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ON ADDITIVITY AND LINEARITY IN GENERAL SYSTEMS THEORY

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ABSTRACT

This paper shows that many properties of linear time-invariant systems are due to the underlying group structure, even if not commutative. Within a general time-systems framework some properties of additive time-varying systems are examined : attainability, observability, reachability, controllability, realizability and state-space construction. These are regarded from the viewpoint of mathematical foundations of general time-systems theory.

In Windeknecht's general time-systems theory (1) a system is defined as a relation between input and output signals (2). Another distinct concept is that of realization, which involves internal states (3). The problems of analysis and synthesis can be described as that of correlating a system with a realization with the same input-output behavior. Thus, non-anticipatory systems represent the behavior of deterministic realizations. This paper employs this framework extended to time-varying systems in order to investigate the role played by linearity. The aim is showing that several properties usually attributed to linearity are actually due to a group structure, which does not have to be commutative (4).

We shall fix a <u>time domain</u> $T \neq \emptyset$ ordered by $<$ together with a <u>output-observation domain</u> $\emptyset \neq J \subseteq T$. Also fixed are the groups I and Z (input and output alphabets) and subgroups U of functions from T into I and Y of functions from J into Z (input and output spaces). We shall be dealing with (sets of) functions with domain included in T and using the suffixes *t and §t to denote restriction to the set of instants prior to t , resp. from t onwards (including t).

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In order to simplify the treatment we shall assume the input space U to have a closure property : for all t in J and all u and v in U $u\underline{t}v$ = u*t $\cup$ v§t is in U . Because of this, we have a commutative separation : for all u and v in U $u\underline{t}0 + 0\underline{t}v = u\underline{t}v = 0\underline{t}v + u\underline{t}0$. This will be used frequently in the sequel.

By an <u>additive t-register</u> $\mathcal{R}$ we mean a state group Q together with a pair of group homomorphisms m*t : U*t $\longrightarrow$ Q (memory map) and n§t : QxU§t $\longrightarrow$ Y§t (output map). Because of the separation, we get an equivalent definition of additive t-register via the representation n§t(q,u§t) = c§t(q) + d§t(u§t) , where we have two homomorphisms c§t : Q $\longrightarrow$ Y§t and d§t : U§t $\longrightarrow$ Y§t .

The set of t-attainable states is a subgroup , the image of m*t , whereas the set of states t-equivalent to the zero state is a normal subgroup kerc§t of Q. Since q is t-equivalent to q' iff (q-q') is in kerc§t , we can transform $\mathcal{R}$ into its reduction $\bar{\mathcal{R}}$, which is completely t-attainable and t-observable. This reduction is minimal among all t-registers having the (input-output) <u>behavior</u> n§t(m*t(u*t),u§t) of $\mathcal{R}$, in the sense of being a homomorphic image of a subgroup.

A register embodies the idea of the state space being used to store information with part of the input signal setting the initial condition. (Notice that T is not assumed to have minimal elements.) The next definition starts incorporating state structures for several instants of time.

An <u>additive transition structure</u> consists of a family Q(j) of state groups together with homomorphisms M(j,i) from Q(i)xU§i*j into Q(j) for $i \leq j$ in J, satisfying the identity and semigroup composition properties. Equivalently, because of separation, M(j,i)(q,u§i*j) = A(j,i)q + B(j,i)u§i*j , where A(j,i) and B(j,i) are homomorphims such that A(i,i)q=q , B(i,i)u§i*i=0 , A(j,i)q= =A(j,t)A(t,i)q , B(j,i)u§i*j = A(j,t)B(t,i)u§i*t + B(j,t)u§t*j , for $i \leq t \leq j$ in J .

Now consider the sets Rch(j,i) of states <u>reachable</u> at j from the state 0 at i and Ctr(j,i) of states at i <u>controllable</u> to 0 at j. Clearly, Rch(j,i)=ImB(j,i) is a subgroup of Q(j) and Ctr(j,i) = = $A(j,i)^{-1}$[ImB(j,i)] is a subgroup of Q(i) . The next result generalizes a condition for a finite-dimensional linear time-invariant system to have all its reachable states controllable (7).

<u>Proposition</u> - Let $i \leq t \leq t' \leq j \leq k$ in J and assume Rch(k,i) $\subseteq$ Rch(k,j) . Then Rch(t',t) $\subseteq$ Ctr(k,t'), thus Rch(j,i) $\subseteq$ Ctr(k,j) .
<u>Proof</u>. Follows from $q \in$ Ctr(k,t') iff A(k,t')q $\in$ Rch(k,t') . QED

From an additive i-register and an additive j-i transition function one can obtain naturally an additive j-register. However, the output of non-anticipatory systems can generally be determined pointwise, which suggests the following definition.

An <u>additive machine</u> $\mathcal{M}$ consists of an additive transition structure together with a homomorphism N(t) : Q(t)xU(t) $\longrightarrow$ Z , for each t in J. Equivalently, the instantaneous output map can be represented as N(t)(q,u(t))=C(t)q+D(t)u(t) , with homomorphisms C(t) : Q(t) $\longrightarrow$ Z and D(t) : U(t) $\longrightarrow$ Z . For each i in J we have an induced output map n§i : Q(i)xU§i — Y§i given by, for j=i in J, n§i(q,u§i)(j)=N(j)(M(j,i)(q,u§i*j),u(j)) , with similar expressions for c§i and d§i . Thus kerc§i = $\bigcap_{j \geq i}$ ker(C(j)A(j,i)) and for j$\geq$i A(j,i)kerc§i $\subseteq$ kerc§j .

<u>Proposition</u> - Given $\mathcal{M}$ there exists a unique machine $\tilde{\mathcal{M}}$, which is completely observable on J, such that for j$\geq$i in J, $\tilde{A}$(j,i)p(i) = = p(j)A(j,i) , $\tilde{B}$(j,i)=p(j)B(j,i) , $\tilde{C}$(j)p(j)=C(j) , $\tilde{D}$(j)=D(j) ; where p(t) denotes the canonical projection onto Q(t)/kerc§t .

The idea of using a register together with a transition structure is taken up in the following definition.

Given t in J, an <u>additive realization</u> $\mathcal{P}$ consists of an additive machine together with a memory homomorphism m*t : U*t — Q(t) . Notice that for each i$\geq$t in J $\mathcal{P}$ gives rise to an additive i-register, thus being reducible to a completely observable and attainable $\tilde{\mathcal{P}}$ with the same behavior, which is, for j$\geq$t in J, C(j)A(j,t)m*t(u*t) + C(j)B(j,t)u§t*j + D(j)u(j) .

By an <u>additive system</u> we mean an additive relation S from U to Y , i.e. a subgroup of UxY, with domain U .

Given an instant t in T we define the <u>t-section</u> S§t of S to consist of those (u,y§t) with (u,y) in S and call S <u>t-functional</u> iff S§t is a function, which is then a homomorphism such that for all u in U S§t(u) = S§t(u<u>t</u>0) + S§t(0<u>t</u>u) .

Now, the behavior of an additive t-register gives a t-section of a t-functional additive system. We shall now extend Nerode's construction to obtain the converse.

<u>Theorem</u> - Any t-section of a t-functional additive system S is the behavior of an additive t-register $\mathcal{R}$, which is minimal.
<u>Proof</u>. As kerS§t(.<u>t</u>0) is a normal subgroup of U*t we can take Q to be the corresponding quotient and use the canonical projection to define m*t and n§t in a natural way. QED

Now consider for i in J the relation S£i consisting of those ((u*i,u(i)),y(i)) with (u,y) in S . We shall call S <u>t-J-causal</u> iff for each i$\geq$t in J S£i is a function, whence a homomorphism from U*ixU(i) into Z. With these concepts we can characterize which systems have a realization.

<u>Theorem</u> - A t-section of an additive system S is the behavior of an additive t-J- realization iff S is t-J-causal.
<u>Proof</u>. If S is t-J-causal then for each i≥t in J S is i-functional and the preceding theorem gives an additive i-register with Q(i)= U*i/K(i) where K(i)=kerS§i(.t0) . So we can define , for j≥i in J, M(j,i)(u*i+K(i),v§i*j) = uiv*j + K(j) and N(j,i)(u*i+K(i),v(i)) = = S£i(u*i,v(i)) . The other direction follows from previous remarks on the behavior.QED

We have examined the role played by additivity as opposed to linearity with respect to some basic concepts and results of general time-systems theory. These results have been derived from simple facts about groups. In contrast to the commutative additive machines of (3) we did not assume commutativity, for whatever should commute does commute and we only need quotients by kernels of homomorphisms. Of course, linearity is a very useful, and frequently natural, assumption, especially in providing convenient representations. But we can at least get started simply with additivity, which is interesting from the viewpoint of foundations.

Some points deserve comments. We have used heavily the closure property assumed for the input space. Had we not assumed this all the results would still hold at the expense of considering the sets of continuations of input signals. Also, by extending Nerode's construction what we actually get is a phase space for each instant. These can then be merged into a single attainable and observable state space when some extra machinery is applicable.

Finally, most definitions make sense with the word 'additive' omitted or changed to, say, 'continuous'. This translates the concepts to another (concrete) category, where most proofs can be suitably adapted, in general.

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