias tradicionais para a eliminação do corte não servem para S5, pois o calculo de sequentes para S5, não é fechado quanto as reduções permutativas. Este problema é resolvido pelo menos por uma formalização em Dedução Natural, onde as restrições estruturais nas formas dos ramos das derivações, garantem o resultado da normalização [2]. Logo, o principal objetivo deste trabalho é definir um Cálculo de Sequentes, (SC_5) para S5, onde estas fortes restrições estruturais são simuladas.

Palavras-Chave: Cálculo de Sequente, Teoria Modal, S5.

1 Introduction

The definition of a cut-free sequent calculus for S5 is a well-known problem. Traditional cut-elimination strategies do not seem to work for S5 and the main reason for this fact is that sequent calculus systems for S5 do not seem to be closed under "permutative" reductions. On the other hand, this problem is already solved for at least one Natural Deduction formalization of S5, where strong structural restrictions on the form of the branches of derivations guarantee a normalization result [2].

The aim of the present work is to define a new sequent calculus, (SC_5) for S5 where these strong global structural restrictions are simulated. The new restrictions are defined through the concepts of *Deductive Determinant* ([1]) and *Left /Right* parts of formulas. The concepts of *Left /Right* parts can be considered as a natural extension to a modal setting of the concepts of Negative and Positive parts described in [3]. We show that SC_5 is sound and complete and we prove the cut-elimination theorem for SC_5 using Gentzen's standard procedure.

This approach to a cut-free formalization of $S5$ can be seen as dual to the analytic-cut approach. While the latter restricts the form of the cut-rule in order to have cut-elimination, the former restricts relationships among the subformulas of formulas present in applications of some modal rules in order to ensure that permutation of rules do not affect the standard cut-elimination procedure.

In section two, the Sequent Calculus for $S5$, is presented the completeness and correctness of this systen are shown. In section three the cut-elimination property is proved.

2 A Sequent Calculus for $S5$ - SC_5

A sequent is an ordered pair $<\Gamma, \Sigma>$ or $\Gamma \Rightarrow \Sigma$ where Γ and Σ are finite bags of formulas. Γ is called antecedent and Σ, succedent.

Intuitively, if $\Gamma = \{A_1, ..., A_m\}$, and $\Sigma = \{B_1, ..., B_n\}$ where $m, n \geq 1$,then $\Gamma \Rightarrow \Sigma$ means:

$$A_1 \wedge ... \wedge A_m \;\Rightarrow\; B_1 \vee ... \vee B_n.$$

If $n = 0$, $\Gamma \Rightarrow \Sigma$ means that $A_1 \wedge ... \wedge A_m$ yields a contradiction, and if $m = 0$, it means that $B_1 \vee ... \vee B_n$ holds.

We consider the Semantic of Possible Worlds, which may be extended with the following definition:

Definition 2.1 *Let* $\Gamma \Rightarrow \Sigma$ *be a sequent.*

$\Gamma \Rightarrow \Sigma$ *is valid for all frames* W *(*$W \models \Gamma \Rightarrow \Delta$*) iff for all* $w \in W$*, if* $V(\bigwedge \Gamma, w) = 1$ *then* $V(\bigvee \Sigma, w) = 1$*.*

To define this new Sequent Calculus - SC_5, we consider an original system of sequents as shown in Gentzen for propositional logic and add rules for modal operator of necessity ($\Box$) and rules for modal operator of possibility ($\Diamond$). The restrictions are defined through the concepts of Deductive Determinant ([1]) and *Left /Right* Parts of formulas. The concepts of *Left /Right* parts can be considered as a natural extension to a modal setting of the concepts of Negative and Positive parts described in [3].

2.1 Left Part, Right Part and Deductive Determinant

Definition 2.2 *Let F be a formula.*

The Left Part of F,(LP(F))is:

1. *LP(F)* $= \emptyset$ *if F is atomic*
2. *LP(F)* $= \{F_1\}$ *if* $F = \neg F_1$
3. *LP(F)* $= \{F_1\} \cup LP(F_2)$ *if* $F = F_1 \rightarrow F_2$
4. *LP(F)* $= LP(F_1) \cup LP(F_2)$ *if* $F = F_1 \vee F_2$
5. *LP(F)* $= LP(F_1) * LP(F_2)$ *if* $F = F_1 \wedge F_2$
6. *LP(F)* $= \Diamond LP(F_1)$ *if* $F = \Box F_1$
7. *LP(F)* $= \Box LP(F_1)$ *if* $F = \Diamond F_1$

Where:

$$\{A_1, \ldots, A_k\} * \{B_1, \ldots B_n\} = \{A_1 \wedge B_1, A_1 \wedge B_2, \ldots A_1 \wedge B_n, \ldots, A_k \wedge B_n\}$$

We may consider that the Left Part of a formula F, $LP(F)$, is a set of subformulas of F, $A_1, ... A_n$, such that for each A_i,

$$If \quad \vdash_{S5} \;\Rightarrow \neg A_i \qquad then \qquad \vdash_{S5} \;\Rightarrow \; F$$

Definition 2.3 *Let F be a formula.*
The Right Part of F,(RP(F)) is:

1. $RP(F) = \{F\}$ *if F is atomic*
2. $RP(F) = \perp$ *if* $F = \neg F_1$
3. $RP(F) = RP(F_2)$ *if* $F = F_1 \rightarrow F_2$
4. $RP(F) = RP(F_1) \ \cup \ RP(F_2)$ *if* $F = F_1 \vee F_2$
5. $RP(F) = RP(F_1) \ * \ RP(F_2)$ *if* $F = F_1 \wedge F_2$
6. $RP(F) = \Box RP(F_1)$ *if* $F = \Box F_1$
7. $RP(F) = \Diamond RP(F_1)$ *if* $F = \Diamond F_1$

We may consider that the Right Part of a formula F, $RP(F)$, is a set of subformulas of F $A_1, ...A_n$, such that for each A_i,

$$If \quad \vdash_{S5} \ \Rightarrow \ A_i \qquad then \qquad \vdash_{S5} \ \Rightarrow \ F$$

Definition 2.4 *Let F be a formula.*
The Deductive Determinant of F, $(DD(F))$ *is:*

1. $DD(\top) = \{\top\}$
2. $DD(\perp) = \emptyset$
3. $DD(F) = \{F\}$ *if F is literal*
4. $DD(F) = \{F\} \ \cup \ DD(\neg F_1) \cup DD(F_2)$ *if* $F = F_1 \ \rightarrow \ F_2$
5. $DD(F) = \{F\} \ \cup \ DD(F_1) \ \cup \ DD(F_2)$ *if* $F = F_1 \ \vee \ F_2$
6. $DD(F) = \{F\} \ \cup \ DD(F_1) \ * \ DD(F_2)$ *if* $F = F_1 \ \wedge \ F_2$
7. $DD(F) = \{F\} \ \cup \ \Box DD(F_1)$ *if* $F = \Box F_1$
8. $DD(F) = \{F\} \ \cup \ \Diamond DD(F_1)$ *if* $F = \Diamond F_1$
9. $DD(\neg F) =$

- $\{\neg F\} \cup DD(F_1) * DD(\neg F_2)$ *if* $F = F_1 \to F_2$
- $\{\neg F\} \cup DD(\neg F_1) * DD(\neg F_2)$ *if* $F = F_1 \vee F_2$
- $\{\neg F\} \cup DD(\neg F_1) \cup DD(\neg F_2)$ *if* $F = F_1 \wedge F_2$
- $\{\neg F\} \cup \Diamond DD(\neg F_1)$ *if* $F = \Box F_1$
- $\{\neg F\} \cup \Box DD(\neg F_1)$ *if* $F = \Diamond F_1$

The Deductive Determinant of a formula F, $DD(F)$, is a set of formulas α, built with subformulas of F such that, if α is provable then a proof of F can be built.

Definition 2.5 *Let* Γ *be a bag of formulas.*

- $DD(\Gamma) = DD(\bigwedge \Gamma)$
- $RP(\Gamma) = RP(\bigwedge \Gamma)$
- $LP(\Gamma) = LP(\bigwedge \Gamma)$

Lemma 2.1 $\vdash_{S5} \quad \Gamma \wedge LP(\Gamma) \Rightarrow RP(\Gamma)$

The proof of the proposition above proceeds by induction on the length of the deduction.

We will define relations between two (bags of) formulas, induced by DD, LP and RP of these (bags of) formulas.

Definition 2.6 *Let F and G be formulas:*

$$F \prec_{DD} G \ iff$$

For all X, if $X \in DD(F) - \{F\}$ *then* $X \in DD(G)$ *or* $X = X_1 \wedge ... \wedge X_n \in DD(F)$ *and for some* i, $X_i \in DD(G)$ *or F is* $\Box A$ *and G is A or F is A and G is* $\Diamond A$.

Definition 2.7 *Let F and G be formulas:*

$$F \prec_{RP} G \ iff$$

For all X, if $X \in RP(F)$ *then* $X \in RP(G)$ *or* $X = X_1 \wedge ... \wedge X_n \in RP(F)$ *and for some* i, $X_i \in RP(G)$ *or F is* $\Box A$ *and G is A or F is A and G is* $\Diamond A$.

Definition 2.8 *Let F and G be formulas*

$$F \prec_{RPDD} G \ iff$$

For all X, if $X \in RP(F)$ *then* $X \in DD(G)$ *or* $X = X_1 \wedge ... \wedge X_n \in RP(F)$ *and for some i,* $X_i \in DD(G)$ *or F is* $\Box A$ *and G is A or F is A and G is* $\Diamond A$*.*

Definition 2.9 *Let F and G be formulas:*

$$F \prec_{LP} G \ iff$$

For all X, if $X \in LP(F)$ *then* $X \in LP(G)$ *or* $X = X_1 \wedge ... \wedge X_n \in LP(G)$ *and for some i,* $X_i \in LP(F)$ *or F is* $\Diamond A$ *and G is A or F is A and G is* $\Box A$*.*

Definition 2.10 *Let* Γ *and* Δ *be bags of formulas.*

1. $\Gamma \prec_{DD} \Delta$ *iff* $\bigwedge \Gamma \prec_{DD} \bigwedge \Delta$
2. $\Gamma \prec_{RP} \Delta$ *iff* $\bigwedge \Gamma \prec_{RP} \bigwedge \Delta$
3. $\Gamma \prec_{LP} \Delta$ *iff* $\bigwedge \Gamma \prec_{LP} \bigwedge \Delta$

Based in the established definition we add the notation of ($\prec_{DDLP}$):

$$\Gamma \prec_{DDLP} \Delta \quad iff \quad \Gamma \prec_{DD} LP(\Delta)$$

Using these definitions we may prove:

Results 2.1 *If A and B are formulas we have:*

1. $\Box A \prec_{DD} A \prec_{DD} \Diamond A$, $\Box A \prec_{RP} A \prec_{RP} \Diamond A$ *and* $\Diamond A \prec_{LP} A \prec_{LP} \Box A$.
2. $A \prec_{DD} A \vee B$, $B \prec_{DD} A \vee B$, $A \prec_{RP} A \vee B$, $B \prec_{RP} A \vee B$, $A \vee B \prec_{LP} A$ *and* $A \vee B \prec_{LP} B$.
3. $A \wedge B \prec_{DD} A$, $A \wedge B \prec_{DD} B$, $A \wedge B \prec_{RP} A$, $A \wedge B \prec_{RP} B$, $A \prec_{LP} A \wedge B$, *and* $A \prec_{LP} A \wedge B$.
4. *If* $A \prec_{DD} B \vee C$ *then* $A \wedge \neg B \prec_{DD} C$ *and* $A \wedge \neg C \prec_{DD} B$.
5. *If* $A \wedge B \prec_{DD} C$ *then* $A \prec_{DD} C \vee \neg B$ *and* $B \prec_{DD} C \vee \neg A$.

6. *If* $A \prec_{DD} B$ *then* $\neg B \prec_{DD} \neg A$,*if* $A \prec_{RP} B$ *then* $\neg B \prec_{RP} \neg A$ *and if* $A \prec_{LP} B$ *then* $\neg B \prec_{LP} \neg A$.

7. *If* $A \prec_{RP} RP(B \vee C)$ *then* $A \wedge \neg B \prec_{RP} C$ *and* $A \wedge \neg C \prec_{RP} B$.

8. *If* $A \wedge B \prec_{RP} C$ *then* $A \prec_{RP} C \vee \neg B$ *and* $B \prec_{RP} C \vee \neg A$.

9. *If* $B \vee C \prec_{LP} A$ *then* $B \prec_{LP} A \vee \neg C$ *and* $C \prec_{LP} A \vee \neg B$.

10. *If* $C \prec_{LP} A \wedge B$ *then* $C \vee \neg A \prec_{LP} B$ *and* $C \vee \neg B \prec_{LP} A$.

2.2 The System

Definition 2.11 *The Sequent Calculus for S5,* $S5_{SC}$ *has:*

1. Initial Sequents: All sequents of the form :
   - $\mathbf{A} \Rightarrow \mathbf{A}$
   - $\mathbf{\Gamma}, \bot \Rightarrow \mathbf{\Sigma}$
   - $\mathbf{\Gamma} \Rightarrow \mathbf{\Sigma}, \top$

2. Operational Rules:

$$\frac{\Gamma_1 \Rightarrow A, \Sigma_1 \quad \Gamma_2 \Rightarrow B, \Sigma_2}{\Gamma_1, \Gamma_2 \Rightarrow A \wedge B, \Sigma_1, \Sigma_2}\ (\Rightarrow \wedge) \qquad \frac{\Gamma_1, \Gamma_2, A, B \Rightarrow \Sigma}{\Gamma_1, \Gamma_2, A \wedge B \Rightarrow \Sigma}\ (\wedge \Rightarrow)$$

$$\frac{\Gamma \Rightarrow A, B, \Sigma}{\Gamma \Rightarrow A \vee B, \Sigma}\ (\Rightarrow \vee) \qquad \frac{\Gamma_1, A \Rightarrow \Sigma_1 \quad \Gamma_2, B \Rightarrow \Sigma_2}{\Gamma_1, \Gamma_2, A \vee B \Rightarrow \Sigma_1, \Sigma_2}\ (\vee \Rightarrow)$$

$$\frac{\Gamma, A \Rightarrow \Sigma}{\Gamma \Rightarrow \neg A, \Sigma}\ (\Rightarrow \neg) \qquad \frac{\Gamma \Rightarrow A, \Sigma}{\Gamma, \neg A \Rightarrow \Sigma}\ (\neg \Rightarrow)$$

$$\frac{\Gamma, A \Rightarrow B, \Sigma}{\Gamma \Rightarrow A \to B, \Sigma}\ (\Rightarrow \to) \qquad \frac{\Gamma_1 \Rightarrow A, \Sigma_1 \quad \Gamma_2, B \Rightarrow \Sigma_2}{\Gamma_1, \Gamma_2, A \to B \Rightarrow \Sigma_1, \Sigma_2}\ (\to \Rightarrow)$$

$$\frac{\Gamma \Rightarrow A, \Sigma}{\Gamma \Rightarrow \Box^i A, \Sigma}\ (\Rightarrow \Box)(\star) \qquad \frac{\Gamma, A \Rightarrow \Sigma}{\Gamma, \Box A \Rightarrow \Sigma}\ (\Box \Rightarrow)$$

$$\frac{\Gamma \Rightarrow A, \Sigma}{\Gamma \Rightarrow \Diamond A, \Sigma}\ (\Rightarrow \Diamond) \qquad \frac{\Gamma, A \Rightarrow \Sigma}{\Gamma, \Diamond^i A \Rightarrow \Sigma}\ (\Diamond \Rightarrow)(\star\star)$$

($\star$) To apply the rule ($\Rightarrow \Box$), we have the following restrictions:

- The formulas of Γ and Σ are modals or negative of modal formulas, ($\Gamma \equiv \Gamma^m$ and $\Sigma \equiv \Sigma^m$) or
- A is of the form $\Diamond A'$ or $\neg\Box A'$, or
- Γ is of the form Γ', Δ and :

$$\Delta \wedge \neg\Sigma \ \prec_{DDLP} \ \Gamma' \ \textit{ and } \ \Gamma' \ \prec_{RPDD} \ \Box A$$

The rule will be denoted in each case above by ($\Rightarrow \Box$), $(\Rightarrow \Box)_1$ and $(\Rightarrow \Box)_2$.

($\star\star$) To apply the rule of ($\Diamond \Rightarrow$), we have the following restrictions:

- The formulas of Γ and Σ are modals or negative of modal formulas, ($\Gamma \equiv \Gamma^m$ and $\Sigma \equiv \Sigma^m$) or
- A is of the form $\Box A'$ or $\neg\Diamond A'$, or
- Γ is of the form Γ', Δ and :

$$\Delta \wedge \Diamond A \ \prec_{DDLP} \Gamma' \ \textit{ and } \ \Gamma' \ \prec_{RPDD} \ \Sigma$$

The rule will be denoted in each case above by ($\Diamond \Rightarrow$), $(\Diamond \Rightarrow)_1$ and $(\Diamond \Rightarrow)_2$.

3. Structural Rules:

$$Cut \quad \frac{\Gamma_1 \ \Rightarrow \ A, \Sigma_1 \quad \Gamma_2, A \ \Rightarrow \ \Sigma_2}{\Gamma_1, \Gamma_2 \ \Rightarrow \ \Sigma_1, \Sigma_2}$$

$$Contraction \ \ (cont) \quad \frac{\Gamma, B, B \ \Rightarrow \ \Sigma}{\Gamma, B \ \Rightarrow \ \Sigma} \quad \frac{\Gamma \ \Rightarrow \ B, B, \Sigma}{\Gamma \ \Rightarrow \ B, \Sigma}$$

$$Thinning \ \ (thin) \quad \frac{\Gamma \ \Rightarrow \ \Sigma}{\Gamma, C \ \Rightarrow \ \Sigma} \qquad \frac{\Gamma \ \Rightarrow \ \Sigma}{\Gamma \ \Rightarrow \ C, \Sigma}$$

We may observe that in the Structural rules above:

- In the cut rule A is called the cut-formula. A proof without the cut rule is called cut-free.
- In the contraction rule B is called the contracted-formula.
- In the thinning rule, C is called the thin-formula.
- The contraction and thinning rules will be referred to as "weak inferences" while all others will be called "strong inferences".

Definition 2.12 $\vdash_{SC_5} \quad \Gamma \Rightarrow_{Thin-free} \Sigma$ *if and only if* $\vdash_{SC_5} \quad \Gamma \Rightarrow \Sigma$ *and in this deduction the Thin rule (right) is not used.*

Proposition 2.1 *If* $\Gamma \Rightarrow \Sigma$ *then, there are* $\Gamma' \subseteq \Gamma$ *and* $\Sigma' \subseteq \Sigma$ *such that* $\Gamma' \Rightarrow_{thin-free} \Sigma'$ *and:*

1. $\bigwedge \Gamma' \prec_{DD} \bigvee \Sigma'$
2. $\bigwedge \Gamma' \prec_{RP} \bigvee \Sigma'$
3. $\bigvee \Sigma' \prec_{LP} \bigwedge \Gamma'$

The proof of this proposition proceeds by induction on the lenght of the deduction.

One can prove that the initial sequents are valid and the rules preserve validity, so we have:

Proposition 2.2 Correctness *Let* $\Gamma \Rightarrow \Sigma$ *be a sequent*
If $\Gamma \Rightarrow \Sigma$ *is a theorem of* SC_5*, then* $\Gamma \Rightarrow \Sigma$ *is valid in S5.*

One can also prove that the axioms, and rules of S5 may be proved in SC_5; thus S5 is complete.

Proposition 2.3 Completeness *Let* $\Gamma \Rightarrow \Sigma$ *be a sequent*
If $\Gamma \Rightarrow \Sigma$ *is valid in S5 then* $\Gamma \Rightarrow \Sigma$ *is a theorem of* SC_5*.*

3 The Cut-elimination Theorem

The purpose of this section is to prove the cut-elimination theorem. We shall follow Takeuti's proof

To prove this theorem, first we introduce a new rule of inference, the mix rule, and show that the mix rule and the cut are equivalent. After, we define two ways of measuring the complexity of a proof (the degree and the rank).

The operation of cut-elimination (or mix-elimination) consists in taking each cut (or mix) which has no cut (or mix) above, and successively pushing it upward until it disappears at the top. We show that this pushing upwards ought be done through any rule or initial sequent that may occur in the proof, and that each time that we push it up we must find a new deduction with the complexity less than the other.

Definition 3.1 *Let A be a formula. An inference of the following form is called a mix (with respect to A):*

$$\frac{\Gamma_1 \Rightarrow \Sigma_1 \quad \Gamma_2 \Rightarrow \Sigma_2}{\Gamma_1, \Gamma_2^{(-A)} \Rightarrow \Sigma_1^{(-A)}, \Sigma_2}\ (A)$$

if both Σ_1 and Γ_2 contain the formula A, and $\Sigma_1^{(-A)}$ and $\Gamma_2^{(-A)}$ are obtained from Σ_1 and Γ_2 respectively by deleting all the occurrences of A in them.

We may observe that the mix formula A of this inference is normally indicated in parentheses (as in the definition).

Let us call the system which is obtained from SC_5 by replacing the cut rule by mix rule, SC_{mix}.

Lemma 3.1 *SC_5 and SC_{mix} are equivalent.*

The proof of this lemma is done showing that the cut is proved in SC_{mix}, and the mix is proved in SC_5.

So, by this lemma, in order to prove the cut elimination it suffices to show that the mix rule is redundant in SC_{mix}, i.e., every deduction in SC_{mix} may be proved without a mix, because the proof of SC_{mix} without a mix is, at the same time, a proof in SC_5 without a cut.

But, in order to prove the mix-elimination, will be necessary first to define two ways of measuring the complexity of a proof:

Definition 3.2 *Let A be a formula. The degree of A (denoted by $g(A)$) is the number of logical symbols contained in A.*

Definition 3.3 *The degree of a mix is the degree of the mix formula.*

Definition 3.4 *Let Π be a deduction such that Π has only one mix as the last inference. The degree of Π (denoted by $g(\Pi)$) is the degree of this mix.*

Let Π be a proof which has only one mix as the last inference, so Π is of the form:

$$\frac{\Gamma_1 \Rightarrow \Sigma_1 \quad \Gamma_2 \Rightarrow \Sigma_2}{\Gamma_1, \Gamma_2^{(-A)} \Rightarrow \Sigma_1^{(-A)}, \Sigma_2} \quad (A)$$

In a deduction Π, we refer to the left and right upper sequents as S_1 and S_2 (or S_1 if there is only one upper sequent) and to the lower sequents of Π, as S.

Definition 3.5 *A left (right) thread in Π is a thread in Π which contains the left (right) upper sequent of the mix.*

The rank of a thread $\mathcal{F}$ in Π (denoted by rank ($\mathcal{F}$, Π)) is defined as follows:

Definition 3.6 *If $\mathcal{F}$ is a left (right) thread, then the rank of $\mathcal{F}$ is the number of consecutive sequents, counting upward from the left (right) upper sequent of the mix, that contains the mix formula in its succedent (antecedent).*

We may observe that since the left (right) upper sequent always contains the mix formula, the rank of a thread in Π is at least 1.

Definition 3.7 *Let Π be a deduction*

- If $\mathcal{F}$ ranges over all the left threads in Π,
  $rank_l(\Pi) = max_{\mathcal{F}}(rank(\mathcal{F}, \Pi))$
- If $\mathcal{F}$ ranges over all the right threads in Π,
  $rank_r(\Pi) = max_{\mathcal{F}}(rank(\mathcal{F}, \Pi))$
- The rank of Π is defined as: $rank(\Pi) = rank_l(\Pi) + rank_r(\Pi)$

We may observe that rank(Π) is always ≥ 2.

Proposition 3.1 *If* Π *is a deduction of* $\Gamma \Rightarrow \Sigma$ *which contains only one mix occurring as the last inference, then* $\Gamma \Rightarrow \Sigma$ *is provable without a mix.*

Proof: This proof will be done by double induction on the degree (g) and the rank of the proof.

(I) If rank(Π) = 2, we have the following cases:

a) The left upper sequent S_1 is an initial sequent. In this case Π is of the form:

$$\frac{A \Rightarrow A \quad \Gamma_2 \Rightarrow \Sigma_2}{A, \Gamma_2^{(-A)} \Rightarrow \Sigma_2} \ (A)$$

We can eliminate the mix, only using S_2 and some structural rules.

$$\cfrac{\cfrac{\cfrac{\Gamma_2 \Rightarrow \Sigma_2}{\vdots}}{A, ..., A, \Gamma_2^{(-A)} \Rightarrow \Sigma_2}}{\cfrac{\vdots}{A, \Gamma_2^{(-A)} \Rightarrow \Sigma_2} \ (contr.)}$$

b) If the right upper sequent S_2 is an initial sequent then we proced similarly to a).

c) If neither S_1 nor S_2 are initial sequents.

(c-i) Suppose that S_1 is the lower sequent of a structural inference. Since $rank_l(\Pi) = rank_r(\Pi) = 1$, then the mix formula A can not appear in the succedent of the upper sequent of this rule . So Π is of the form:

$$\frac{\dfrac{\Gamma_1 \Rightarrow \Sigma_1}{\Gamma_1 \Rightarrow \Sigma_1, A} \ (thin) \quad \Gamma_2 \Rightarrow \Sigma_2}{\Gamma_1, \Gamma_2^{(-A)} \Rightarrow \Sigma_1, \Sigma_2} \ (A)$$

where Σ_1 does not contain A (so $\Sigma_1 \equiv \Sigma_1^{(-A)}$). We can eliminate the mix making a new deduction Π' of the form:

$$\cfrac{\cfrac{\Gamma_1 \Rightarrow \Sigma_1}{\vdots \ (thin)}}{\Gamma_1, \Gamma_2^{(-A)} \Rightarrow \Sigma_1^{(-A)}, \Sigma_2}$$

because Σ_1 is equivalent to $\Sigma_1^{(-A)}$ and using only structural rules.

(c-ii) If S_2 is the lower sequent of a structural inference, the mix elimination is similar to (c-i).

(c-iii) Both S_1 and S_2 are lower sequents of logical inferences. But $rank_l(\Pi) = rank_r(\Pi) = 1$, so the mix formula on each side must be the principal formula of the logical inference. We will use induction on the degree of the mix-formula.

When we push the mix up, we find a new deduction Π', probably still containing a mix, but the degree of the mix-formula of Π is greater than the new mix-formula (of Π').

So, by the induction hypothesis we can obtain a proof which contains no mixes and with the same end-sequents. It is important to observe again that we will only distinguish the modal cases; the others cases are as in [5].

(1) If S_2 is lower sequent of $(\Box \Rightarrow)$, then Π is of the form:

$$\dfrac{\Gamma_1 \Rightarrow \Sigma_1 \qquad \dfrac{\Gamma_2, A \Rightarrow \Sigma_2}{\Gamma_2, \Box A \Rightarrow \Sigma_2}}{\Gamma_1, \Gamma_2^{(-\Box A)} \Rightarrow \Sigma_1^{(-\Box A)}, \Sigma_2}\ (\Box A)$$

but $rank_l(\Pi) = 1$ then suppose that $\Box A$ was introduced in Π_1 by logical inference, then S_1 may be lower sequent of:

1. $(\Rightarrow \Box)$. Then Π is of the form:

$$\dfrac{\dfrac{\Gamma_1^m \Rightarrow A, \Sigma_1^m}{\Gamma_1^m \Rightarrow \Box A, \Sigma_1^m} \qquad \dfrac{\Gamma_2, A \Rightarrow \Sigma_2}{\Gamma_2, \Box A \Rightarrow \Sigma_2}}{\Gamma_1^m, \Gamma_2^{(-\Box A)} \Rightarrow \Sigma_1^{m,(-\Box A)}, \Sigma_2}\ (\Box A)$$

   Such that the only mix in Π is this. We may push the mix up, and obtain Π' of the form:

$$\dfrac{\dfrac{\Gamma_1^m \Rightarrow \Sigma_1^m, A \qquad \Gamma_2, A \Rightarrow \Sigma_2}{\Gamma_1^m, \Gamma_2^{(-A)} \Rightarrow \Sigma_1^{m,(-A)}, \Sigma_2}\ (A)}{\dfrac{\vdots\ (thin)}{\Gamma_1^m, \Gamma_2^{(-\Box A)} \Rightarrow \Sigma_1^{m,(-\Box A)}, \Sigma_2}}$$

   Π' has degree less than Π. So we can obtain a proof which contains no mixes.

2. $(\Rightarrow \Box)_1$ Then Π is of the form:

$$\dfrac{\dfrac{\Gamma_1 \;\Rightarrow\; \Sigma_1, \Diamond A}{\Gamma_1 \;\Rightarrow\; \Sigma_1, \Box\Diamond A} \qquad \dfrac{\Gamma_2, \Diamond A \;\Rightarrow\; \Sigma_2}{\Gamma_2, \Box\Diamond A \;\Rightarrow\; \Sigma_2}}{\Gamma_1, \Gamma_2^{(-\Box\Diamond A)} \;\Rightarrow\; \Sigma_1^{(-\Box\Diamond A)}, \Sigma_2} \; (\Box\Diamond A)$$

We may push the mix up, and obtain Π' of the form:

$$\dfrac{\dfrac{\dfrac{\Gamma_1 \;\Rightarrow\; \Sigma_1, \Diamond A \qquad \Gamma_2, \Diamond A \;\Rightarrow\; \Sigma_2}{\Gamma_1, \Gamma_2^{(-\Diamond A)} \Rightarrow \Sigma_1^{(-\Diamond A)}, \Sigma_2} \; (\Diamond A)}{\vdots}}{\Gamma_1, \Gamma_2^{(-\Box\Diamond A)} \;\Rightarrow\; \Sigma_1^{(-\Box\Diamond A)}, \Sigma_2} \; (thin)$$

Π' has degree less than Π. So we can obtain a proof which contains no mixes.

3. $(\Box \Rightarrow)_2$. Can not be used if the $rank(\Pi) = 2$.

(2) If S_1 is the lower sequent of $(\Rightarrow \Diamond)$, then S_2 may be a lower sequent of $(\Diamond \Rightarrow)$, $(\Diamond \Rightarrow)_1$, we may push the mix up, and obtain Π' without mix, similar to 1.1 or 1.2, and $(\Diamond \Rightarrow)_2$ can not be used like 1.3.

(II) If $r \geq 2$; then $rank_l(\Pi) \geq 1$ and/or $rank_r(\Pi) \geq 1$. In this case the induction hypothesis is that from every proof Π' which contains a mix only as the last inference, and which satisfies either

$$g(\Pi') \leq g(\Pi) \;\; or \;\; g(\Pi') = g(\Pi) \;\; and \;\; rank(\Pi') \leq rank(\Pi)$$

we can eliminate the mix.

(A) Suppose that $rank_r(\Pi) \geq 1$ and $rank_l(\Pi) = 1$. We may have the following cases:

a) Σ_2 contains A. We will get a mix-free proof only using $\Gamma_1 \Rightarrow \Sigma_1$ and structural rules.

$$\dfrac{\dfrac{\dfrac{\dfrac{\Gamma_1 \Rightarrow \Sigma_1}{\vdots} \; cont.}{\Gamma_1 \Rightarrow \Sigma_1^{(-A)}, A}}{\vdots} \; thin}{\Gamma_1, \Gamma_2^{(-A)} \Rightarrow \Sigma_1^{(-A)}, \Sigma_2}$$

b) S_2 is the lower sequent of a non logical inference whose principal formula is A. Π is of the form:

$$\dfrac{\Gamma_1 \Rightarrow A, \Sigma_1 \qquad \dfrac{\dfrac{\Gamma_2', A \Rightarrow \Sigma_2'}{\vdots}}{\Gamma_2, A \Rightarrow \Sigma_2}}{\Gamma_1, \Gamma_2^{(-A)} \Rightarrow \Sigma_1^{(-A)}, \Sigma_2} \; (A)$$

where this is the only mix in the proof. The rule used in the deduction of $\Gamma_2, A \Rightarrow \Sigma_2$ is a structural rule, then Γ_2' contains at least one A. So we may push the mix up and we obtain Π' of the form:

$$\dfrac{\dfrac{\dfrac{\Gamma_1 \Rightarrow A, \Sigma_1 \qquad \Gamma_2', A \Rightarrow \Sigma_2}{\Gamma_1, \Gamma_2^{(-A)'} \Rightarrow \Sigma_1^{(-A)}, \Sigma_2} \; (A)}{\vdots}}{\Gamma_1, \Gamma_2^{(-A)} \Rightarrow \Sigma_1^{(-A)}, \Sigma_2}$$

And Π' is such that g(Π) = g(Π'), but $rank_l(\Pi) \geq rank_l(\Pi')$

c) Σ_2 contains no A's and S_1 is the lower sequent of a logical inference whose principal formula is A. (We are only treating the modal cases.)

(•) Suppose that S_1 is the lower sequent of a right inference, then we have the following cases:

1. Π_1 ends with $(\Rightarrow \Box)$. So Π is of the form:

$$\dfrac{\dfrac{\Gamma_1'^m \;\Rightarrow\; A, \Sigma_1^m}{\Gamma_1'^m \;\Rightarrow\; \Box A, \Sigma_1^m} \; (\Rightarrow \Box) \qquad \dfrac{\dfrac{\Gamma_2', \Box A \Rightarrow \Sigma_2'}{\vdots}}{\Gamma_2, \;\; \Box A \;\; \Rightarrow \;\; \Sigma_2}}{\Gamma_1'^m, \Gamma_2^{(-\Box A)} \;\;\Rightarrow\;\; \Sigma_1^{m,(-\Box A)}, \Sigma_2} \; (\Box A)$$

   We have to search where $\Box A$ was introduced in Π_2. If it was introduced by thinning the mix may be eliminated only using thinning.

   If Π_2 ends with $(\Box \Rightarrow)$, so Π is of the form:

$$\dfrac{\dfrac{\Gamma_1'^m \;\Rightarrow\; A, \Sigma_1^m}{\Gamma_1'^m \;\Rightarrow\; \Box A, \Sigma_1^m} \qquad \dfrac{\Gamma_2, \;\; A \;\; \Rightarrow \;\; \Sigma_2}{\Gamma_2, \;\; \Box A \;\; \Rightarrow \;\; \Sigma_2}}{\Gamma_1^m, \Gamma_2^{(-\Box A)} \;\;\Rightarrow\;\; \Sigma_1^m, \Sigma_2} \; (\Box A)$$

we may obtain Π' of the form:

$$\cfrac{\Gamma_1'^m \Rightarrow \Box A, \Sigma_1^m \qquad \cfrac{\Gamma_1'^m \Rightarrow \Box A, \Sigma_1^m \quad \Gamma_2, A \Rightarrow \Sigma_2}{\Gamma_1'^m, \Gamma_2^{(\Box A)}, A \Rightarrow \Sigma_1^m, \Sigma_2}(\Box A)}{\cfrac{\Gamma_1^m, \Gamma_1^m, \Gamma_2^{(-\Box A)} \Rightarrow \Sigma_1^m, \Sigma_1^m, \Sigma_2}{\Gamma_1^m, \Gamma_2^{(-\Box A)} \Rightarrow \Sigma_1^m, \Sigma_2} cont}(\Box A)$$

$$\begin{array}{c} \Gamma_1'^m, \Gamma_2^{(\Box A)}, \Box A \Rightarrow \Sigma_1^m, \Sigma_2 \end{array}$$

If Π_2 does not end with $(\Box \Rightarrow)$, then we have to observe what rules were used after $(\Box \Rightarrow)$; they may be:

(a) $(\Rightarrow \Box)$ So Π_2 is of the form:

$$\cfrac{\cfrac{\Gamma_2^m, \quad A \quad \Rightarrow \quad C, \Sigma_2^m}{\Gamma_2^m, \quad \Box A \quad \Rightarrow \quad C, \Sigma_2^m} (\Box \Rightarrow)}{\Gamma_2^m, \quad \Box A \quad \Rightarrow \quad \Box C, \Sigma_2^m} (\Rightarrow \Box)$$

where $\Gamma_2 = \Gamma_2^m, A$ and $\Sigma_2 = \Box C, \Sigma_2^m$

Then we may push the mix up, and then we use $(\Rightarrow \Box)$. So Π will be transformed into:

$$\cfrac{\cfrac{\Gamma_1^m \quad \Rightarrow \quad A, \Sigma_1^m \qquad \Gamma_2^m, \quad A \quad \Rightarrow \quad C, \Sigma_2^m}{\Gamma_1^m, \quad \Gamma_2^{m,(-A)} \quad \Rightarrow \quad C, \; \Sigma_1^{m,(-A)}, \; \Sigma_2^m} (A)}{\Gamma_1^m, \Gamma_2^{m,(-A)} \quad \Rightarrow \quad \Box C, \Sigma_1^{m,(-A)}, \Sigma_2^m} (\Rightarrow \Box)$$

$$\cfrac{\cfrac{}{\vdots}}{\Gamma_1^m, \quad \Gamma_2^{m,(-\Box A)} \quad \Rightarrow \quad \Box C, \Sigma_1^{m,(-\Box A)}, \Sigma_2^m}$$

(b) $(\Rightarrow \Box)_2$ So Π_2 is of the form:

$$\cfrac{\cfrac{\Gamma_2, A \quad \Rightarrow \quad C, \Sigma_2}{\Gamma_2, \Box A \quad \Rightarrow \quad C, \Sigma_2} (\Box \Rightarrow)}{\Gamma_2, \Box A \quad \Rightarrow \quad \Box C, \Sigma_2} (\Rightarrow \Box)_2$$

$$If \qquad \Box A \wedge \neg \Sigma_2 \prec_{DD} \Gamma_2 \quad and \quad \Gamma_2 \prec_{RPDD} \Box C$$

When we try to push the mix up, we construct a new deduction Π' of the form:

$$\frac{\Gamma_1^m \Rightarrow A, \Sigma_1^m \qquad \Gamma_2, A \Rightarrow C, \Sigma_2}{\Gamma_1^m, \Gamma_2^{(-A)} \Rightarrow C, \Sigma_1^{m,(-A)}, \Sigma_2}\ (A)$$

And now, we have to analyse the conditions of $(\Rightarrow \Box)_2$. But by the proposition 2.1, there exist $\Gamma_1'^m$ and $\neg\Sigma_1'^m$ such that:

$$\Gamma_1'^m \wedge \neg\Sigma_1'^m \prec_{DD} \Box A, \qquad \Gamma_1'^m \wedge \neg\Sigma_1'^m \prec_{RP} \Box A, \qquad \textit{and}$$

$$\Box A \prec_{LP} \Gamma_1'^m \wedge \neg\Sigma_1'^m$$

So using these results in the condition above, we may obtain the new conditions to use $(\Rightarrow \Box)_2$ after the mix.

(c) If Π_2 ends with $(\Rightarrow \Box)_1$, $(\Diamond \Rightarrow)$, or $(\Diamond \Rightarrow)_1$, the mix-elimination is analogous to (a). We may push the mix up and then use these rules. If Π_2 ends with $(\Diamond \Rightarrow)_2$ the mix-elimination is analogous to (b).

2. $(\Rightarrow \Box)_1$ If Π_1 ends with $(\Rightarrow \Box)_1$, then Π is of the form:

$$\frac{\dfrac{\Gamma_1 \ \Rightarrow\ \Diamond A, \Sigma_1}{\Gamma_1 \ \Rightarrow\ \Box^i\Diamond A, \Sigma_1}\ (\Rightarrow \Box)_1 \qquad \Gamma_2', \ \Box^i\Diamond A \ \Rightarrow\ \Sigma_2'}{\Gamma_1, \ \Gamma_2'^{(-\Box^i\Diamond A)} \ \Rightarrow\ \Sigma_1^{(-\Box^i\Diamond A)}, \Sigma_2'}\ (\Box^i\Diamond A)$$

We have to verify how $\Box^i\Diamond A$ was introduced in Π_2. If it was introduced by thinning the mix may be eliminated only using thinning in Π_2. Suppose that it was introduced i times by $(\Box \Rightarrow)$, we have the following cases:

If Π_2 ends with $(\Box \Rightarrow)$, So Π is of the form:

$$\frac{\dfrac{\Gamma_1 \Rightarrow \Diamond A, \Sigma_1}{\Gamma_1 \Rightarrow \Box\Diamond A, \Sigma_1} \qquad \dfrac{\Gamma_2', \Diamond A \Rightarrow \Sigma_2'}{\Gamma_2', \Box\Diamond A \Rightarrow \Sigma_2'}}{\Gamma_1, \Gamma_2'^{(\Box\Diamond A)} \Rightarrow \Sigma_1, \Sigma_2'}(\Box\Diamond A)$$

we may obtain Π' of the form:

$$\dfrac{\Gamma_1 \Rightarrow \Diamond\Box A, \Sigma_1 \qquad \dfrac{\dfrac{\Gamma_1 \Rightarrow \Box\Diamond A, \Sigma_1 \quad \Gamma_2 \Diamond A \Rightarrow \Sigma_2}{\Gamma_1, \Gamma_2^{(-\Box\Diamond A)}, \Diamond A \Rightarrow \Sigma_1, \Sigma_2}(\Box\Diamond A)}{\Gamma_1, \Gamma_2^{(-\Box\Diamond A)}, \Box\Diamond A \Rightarrow \Sigma_1, \Sigma_2}}{\dfrac{\Gamma_1, \Gamma_1, \Gamma_2^{(-\Box\Diamond A)} \Rightarrow \Sigma_1, \Sigma_1, \Sigma_2}{\Gamma_1, \Gamma_2^{(-\Box\Diamond A)} \Rightarrow \Sigma_1, \Sigma_2} Cont.}(\Box\Diamond A)$$

If Π_2 does not end with $(\Box \Rightarrow)$, we have to examine what are the rules that may be used after each $(\Box \Rightarrow)$, and they are: (we will consider only the case of n = 1).

(a) $(\Diamond \Rightarrow)$ If Π_2 is of the form:

$$\dfrac{\dfrac{\Gamma_2^m, \Diamond A, C \Rightarrow \Sigma_2^m}{\Gamma_2^m, \Box\Diamond A, C \Rightarrow \Sigma_2^m}(\Box \Rightarrow)_1}{\Gamma_2^m, \Box\Diamond A, \Diamond C \Rightarrow \Sigma_2^m}(\Diamond \Rightarrow)$$

We may push the mix up and then use $(\Diamond \Rightarrow)_2$, if the condition is first met. But by the proposition 2.1, there exist Γ_1' and $\neg\Sigma_1'$ such that:

$$\Gamma_1' \wedge \neg\Sigma_1' \prec_{DD} \Box\Diamond A, \quad \Gamma_1' \wedge \neg\Sigma_1' \prec_{RP} \Box\Diamond A, \ \ and$$
$$\Box\Diamond A \prec_{LP} \Gamma_1' \wedge \neg\Sigma_1'.$$

But in Π_2 by the proposition 2.1, there exist $\Gamma_2'^m$ and $\Sigma_2'^m$ such that:

$$\Gamma_2'^m \wedge \Box\Diamond A \wedge \Diamond C \prec_{DD} \Sigma_2'^m, \quad \Gamma_2'^m \wedge \Box\Diamond A \wedge \Diamond C \prec_{RP} \Sigma_2'^m, \ \ and$$
$$\Sigma_2'^m \prec_{LP} \Gamma_2'^m \wedge \Box\Diamond A \wedge \Diamond C.$$

Using these results we may obtain the condition to use $(\Diamond \Rightarrow)_2$.

(b) $(\Rightarrow \Box)_2$ Π_2 is of the form:

$$\dfrac{\dfrac{\Gamma_1 \Rightarrow \Diamond A, \Sigma_1}{\Gamma_1 \Rightarrow \Box^i\Diamond A, \Sigma_1} \qquad \dfrac{\dfrac{\Gamma_2, \Diamond A \Rightarrow C, \Sigma_2}{\Gamma_2, \Box^i\Diamond A \Rightarrow C, \Sigma_2}}{\Gamma_2, \Box^i\Diamond A \Rightarrow \Box C, \Sigma_2}}{\Gamma_1, \Gamma_2^{(-\Box\Diamond A)} \Rightarrow \Box C, \Sigma_1^{(-\Box\Diamond A)}, \Sigma_2}$$

If we have one of these conditions:

i. $\Gamma_2 \wedge \neg\Sigma_2 \prec_{DDLP} \Box\Diamond A$ and $\Box\Diamond A \prec_{RPDD} \Box C$, or

ii. $\Box\Diamond A \wedge \neg\Sigma_2 \prec_{DDLP} \Gamma_2$ and $\Gamma_2 \preceq_{RPDD} \Box C$

But by the proposition 2.1, there exist Γ'_1 and $\neg\Sigma'_1$ such that:

$$\Gamma'_1 \wedge \neg\Sigma'_1 \prec_{DD} \Box\Diamond A, \quad \Gamma'_1 \wedge \neg\Sigma'_1 \prec_{RP} \Box\Diamond A, \quad \textit{and}$$

$$\Box\Diamond A \prec_{LP} \Gamma'_1 \wedge \neg\Sigma'_1.$$

So using this result in the both conditions we may obtain the new condition to use $(\Rightarrow \Box)_2$, before pushing the mix up.

(c) If Π_2 ends with $(\Rightarrow \Box)_1$ or $(\Diamond \Rightarrow)_1$, we may push the mix up and then use these rules; If Π_2 ends with $(\Rightarrow \Box)$ the mix-elimination is like (a); If Π_2 ends with $(\Diamond \Rightarrow)_2$ it is like (b).

3. $(\Rightarrow \Box)_2$ Π_1 ends with $(\Rightarrow \Box)_2$. So Π is of the form:

$$\dfrac{\dfrac{\Gamma_1, \Delta \ \Rightarrow \ A, \Sigma_1}{\Gamma_1, \Delta \ \Rightarrow \ \Box A, \Sigma_1}(\Rightarrow \Box)_2 \qquad \Gamma_2,\ \Box A \ \Rightarrow \ \Sigma_2}{\Gamma_1,\ \Gamma_2^{(-\Box A)} \ \Rightarrow \ \Sigma_1^{(-\Box A)}, \Sigma_2}(\Box A)$$

If we have this condition:

$$\Gamma_1 \wedge \neg\Sigma_1 \prec_{DDLP} \Delta \ \textit{ and } \ \Delta \prec_{RPDD} \Box A,$$

We have to examine how $\Box A$ was introduced in Π_2. If by thinning the mix may be eliminated only using thinning in Π_2. If was not by thinning, then it was introduced by $(\Box \Rightarrow)$. Then we have two cases:

If Π_2 ends with $(\Box \Rightarrow)$, then Π is of the form:

$$\dfrac{\dfrac{\Gamma_1, \Delta \Rightarrow A, \Sigma_2}{\Gamma_1, \Delta \Rightarrow \Box A, \Sigma_2} \quad \dfrac{\Gamma_2, A \Rightarrow \Sigma_2}{\Gamma_2, \Box A \Rightarrow \Sigma_2}}{\Gamma_1, \Delta, \Gamma_2^{(-\Box A)} \Rightarrow \Sigma_1, \Sigma_2}(\Box A)$$

we may obtain Π' of the form:

$$\dfrac{\dfrac{\Gamma_1, \Delta \Rightarrow A, \Sigma_1}{\Gamma_1, \Delta \Rightarrow \Box A, \Sigma_1} \quad \dfrac{\dfrac{\Gamma_1, \Delta \Rightarrow \Box A, \Sigma_1 \quad \Gamma_2, A \Rightarrow \Sigma_2}{\Gamma_1, \Delta, \Gamma_2^{(-\Box A)}, A \Rightarrow \Sigma_1, \Sigma_2}(\Box A)}{\Gamma_1, \Delta, \Gamma_2^{(-\Box A)}, \Box A \Rightarrow \Sigma_1, \Sigma_2}}{\dfrac{\Gamma_1, \Delta, \Gamma_1, \Delta, \Gamma_2^{(-\Box A)}, \Box A \Rightarrow \Sigma_1, \Sigma_1, \Sigma_2}{\Gamma_1, \Delta, \Gamma_2^{(-\Box A)}, \Box A \Rightarrow \Sigma_1, \Sigma_2} Cont.}(\Box A)$$

If Π_2 does not end with $(\Box \Rightarrow)$. We have to analyse the rules used after this introduction.

(a) $(\Rightarrow \Box)$, If Π_2 is of the form:

$$\dfrac{\dfrac{\Gamma_2^m, A \Rightarrow C, \Sigma_2^m}{\Gamma_2^m, \Box A \Rightarrow C, \Sigma_2^m}\ (\Box \Rightarrow)}{\Gamma_2^m, \Box A \Rightarrow \Box C, \Sigma_2^m}\ (\Rightarrow \Box)$$

We may push the mix up, analysing the condition first and then using $(\Rightarrow \Box)_2$. But using the proposition 2.1 in Π_2 we have that there exist $\Gamma_2'^m, \neg\Sigma_2'^m$ and $\Sigma_2'^m$ such that:

$$\Gamma_2'^m \wedge \Box A \wedge \neg\Sigma_2'^m \prec_{DD} \Box C, \quad \Gamma_2'^m \wedge \Box A \wedge \neg\Sigma_2'^m \prec_{RP} \Box C), \quad \textit{and}$$
$$\Box C \prec_{LP} \Gamma_2'^m \wedge \Box A \wedge \neg\Sigma_2'^m.$$

But in Π_1, using the same proposition, there exist Γ_1', Δ' and Σ_1' such that: $\Gamma_1' \wedge \Delta \wedge \neg\Sigma_1' \prec_{DD} \Box A$.
So, $\Gamma_1' \wedge \Delta' \wedge \Gamma_2'^m \wedge \neg\Sigma_1' \wedge \neg\Sigma_2'^m \prec_{DD} \Box C$
Then we may have:

i. $\Delta' \wedge \Gamma_2'^m \wedge \neg\Sigma_1' \wedge \Sigma_2'^m \prec_{DDLP} \Gamma_1'$ and $\Gamma_1' \prec_{RPDD} \Box C$, or
ii. $\Gamma_1' \wedge \Gamma_2'^m \wedge \neg\Sigma_1' \wedge \Sigma_2'^m \prec_{DDLP} \Delta$ and $\Delta' \prec_{RPDD} \Box C$, or
iii. $\Gamma_1' \wedge \Delta' \wedge \neg\Sigma_1' \wedge \Sigma_2'^m \prec_{DDLP} \Gamma_2'^m$ and $\Gamma_2'^m \preceq_{RPDD} \Box C$.

So we may use $(\Rightarrow \Box)_2$ after the mix.

(b) $(\Rightarrow \Box)_2$ Π_2 is of the form:

$$\dfrac{\dfrac{\Gamma_2, A \Rightarrow C, \Sigma_2}{\Gamma_2, \Box A \Rightarrow C, \Sigma_2}\ (\Box \Rightarrow)}{\Gamma_2, \Box A \Rightarrow \Box C, \Sigma_2}\ (\Rightarrow \Box)_2$$

If we have one of these cases:

- $\Gamma_2 \wedge \neg\Sigma_2 \prec_{DDLP} \Box A$ and $\Box A \prec_{RPDD} \Box C$, or
- $\Box A \wedge \neg\Sigma_2) \prec_{DDLP} \Gamma_2$ and $\Gamma_2 \prec_{RPDD} \Box C$.

We may push the mix up, analysing the conditions first and then using $(\Rightarrow \Box)_2$. But in Π_1, by the proposition 2.1, we have that there exist $\Gamma_1', \Delta', \neg\Sigma_1'$ such that:

$$\Gamma_1' \wedge \Delta' \wedge \neg\Sigma_1' \prec_{DD} \Box A, \;\; \Gamma_1' \wedge \Delta' \wedge \neg\Sigma_1' \prec_{RP} \Box A), \;\; \textit{and}$$

$$\Box A \prec_{LP} \Gamma_1' \wedge \Delta' \wedge \neg\Sigma_1'.$$

But in Π_2 if $\Gamma_2 \wedge \neg\Sigma_2 \prec_{DDLP} \Box A$ and $\Box A \prec_{RPDD} \Box C$, then $\Gamma_2 \wedge \neg\Sigma_2 \prec_{DDLP} \Gamma_1' \wedge \Delta' \wedge \neg\Sigma_1'$ and $\Gamma_1' \wedge \Delta' \wedge \neg\Sigma_1' \prec_{RP} \Box A$, and $\Box A \wedge \neg\Sigma_2 \prec_{DDLP} \Gamma_2$ and $\Gamma_2 \prec_{RPDD} \Box C$.

But $\Gamma_1' \wedge \Delta' \wedge \neg\Sigma_1' \prec_{DD} \Box A$, Then, $\Gamma_1' \wedge \Delta' \wedge \neg\Sigma_1' \wedge \neg\Sigma_2 \prec_{DDLP} \Gamma_2$ and $\Gamma_2 \prec_{RPDD} \Box C$.

So, we have the new condition.

(c) If Π_2 ends with $(\Rightarrow \Box)_1$ or $(\Diamond \Rightarrow)_1$. We may push the mix up and then use $(\Rightarrow \Box)_1$, or $(\Diamond \Rightarrow)_1$; If Π_2 ends with $(\Diamond \Rightarrow)$ the mix-elimination is like (a); If Π_2 ends with $(\Diamond \Rightarrow)_2$ it is like (b).

4. If Π_1 ends with $(\Rightarrow \Diamond)$ then Π is of the form:

$$\dfrac{\dfrac{\Gamma_1 \;\Rightarrow\; A, \Sigma_1}{\Gamma_1 \;\Rightarrow\; \Diamond A, \Sigma_1}(\Rightarrow \Box)_2 \qquad \Gamma_2, \;\Diamond A \;\Rightarrow\; \Sigma_2}{\Gamma_1, \;\Gamma_2^{(-\Diamond A)} \;\Rightarrow\; \Sigma_1^{(-\Diamond A)}, \Sigma_2} \quad (\Diamond A)$$

We have to examine how $\Diamond A$ was introduced in Π_2. If by thinning the mix may be eliminated only using thinning in Π_2. If was not by thinning, we habe two cases to be analysed:

If Π_2 ends with $(\Diamond \Rightarrow)$, $(\Diamond \Rightarrow)_1$, or $(\Diamond \Rightarrow)_2$. We will only analyse the case of $(\Diamond \Rightarrow)_2$. So Π is of the form:

$$\dfrac{\dfrac{\Gamma_1 \Rightarrow A, \Sigma_1}{\Gamma_1 \Rightarrow \Diamond A, \Sigma_1} \qquad \dfrac{\Gamma_2, A \Rightarrow \Sigma_2}{\Gamma_2, \Diamond A \Rightarrow \Sigma_2}}{\Gamma_1, \Gamma_2^{(-\Diamond A)} \Rightarrow \Sigma_1, \Sigma_2}(\Diamond A)$$

we may obtain Π' of the form:

$$\dfrac{\Gamma_1 \Rightarrow A, \Sigma_1 \qquad \dfrac{\Gamma_1 \Rightarrow \Diamond A, \Sigma_1 \quad \Gamma_2, A \Rightarrow \Sigma_2}{\Gamma_1, \Gamma_2^{(-\Diamond A)}, A \Rightarrow \Sigma_1, \Sigma_2}(\Diamond A)}{\dfrac{\dfrac{\Gamma_1, \Gamma_1, \Gamma_2^{(-\Diamond A),(-A)} \Rightarrow \Sigma_1^{(-A)}, \Sigma_1, \Sigma_2}{\Gamma_1, \Gamma_1, \Gamma_2^{(-\Diamond A)} \Rightarrow \Sigma_1, \Sigma_1, \Sigma_2} thin}{\Gamma_1, \Gamma_2^{(-\Diamond A)} \Rightarrow \Sigma_1, \Sigma_2} cont.}(A)$$

If Π_2 does not end with $(\Diamond \Rightarrow)$, $(\Diamond \Rightarrow)_1$, or $(\Diamond \Rightarrow)$ then we have to analyse the rules used in this introduction. They may be:

1. $(\Diamond \Rightarrow)$ $\Diamond A$ was introduced in Π_2 by $(\Diamond \Rightarrow)$. So Π_2 is of the form:

$$\dfrac{\dfrac{\dfrac{\Gamma_2'^m, A \Rightarrow \Sigma_2'^m}{\Gamma_2'^m, \Diamond A \Rightarrow \Sigma_2'^m}}{\vdots}}{\Gamma_2^m, \Diamond A \Rightarrow \Sigma_2^m}$$

Now we have to study what rules may be used after $(\Diamond \Rightarrow)$.

(a) If only structural rules, were used after $(\Diamond \Rightarrow)$, we may push the mix up and then use the structural rules.

(b) Operational rules. If only operational rules are used the case is analogous to the other case.

We will study what rules may be used after $(\Diamond \Rightarrow)$ in Π_2. They are:

(a) $(\Rightarrow \Box)_2$, So Π_2 is of the form:

$$\dfrac{\dfrac{\Gamma_2, A \Rightarrow C, \Sigma_2}{\Gamma_2, \Diamond A \Rightarrow C, \Sigma_2}}{\Gamma_2, \Diamond A \Rightarrow \Box C, \Sigma_2}$$

$$If \quad \Diamond A \wedge \neg\Sigma_2 \prec_{DDLP} \Gamma_2 \;\; and \;\; \Gamma_2^m \prec_{RPDD} \Box C, \;\; or$$

$$\Gamma_2 \wedge \neg\Sigma_2 \prec_{DDLP} \Diamond A \;\; and \;\; \Diamond A \prec_{RPDD} \Box C.$$

We may push the mix up and then use $(\Rightarrow \Box)_2$, if the conditions is met.

But by the proposition 2.1 in Π_1 we have that there exist Γ_1', and Σ_1' such that: $\Gamma_1' \wedge \neg\Sigma_1' \prec_{DD} \Diamond A$, $\Gamma_1' \wedge \neg\Sigma_1' \prec_{RP} \Diamond A$ and

$$\Diamond A \prec_{LP} \Gamma_1' \wedge \neg\Sigma_1'$$

So we may obtain the condition to use $(\Rightarrow \Box)_1$.

(b) $(\Rightarrow \Box)$, So Π_2 is of the form:

$$\dfrac{\dfrac{\Gamma_2^m, A \Rightarrow B^m, \Sigma_2^m}{\Gamma_2^m, \Diamond A \Rightarrow B^m, \Sigma_2^m}}{\Gamma_2^m, \Diamond A \Rightarrow \Box(B^m), \Sigma_2^m}$$

We may push the mix up and then use the proposition 2.1 in Π_1 and Π_2 to use $(\Rightarrow \Box)_2$ instead of $(\Rightarrow \Box)$, if B is a necessary formula or we use $(\Rightarrow \Box)_1$ is B is a possible formula.

(c) If $(\Box \Rightarrow)$ or $(\Rightarrow \Diamond)$ are used after $(\Diamond \Rightarrow)$, we may push the mix up and then use these rules, because they have no restrictions; If after $(\Diamond \Rightarrow)$ are used $(\Diamond \Rightarrow)_1$ or $(\Rightarrow \Box)_1$, we may push the mix up, and then use these rules; If after $(\Diamond \Rightarrow)$ $(\Diamond \Rightarrow)_2$ is used, the case is like (a); If after $(\Diamond \Rightarrow)$ $(\Diamond \Rightarrow)$is used, the case is like (b).

The other cases will be analogous to the first.

(B) Suppose $rank_r(\Pi) = 1$ and $rank_l(\Pi) \geq 1$. The case will be analogous to the case (A)

(C) Suppose $rank_r(\Pi) \geq 1$ and $rank_l(\Pi) \geq 1$. The case is proved in the same way as (A) and (B).

Proposition 3.2 *If a sequent is provable in SC_{mix} then it is provable in SC_{mix} whithout a mix.*

Proof: The proof of this proposition is an immediate consequence of the previous lemma.

4 Conclusion

So, we define a Sequent Calculus for S5 (SC_5), and we prove the cut-elimination theorem for SC_5. This approach to a cut-free formalization of S5 can be seen as dual to the analytic-cut approach. While the latter restricts the form of the cut-rule in order to have the cut-elimination, the former restricts relationshipa among the subformulas of formulas present in applications of some modal rules in order to ensure that permutation of tules do not affect the standard cut-elimination procedure.

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