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ON THE STRUCTURE OF MULTILINEAR SYSTEMS

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ABSTRACT

The structure of a class of multilinear systems is investigated from a general system-theoretic viewpoint by algebraic methods. This class encompasses not only stationary and time-varying systems but also systems with discrete or continuous time.
Multilinear systems may be regarded as generalizations of linear systems, their nonlinearity being of a special kind. They occur in such diverse areas as pattern recognition, nuclear reactor control, biological modeling.
It is shown that many multilinear systems can be decomposed into parallel linear systems interconnected by a tensor-product block followed by a linear system, thus indicating that the multilinearity can be concentrated.

1. INTRODUCTION

In a linear system the inputs act linearly on the output. This is no longer the case for multilinear systems, where this dependence is multilinear rather than linear. A bilinear systems has two input lines and the influence of each input on the output is linear provided the other input is held fixed. The combined effect on the output of variations on both input lines need not be linear. Thus, a multilinear system presents a non-linearity of a particular kind, being somehow almost linear.

Multilinear systems are involved in the modeling of phenomena from several areas, such as pattern recognition, nuclear reactor control, biological population growth, etc.

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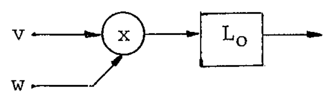
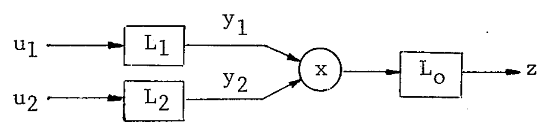
Imagine a potentiometer, say on an analog computer, the value of which is linearly controlled by a signal w(t). Its response to input v(t) will have the form k.w(t).v(t). If its output line is fed into a linear system L_o we obtain a bilinear system



Another natural way to obtain a bilinear system consists in connecting two linear systems L_1 and L_2 in parallel via a multiplier. The result of combining both methods will be a bilinear system,



Notice that the non-linearity is concentrated in a memoryless multiplier block, all the other components being linear. It will be shown that many bilinear systems can be obtained this way.

We shall employ a generalization of Windehnecht's (1) general time-systems framework, designed to allow for more flexibility in the time domains. We shall need some standard algebraic terminology and basic results (2), (3).

2. SYSTEMS

A system is frequently regarded as a relation between input and output signals. Here we shall imagine that input is fed into the system at certain instants $p \in P$ and its output is observed at instants $t \in T$, considering its input-output behavior as basic.

We shall define a system as a 7-tuple

$S = \langle P, T, E, Z, U, Y, S\rangle$, where

. P and T are non-empty sets (input and output time domains);
. E and Z are non-empty sets (input and output alphabets);
. U is a non-empty subset of E^P (input space);
. Y is a non-empty subset of Z^T (output space);
. S is a relation from U to Y with domain U.

An important property of many physical systems is non-anticipation. For instance the output over the interval $[t_0,t_1)$ for real numbers $t_0<t_1$ does not depend on the input over $[t_1, +\infty)$ only on the part over $(-\infty, t_o)$. This motivates the next definition.

Given subsets I of P and J of T consider the relation (with '|' denoting restriction)

$${}_I S_J = \{(u|I, y|J) \ / \ (u,y) \in S\}$$

and call S I-J-deterministic iff ${}_I S_J$ is a function.

From now on we fix a positive natural n and a commutative ring K with unit.

We shall a system S n-linear (over K) iff

. $E = E_1 \times E_2 \times \ldots \times E_n$, with each E_j a K-module;
. $U = U_1 \times U_2 \times \ldots \times U_n$, with each U_j a K-submodule of E_j^P ;
. Z is a K-module and Y a K-submodule of Z^T.
. the relation S is n-linear over K, i.e.,
for each $i = 1,\ldots,n$ and all $u_1 \in U_1,\ldots,u_{i-1} \in U_{i-1}$; $u_{i+1} \in U_{i+1},\ldots,u_n \in U_n$ the induced relation

$$\{(u_i,y) \ / \ (u_1,\ldots,u_i,\ldots,u_n) \ Sy\}$$

is a K-submodule of $U_i \times Y$.

Notice that the usual linear systems are exactly the 1-linear ones. Thus, multilinear systems generalize the linear ones.

Clearly, a system S is n-linear and I-J-deterministic iff

$${}_I S_J : (U_1 \times \ldots \times U_n)|I \to Y|J$$

is an n-linear function, which becomes a linear function in the case of linear system.

Some remarks about our concept of system may be in order. Firstly, it is possible to consider P and T as subsets of a common time domain, which may be ordered (but it is important to notice that this ordering need have a minimal element). Secondly, it is not necessary to take P and T as related to "physical time". In fact, it may be more convenient in some cases not to do so. We shall now give some examples to illustrate these points.

For a continuous-time system (say, simulated on an analog computer by means of integrators, potentiometers, multipliers, switches, etc), we may take P to be the reals and T the non-negative reals. For a discrete-time system (say, described by difference equations) we may consider P to be the integers and T the naturals. In both cases we do not care to observe the negative part, regarding it as setting the initial condition.

gives a linear I-I-deterministic system L_i. Now define R by

$$R = \{((u_1 + N_1,\dots,u_n + N_n)\ ,\ y)\ /\ (u_1,\dots,u_n)\ My\}$$

Notice that R is n-linear, inheriting I-J-determinism from M QED.

Notice that if system M in the above lemma has input and output time domains P and T, resp., them $L_1,\dots,L_n$ have P as both time domains. In the next proposition, R will be further decomposed, yielding L_o with T as both time domains.

Proposition. An n-linear I-J-deterministic system M can be decomposed into a parallel interconnection of n linear I-I-deterministic systems by a P-T-deterministic tensor product $\bar{R}$ followed by a linear J-J-deterministic system L_o.

Proof. We have $M = (L_1 \times \dots \times L_n).R$ with an n-linear function ${}_IR_J : (V_1 \times\dots\times V_n)|I \to Y|J$ from the previous lemma. Consider the tensor products over K

$$\bar{R} = V_1 \times\dots\times V_n \to W = V_1 \times \dots \times V_n$$

$$\tilde{R} = V_1 \times\dots\times V_n|I \to X = V_1|I \times \dots \times V_n|I$$

and the tensor product $D : W \to X$ of the restrictions from V_i to $V_i|I$ for $i = 1,\dots,n$. There exists a linear function $L : X \to Y|J$ such that ${}_IR_J = D.L$. Define L_o by

$$L_o = \{(w,y)\ /\ wD.L = y|J\}$$

Then $R = \bar{R}.L_o$ with L_o as required. QED

Deterministic systems form a broad class. A particular case of special interest is that of systems where the output can be determined pointwise as in the following definition.

Call a system S I-J-causal iff for each $t \in J$ there exists $I^t \subseteq I$ such that S is I^t - $\{t\}$-deterministic and $I = \bigcup_{t\in J} I^t$.

For an I-J-causal system we have $y(t)$ as a function of $u|I^t$. But frequently we can obtain more information about the output from the knowledge of $u|I^t$. Indeed, given $t \in T$, consider the family of all $J' \subseteq J$ such that ${}_{I^t}S_{J'}$ is a function. By Zorn's lemma (2), this family has a maximal element J^t, with $t \in J^t \subseteq J$.

Theorem - An n-linear I-J-causal system M can be decomposed as $M = (L_1.\hat{L}_1 \times\dots\times L_n.\hat{L}_n).\hat{R}$ with $L_1,\dots,L_n$ linear I-I-deterministic, $\hat{L}_1,\dots,\hat{L}_n$ linear I-J-causal and $\hat{R}$ n-linear I-J-causal.

Proof - Let $t \in J$ and notice that M is I^t-J^t-deterministic according to the above remarks. So, the previous lemma gives a decomposition $M = (L_1 \times\dots\times L_n).R$ with R n-linear I^t-J^t-deterministic. Thus we have an n-linear function

$$(v_1^t\ ,\dots,v_n^t\ )\ R^t = (t)\ (v_1^t,\dots,v_n^t)\ {}_{I^t}R_{J^t}$$

where $v_i^t = v_i|I^t$, for $i = 1,\dots,n$. Now, we have a K-submodule $N_i(t)$ of V_i^t with canonical projection $\hat{L}_i^t : V_i^t \to W_i^t$, for $i = 1,\dots,n$ and an n-linear function $\hat{R}$ factoring $R^t = (\hat{L}_1^t \times\dots\times \hat{L}_n^t).\hat{R}$. Thus we have the required systems. QED.

Now, for A/D (resp. D/A) converters it may be more convenient to take P=**R** and T=**N** (resp. P=**Z** and T=$\mathbf{R}_+$).

Finally, consider the case of Turing machines with a designated initial state. Here the initial contents of the tape(s) determine the final contents, if any. So, we regard the input as having the form u : **N** → I (initial contents) and the output of the form y : **Z** → Z (final contents), and take P = **N** and T = **Z**. Thus we have an **N**-**Z**-deterministic system (for halting machines), which we would not have had we taken time related to the number of steps in the computation. For the case of one input read-only tape with k working read-write tapes, we can consider P = **N** and T = $\mathbf{Z}^k$.

3. DECOMPOSITIONS

We shall examine the rough structure of multilinear systems. These shall be described in terms of decompositions.

Consider two systems S , R with output components (T,Z,Y) of S coinciding with the corresponding input components of R. By their serial connection S followed by R we mean the system $S.R$ with relation

$$S.R = \{(u,w) \ / \ \text{for some } y \in Y, (u,y) \in S \text{ and } (y,w) \in R\}$$

If the input components of S coincide with the corresponding ones of R and likewise for the output components then we define its direct product S x R to be the system with relation.

$$SxR = \{((u,v),(y,w)) \ / \ (u,y) \in S \text{ and } (v,w) \in R\}$$

Finally, the parallel connection of S and S' by R, assumed compatible, is the system $(S \times S').R$

In decomposing n-linear systems we shall try to isolate the non-linear, so as to, roughly speaking, diminish its dimensionality. We consider first the case of deterministic systems.

Lemma - An n-linear (I-J-deterministic system M can be decomposed into the parallel connection of n linear I-I-deterministic systems by an n-linear system R(which will be I-J-deterministic if M is) so that M = $(L_1 \times \ldots \times L_n).R$.
Proof - Let i = 1,...,n and define

$$N_i|I = \{u_i|I \ / \ (u_1|I,\ldots,u_i|I,\ldots,u_n|I)\,{}_I M_J\, O^J \ , \text{ for all } u_j \in U_j \text{ with } j \neq i\}$$

Notice that $N_i|I$ is a K-submodule of $U_i|I$ inducing a K-submodule $N_i = \{u_i \in U_i \ / \ u_i|I \in N_i|I\}$ of U_i.
The canonical projection L_i of U_i onto the quotient $V_i = U_i/N_i$

We can further decompose $\tilde{R}$ in a manner analogous to the deterministic case.

Corollary - An n-linear I-J-causal system M can be decomposed as $M = (L_1 . \hat{L}_1 \times \ldots \times L_n . \hat{L}_n) . \tilde{R}.\hat{L}_o$ with $L_1,\ldots,L_n,\hat{L}_1,\ldots,\hat{L}_n$ as above. $\tilde{R}$ a P-Q-causal tensor product and $\hat{L}_o$ linear J-I-causal.

A further specialization of interest is that of memoryless systems, such as adders, multipliers, where the input-output transformation is instantaneous. The next definition embodies a slight relaxation of this constraint.

Call a system S I-J-static iff for each $t \in J$ there exists $p_t \in I$ such that S is $\{p_t\}$ - $\{t\}$-deterministic and $I = \{p_t/t \in J\}$. Clearly an I-J-static system is an I-J-causal one where $y(t)$ is a function of the input value $u(p_t)$ at instant $p_t \in I$. The case $p_t = t$ corresponds to combinational switching circuits, whereas the more general case $p_t \leq t$ corresponds to definite machines.

Multilinear static systems have decompositions similar to the other cases.

Proposition - An n-linear I-J-static system M can be decomposed into n linear I-I-static systems in parallel by a P-T-static tensor product followed by an J-J-static linear system, so that $M = (L_1 \times \ldots \times L_n) . R . L_o$

4. CONCLUSION

A very broad definition of time-system was given, accounting for different time behaviors on the input and output sides. Serial and parallel decomposition for such systems were taken as the natural extensions of the familiar concepts. Three increasingly restrictive subclasses were considered, deterministic, causal and static systems, which formalize various notions of non-anticipation and causality.

Multilinear systems over a commutative ring with unity were defined as encompassing the linear ones and then their overall structure described by decomposing them into parallel connections of linear systems via a tensor product followed by another linear system.

These results do not give much information about the fine structure of multilinear systems. This can sometimes be obtained in special cases, e.g. discrete stationary systems (6). However, they do have some interesting consequences. For instance, many, though not all, problems concerning multilinear systems can be attacked by studying the effect of a tensor-product connection of linear systems.

Moreover, the decompositions were obtained without explicit state-space construction, thus allowing unconstrained state assignments to the component systems. As an illustration, only multipliers have to be added to the stock of building blocks for linear systems (adders, scalers, delays and integrators) in order to construct any multilinear system described by differential/difference equations.

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