Nonlinear Finite Element Analysis in Structural Mechanics

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Approximate Solution of Nonlinear Problems in Incompressible Finite Elasticity

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1. INTRODUCTION

The main goal of this paper is to describe some numerical methods for solving nonlinear variational problems in incompressible finite elasticity.

In Sec. 2 we discuss a decomposition principle for a large class of variational problems and then derive several iterative methods of solution from this principle. We give in Sec. 3, the formulation of elastostatics two-dimensional and axisymmetric problems for incompressible Mooney-Rivlin materials.

In Sec. 4 we describe the application of the algorithms of Sec. 2 to the iterative solution of the mechanical problems of Sec. 3. In Sec. 5 we give some brief indication on the finite element approximation of the problems of Sec. 3. Finally some numerical results obtained applying the methods of Sec. 4,5 are presented and discussed in Sec. 6.

2. DECOMPOSITION OF VARIATIONAL PROBLEMS AND ASSOCIATED ALGORITHMS.

2.1. A family of Variational Problems.

Restricting our attention to real Hilbert spaces, we consider two such Hilbert spaces V and H equipped with $\|\cdot\|$ and $|\cdot|$, and inner products $((\cdot,\cdot))$ and $(\cdot,\cdot)$, respectively.

Let $B \in \mathcal{L}(V,H)$ and F,G be two proper, convex, lower semicontinuous functionals from H,V to $\overline{\mathbb{R}}$, respectively, such that

$$\text{dom}(G) \cap \text{dom}(F \circ B) \neq \emptyset \,. \tag{2.1}$$

We associate with the above V,H,B,F,G the following minimization problem

(P) $\underline{\text{Find}}$ $u \in V$ $\underline{\text{such that}}$ $J(u) \leq J(v) \quad \forall\, v \in V$,

where $J : V \to \overline{\mathbb{R}}$ is defined by

$$J(v) = F(Bv) + G(v). \tag{2.2}$$

It appears from (2.2) that $J(\cdot)$ and therefore (P) have very special structures and taking this into account it is then quite natural to design special methods of solution for (P).

Remark 2.1 : Most of the considerations which follow can be applied to variational problems like

$$f \in B^* A_1(Bu) + A_2(u), \tag{2.3}$$

where $f \in V'$ (dual space of V) and where A_1 (resp. A_2) is a monotone operator (possibly multivalued) from H to H' (dual space of H) (resp. V to V') ; the operator $A = B^* \circ A_1 \circ B + A_2$, from $V \to V'$ is not in general the differential (or subdifferential) of a functional J. For many results on these generalizations we refer to P.L. LIONS-B. MERCIER [1] and GABAY [2]. If we suppose that in addition to (2.1) we also have

$$\lim_{\|v\| \to +\infty} J(v) = +\infty \tag{2.4}$$

then (P) has a solution which is unique if J is strictly convex.

Remark 2.2 : The applications in finite elasticity that we have in view are actually related to nonconvex minimization problems.

2.2 A decomposition principle.

Let us define $W \subset V \times H$ by

(2.5) $W = \{\{v,q\} \in V\times H,\ Bv-q = 0\}$.

Problem (P) is then clearly equivalent to

$$(\pi)\quad \begin{cases} \underline{\text{Find}}\ \{u,p\} \in W\ \underline{\text{such that}} \\ j(u,p) \le j(v,q) \quad \forall\ \{v,q\} \in W, \end{cases}$$

where

(2.6) $j(v,q) = F(q) + G(v)$.

Remark 2.3 : The new problem (π) has clearly a mixed formulation "flavor" since the linear relation $Bv-q = 0$ suggests the introduction of a Lagrange multiplier.

Remark 2.4 : Problems (P) and (π) are equivalent, but considering (π) we have in some sense simplified the nonlinear structure of (P), at the expense however of the new variable q and of the relation

(2.7) $Bv-q = 0$.

Actually since (2.7) is a linear relation, very efficient techniques may be used to treat it ; in the following this will be done by using, simultaneously, penalty and Lagrange multipliers, via a convenient augmented lagrangian functional.

2.3. An augmented lagrangian functional associated with (π).

Let $r > 0$; define $\mathcal{L}_r : V\times H\times H \to \overline{\mathbb{R}}$ by

(2.8) $\mathcal{L}_r(v,q,\mu) = F(q)+G(v)+ \frac{r}{2}\ |Bv-q|^2+(\mu,Bv-q)$.

It can be proved that if $\{u,p,\lambda\}$ is a saddle-point of $\mathcal{L}_r$ over $V\times H\times H$, i.e., if

$$(2.9)\quad \begin{cases} \{u,p,\lambda\} \in V\times H\times H \text{ and } \forall\ \{v,q,\mu\} \in V\times H\times H \\ \mathcal{L}_r(u,p,\mu) \le \mathcal{L}_r(u,p,\lambda) \le \mathcal{L}_r(v,q,\lambda) \quad , \end{cases}$$

then $\{u,p\}$ solves (π), i.e., u solves (P) (with $p = Bu$).

2.4. A first algorithm for solving (P).

To solve (P) and (π) we shall compute the saddle-points of $\mathcal{L}_r$ by an algorithm derived from the duality algorithms considered, for example, in GLOWINSKI-LIONS-TREMOLIERES [3,4], Chap. 2. Such an algorithm applied to the solution of (2.9) is

(2.10) $\lambda^o \in H$, given

then if $n \geq 0$, λ^n given, we compute u^n, p^n, λ^{n+1} by

$$(2.11)\quad \begin{cases} \text{Find } \{u^n,p^n\} \in V\times H \text{ such that} \\ \mathcal{L}_r(u^n,p^n,\lambda^n) \leq \mathcal{L}_r(v,q,\lambda^n) \quad \forall \{v,q\} \in V\times H, \end{cases}$$

$$(2.12)\quad \lambda^{n+1} = \lambda^n+\rho(Bu^n-p^n).$$

Concerning the convergence of algorithm (2.10)-(2.12) it can be proved (see [5, Chap. 3] and [6, Chap. 5]) that under very reasonable assumptions on F,G,B and if

$$(2.13)\quad 0 < \rho < 2r,$$

then we have as $n \to +\infty$

(2.14) $u^n \to u$ strongly in V,

(2.15) $p^n \to p = Bu$ strongly in H,

(2.16) $\lambda^n \to \lambda$ weakly in H,

where u is the solution of (P), and where λ is such that $\{u,p,\lambda\}$ is a saddle-point of $\mathcal{L}_r$ over $V\times H\times H$.

Remark 2.5 : The only nontrivial step in the above algorithm is clearly the solution of the minimization problem (2.11). Actually to solve (2.11), taking into account its special structure, it is very convenient to use a functional block relaxation method (like those discussed in CEA-GLOWINSKI [7];

see [5],[6] for more details on the block relaxation solution of (2.11)). If this relaxation method is used, and if in the calculation of $\{u^n,p^n\}$ we only do <u>one</u> inner relaxation, starting from $\{u^{n-1},p^{n-1}\}$, we obtain the algorithm described in Sec. 2.5.

2.5. <u>A second algorithm for solving</u> (P).

The new algorithm is defined by

$$\{u^{-1},\lambda^{o}\} \in V\times H, \text{ given,} \tag{2.17}$$

<u>then for</u> $n \geq 0$, u^{n-1},λ^n <u>given</u>, <u>we compute</u> p^n,u^n,λ^{n+1} <u>by</u>

$$\begin{cases} \text{Find } p^n \in H \text{ such that} \\ \mathcal{L}_r(u^{n-1},p^n,\lambda^n) \leq \mathcal{L}_r(u^{n-1},q,\lambda^n) \quad \forall q \in H, \end{cases} \tag{2.18}$$

$$\begin{cases} \text{Find } u^n \in V \text{ such that} \\ \mathcal{L}_r(u^n,p^n,\lambda^n) \leq \mathcal{L}_r(v,p^n,\lambda^n) \quad \forall v \in V, \end{cases} \tag{2.19}$$

$$\lambda^{n+1} = \lambda^n+\rho(Bu^n-p^n). \tag{2.20}$$

<u>Remark 2.6</u> : Several variants of (2.17)-(2.20) are available. We can for example

(i) Exchange the role of q and v (see also Remark 2.7)

(ii) Update also λ^n between the steps (2.18),(2.19) ; doing so we obtain the following variant (due to GABAY [2]) of (2.17)-(2.20) :

$$\{u^{-1},\lambda^{o}\} \in V\times H, \text{ given,} \tag{2.21}$$

<u>then for</u> $n \geq 0$, u^{n-1},λ^n <u>given</u> <u>we compute</u> $p^n,\lambda^{n+1/2},u^n,\lambda^{n+1}$ <u>by</u>

$$\begin{cases} \mathcal{L}_r(u^{n-1},p^n,\lambda^n) \leq \mathcal{L}_r(u^{n-1},q,\lambda^n) \quad \forall q \in H, \\ p^n \in H, \end{cases} \tag{2.22}$$

$$(2.23)\quad \lambda^{n+1/2} = \lambda^n + \rho(Bu^{n-1} - p^n),$$

$$(2.24)\quad \begin{cases} \mathcal{L}_r(u^n, p^n, \lambda^{n+1/2}) \le \mathcal{L}_r(v, p^n, \lambda^{n+1/2}) \quad \forall v \in V, \\ u^n \in V, \end{cases}$$

$$(2.25)\quad \lambda^{n+1} = \lambda^{n+1/2} + \rho(Bu^n - p^n) \;;$$

q and v play a more symmetrical role in (2.21)-(2.25) than in (2.17)-(2.20).

Remark 2.7 : If one uses algorithm (2.17)-(2.20), it is recommended to solve, in the second step, the problem which has the best properties of ellipticity (for more details, see [5,6]). If one uses (2.21)-(2.25), the character of the problems ellipticity does not matter since q and v play a symmetrical role. ∎

Concerning the convergence of (2.17)-(2.20) it is proved in [5,6] that under vary reasonable assumptions on F,G,B we still have (2.14)-(2.15) if

$$(2.26)\quad 0 < \rho < \frac{1+\sqrt{5}}{2}\, r.$$

If G is linear it follows from GABAY-MERCIER [8] that we can again take $0 < \rho < 2r$.

2.6. Remarks on the choice of ρ and r.

For a given r, the optimal choice of ρ is very close to $\rho = r$, as shown by the various numerical experiments that we have done on the algorithms of Secs. 2.5 and 2.6. The choice of r is a more delicate matter. Theoretically, the larger r is, the faster is the convergence of (2.10)-(2.12). Actually, for large values of r the problem (2.11) will not be well-conditioned and its accurate solution will be a costly operation ; moreover, for very large values of r, round-off errors play a significant(negative) role. As we can see, we have therefore two contradictory behaviors as r increases. The global effect of these phenomena on (2.10)-(2.12) is to produce an algorithm which is not very sensitive to the choice of r and which is very robust.

2.7. Relations with Alternating Directions Methods.

2.7.1. Relations between algorithms (2.17)-(2.20), (2.21)-(2.25) and A.D.I.

We suppose for simplicity that V=H and B=I. We suppose also that F and G have as differentials (or subdifferentials) A_1 and A_2, respectively. Note that A_1 and A_2 are necessarily monotone (possibly multivalued) operators. Then (P) is equivalent to

$$(2.27)\quad A_1(u)+A_2(u) = 0$$

where the equal sign has to be replaced by $\ni$ if A_1 and/or A_2 are multivalued. Suppose that $\rho=r$; then by elimination of λ^n in (2.17)-(2.20) we obtain

$$(2.28)\quad u^{-1} \text{ given,}$$

then for $n \geq 0$,

$$(2.29)\quad rp^n + A_1(p^n) = ru^{n-1} - A_2(u^{n-1}),$$

$$(2.30)\quad ru^n + A_2(u^n) = ru^{n-1} - A_1(p^n).$$

Setting $u^{n+1/2} = p^{n+1}$, we finally have

$$(2.31)\quad ru^{n+1/2} + A_1(u^{n+1/2}) = ru^n - A_2(u^n),$$

$$(2.32)\quad ru^{n+1} + A_2(u^{n+1}) = ru^n - A_1(u^{n+1/2}).$$

We recognize in (2.31),(2.32) a Douglas-Rachford Alternating Direction Implicit (ADI) method (see [9]). Similarly, by elimination of λ^n and $\lambda^{n+1/2}$ we obtain from (2.21)-(2.25) (still supposing $\rho=r$)

$$(2.33)\quad ru^{n+1/2} + A_1(u^{n+1/2}) = ru^n - A_2(u^n)$$

$$(2.34) \quad ru^{n+1} + A_2(u^{n+1}) = ru^{n+1/2} - A_1(u^{n+1/2})$$

which is (see [10]) a Peaceman-Rachford ADI method. Such ADI methods involving fairly general monotone operators A_1 and A_2 have been studied in [1,2].

2.7.2. An initial value problem interpretation of (2.17)-(2.20) and (2.21)-(2.25).

It follows from Sec. 2.7.1 that if $\rho=r$ and $V=H$, $B=I$, then (2.17)-(2.20) and (2.21)-(2.25) can be seen as implicit schemes, based on fractional steps, for the discrete time integration of the initial value problem

$$(2.35) \quad \begin{cases} u(0) = u_o \\ \dfrac{du}{dt} + A(u) = 0 \end{cases}$$

where $A = A_1+A_2$.

From that interpretation r appears as the inverse of a time step Δt (i.e. $r=1/\Delta t$). As shown in [11], this interpretation of the avove algorithms, using the initial value problem (2.35), may be very helpful in obtaining insight concerning the behavior of these algorithms ; for example, the larger r is, the safer the algorithms.

The numerical integration of initial value problems by ADI methods is discussed in [1] under fairly general assumptions on A_1 and A_2.

2.7.3. Further comments.

The solution of problems like (P) by decomposition-coordination methods via augmented lagrangians seems to be due to GLOWINSKI-MARROCCO [12,13,14] (see also POLJAK [15]). For more details and various applications see also [1,2,5,6,8, 11,16-22]. Ref. [11],[22] in particular describe the application of the above algorithms to large displacement calculations of flexible, inextensible, bending pipe lines.

To the best of our knowledge the relations between the algorithms of Sec. 2.5 and ADI methods have been observed for the first time by CHAN-GLOWINSKI [23]. In Sec. 2 we have basically followed the presentation of FORTIN-GLOWINSKI [5] and GLOWINSKI [6].

3. - FORMULATIONS OF ELASTOSTATIC PROBLEMS FOR INCOMPRISSIBLE MOONEY-RIVLIN MATERIALS.

3.1. Notation and mechanical assumptions.

A fundamental problem in nonlinear elasticity is the calculation of the deformations of a solid body made of an homogeneous, incompressible, hyperelastic material, subjected to volumetric forces $\rho_0 \underset{\sim}{f}$ (ρ_o : density in the reference configuration) and superficial forces $\underset{\sim}{S}_o$. In a lagrangian formulation, the related energy functional, corresponding to a displacement field $\underset{\sim}{v}$ is

$$(3.1) \qquad \pi(\underset{\sim}{v}) = \int_{\Omega} \rho_o(\sigma(\underset{\sim}{v}) - \underset{\sim}{f}\cdot\underset{\sim}{v})\,d\underset{\sim}{x} - \int_{\partial\Omega_2} \underset{\sim}{S}_o\cdot\underset{\sim}{v}\,d\Gamma ,$$

where Ω is a bounded domain of $\mathbb{R}^N$ corresponding to the reference configuration ; $\partial\Omega(= \partial\Omega_1 \cup \partial\Omega_2)$ is the boundary of Ω, the body being fixed on $\partial\Omega_1$. We have denoted by $\sigma(\underset{\sim}{v})$ the stored energy functional (per mass unit). For a Mooney-Rivlin material we have

$$(3.2) \qquad \sigma(\underset{\sim}{v}) = E_1(I_1-2) \text{ if } N=2 ,$$

$$(3.3) \qquad \sigma(\underset{\sim}{v}) = E_1(I_1-3)+E_2(I_2-3) \text{ if } N=3$$

with, in (3.2),(3.3), I_i the i-th invariant of the $\underset{\approx}{F}\underset{\approx}{F}^t$ tensor, where

$$(3.4) \qquad \underset{\approx}{F} = \underset{\sim}{\nabla}\underset{\sim}{v}+\underset{\approx}{I}$$

and E_1, E_2 are coefficients which are material dependent. The displacement also has to satisfy the incompressibility condition which here has the following form

$$(3.5) \qquad \det \underset{\approx}{F}(\underset{\sim}{v}) = 1 \text{ a.e. on } \Omega.$$

Remark 3.1 : We have supposed in (3.1) that $\tilde{S}_o$ and $\tilde{f}$ are independent of $\tilde{v}$. This corresponds to the standard simplifying assumption known as the dead loading hypothesis for which $\tilde{f}$ and $\tilde{S}_o$ do not vary during the motion ; if we deal with pressure type forces, then $\tilde{S}_o$ is a function of $\tilde{u}$ (the actual displacement), given by

$$(3.6) \qquad \tilde{S}_o = -q(\tilde{x}+\tilde{u}(\tilde{x}))(\tilde{\tilde{I}}+\tilde{\nabla}\tilde{u})^{-1}\tilde{n} ,$$

where $q(\tilde{x}+\tilde{u}(\tilde{x}))$ is the pressure at the point $\tilde{x}+\tilde{u}(\tilde{x})$ of the actual configuration and $\tilde{n}$ is the unit outward normal at $\partial\Omega$, in the reference configuration.

3.2. Mathematical formulations.

We shall give in this section several possible formulations of the elasto-static problems ; it is still an open problem to prove their equivalence in general (see [24]-[26] for a discussion of these equivalence, and also the comments of Sec. 3.2.4).

3.2.1. Formulation by minimization.

It is reasonable to suppose that those displacements $\tilde{u}$ corresponding to stable equilibrium position obey

$$(3.7) \qquad \begin{cases} \tilde{u} \text{ is a local minimizer over } K \text{ of the functional} \\ \tilde{v} \to \pi(\tilde{v}), \end{cases}$$

with for an incompressible Mooney-Rivlin material

$$(3.8) \qquad \begin{cases} K = \{\tilde{v} \in (H^1(\Omega))^N,\ \tilde{v}=\tilde{0} \text{ on } \partial\Omega_1,\ \det \tilde{\tilde{F}}(\tilde{v}) = 1 \\ \text{a.e.},\ (F^{-1}(\tilde{v}))^t \in (L^2(\Omega))^{N\times N}\}. \end{cases}$$

The existence of solutions for (3.7),(3.8) is proved in [27].

3.2.2. Formulation by equilibrium equations.

The equilibrium positions correspond to the solutions of the following system of nonlinear partial differential equations

$$(3.9)\quad \begin{cases} u \in K, \\ (D\pi(u),v) + \int_\Omega p[u,v]\, dx = 0 \quad \forall\, v \in X, \end{cases}$$

where $D\pi$ is the differential of π (on $(H^1(\Omega))^N$) and where

$$(3.10)\quad [u,v] = \frac{\partial}{\partial u_{i,j}} (\det F(u)) v_{i,j} ,$$

$$(3.11)\quad X = \{v \in (H^1(\Omega))^N,\ v=0 \text{ on } \partial\Omega_1\}$$

(in (3.]0) we have used the standard summation and derivative notations). The above function p, which is clearly a Lagrange multiplier associated with the incompressibility condition (3.5), appears as a pressure.

3.2.3. Formulation by augmented lagrangian.

We proceed as in Sec. 2 by "relaxing" the linear relation (3.4) using an augmented lagrangian. We obtain thus the following formulation :

$$(3.12)\quad \begin{cases} \text{Find } \{u,F,\lambda\} \in W = X \times Y \times (L^2(\Omega))^{N\times N}, \text{ stationary point} \\ \text{over } W \text{ of the augmented lagrangian} \\ \mathcal{L}_R(v,G,\mu) = \pi(v) + \frac{R}{2} \|\nabla v + I - G\|^2_{L^2} - \\ - \int_\Omega \mu \cdot (\nabla v - I - G)\, dx , \end{cases}$$

where, in (3.12), we have $R > 0$ and

$$Y = \{F \in (L^2(\Omega))^{N\times N} ,\ (F^{-1})^t \in (L^2(\Omega))^{N\times N}, \det F = 1 \text{ a.e.}\}.$$

3.2.4. Some relations between formulations (3.7),(3.9) and (3.12).

The following results are proved in [24] :

a) Formulations (3.9) and (3.12) are equivalent.
b) Every "smooth" solution of (3.7) is a solution of (3.9), (3.12),

c) If the functional π is convex (case of a Mooney-Rivlin material if N=2), then every solution of (3.12) is such that u is a minimizer of

$$\underset{\sim}{v} \rightarrow \mathcal{L}_R(\underset{\sim}{v},\underset{\approx}{F},\underset{\approx}{\lambda}) \ ;$$

similarly, for R sufficiently large every solution of (3.12) is such that $\underset{\approx}{F}$ is a minimizer of $\underset{\approx}{G} \rightarrow \mathcal{L}_R(\underset{\approx}{u},\underset{\approx}{G},\underset{\approx}{\lambda})$.

Remark 3.2 : If $\{\underset{\sim}{u},\underset{\approx}{F},\underset{\approx}{\lambda}\}$ is a solution of (3.12), then the condition

$$\partial_{\underset{\sim}{u}}\mathcal{L}_R(\underset{\sim}{u},\underset{\approx}{F},\underset{\approx}{\lambda}) = 0$$

implies

$$(3.13) \qquad -\partial_j\left(\frac{\partial}{\partial u_{i,j}}(\rho_o\sigma)-\lambda_{ij}\right) = \rho_o f_i .$$

From (3.13), $\underset{\approx}{\lambda}$ appears as this part of the first Piola-Kirchoff stress tensor, corresponding to the incompressibility. We observe also that an algorithm solving (3.12) gives the stress field directly.

3.3. Axisymmetric problems.

3.3.1. Formulation of the problems.

We consider the case of an axisymmetric incompressible hyperelastic body subjected to an axisymmetric system of forces. The problem is to find the axisymmetric positions of equilibrium. We denote by $\underset{\sim}{u} = \{u_1,u_2\}$ the displacement, with u_1 the radial displacement and u_2 the axial one. Using an $\{r,z\}$ system of coordinates we have

$$(3.14) \qquad \underset{\approx}{I}+\underset{\sim}{\nabla}\underset{\sim}{u} = \begin{pmatrix} 1+u_{1,1} & u_{1,2} & 0 \\ u_{2,1} & 1+u_{2,2} & 0 \\ 0 & 0 & e(\underset{\sim}{u}) \end{pmatrix} ,$$

where $e(\underset{\sim}{u}) = 1+\frac{u_1}{r}$ is the extension ratio in the circumferential direction, and where

$$u_{j,1} = \frac{\partial u_j}{\partial r} , \quad u_{j,2} = \frac{\partial u_j}{\partial z} .$$

Using the notation of Sec. 3.1, we have (with $e=e(\underset{\sim}{u})$)

$$I_1(\underset{\sim}{u}) = e^2+(\delta_{ij}+u_{i,j})^2 \;,$$

$$I_2(\underset{\sim}{u}) = e^2(\delta_{ij}+u_{i,j})^2+(1+u_{1,1}u_{2,2}-u_{1,2}u_{2,1}+u_{1,1}+u_{2,2})^2,$$

$$\underline{\det}(\underset{\approx}{I}+\nabla\underset{\sim}{u}) = e(1+u_{1,1}u_{2,2}-u_{1,2}u_{2,1}+u_{1,1}+u_{2,2}).$$

For <u>incompressible</u> materials we have

$$\underline{\det}(\underset{\approx}{I}+\nabla\underset{\sim}{u}) = 1 \text{ a.e.}$$

implying in turn

$$I_2(\underset{\sim}{u}) = e^2(\delta_{ij}+u_{i,j})^2+ 1/e^2.$$

3.3.2. <u>Lagrangian formulation</u>.

Let Ω be the <u>half meridian section</u> of the reference domain ; we associate to Ω the following spaces (with $1 \leq p < +\infty$)

$$\mathscr{L}^p = \{\phi \mid \int_\Omega |\phi(x)|^p \, r dr\, dz < +\infty\} \;,$$
$$\mathcal{H}^1 = \{\phi \mid \phi, \frac{\partial\phi}{\partial r}, \frac{\partial\phi}{\partial z} \in \mathscr{L}^2\} \;;$$

we shall use in the sequel the notation $d\underset{\sim}{x} = r dr dz$.

With respect to the general case we do the following modifications :

a) <u>Spaces</u>.

$$X = \{\underset{\sim}{v} \in (\mathcal{H}^1)^2 \;,\; \underset{\sim}{v}=\underset{\sim}{0} \;\underline{\text{on}}\; \partial\Omega_1\} \;,$$

and with $1 \leq i,j \leq 2$

$$Y = \{\underset{\approx}{G} = \{g_o, g\} \in L^\infty(\Omega) \times (\mathscr{L}^2)^4,\; g_o > 0,\; g_o \underline{\det}\, \underset{\approx}{g}=1 \text{ a.e.}\},$$

$$Z = \{\underset{\approx}{\mu} = \{\mu_o, \mu_{ij}\} \in \mathscr{L}^1 \times (\mathscr{L}^2)^4\} \;.$$

b) <u>Augmented lagrangian</u>

$$(3.15) \quad \begin{cases} \mathscr{L}_R(\underset{\sim}{v},\underset{\approx}{G},\underset{\approx}{\mu}) = \pi(\underset{\sim}{v}) + \frac{R}{2}\int_\Omega (\delta_{ij}+v_{i,j}-g_{ij})^2 d\underset{\sim}{x} \\ -\int_\Omega \mu_{ij}(\delta_{ij}+v_{ij}-g_{ij})d\underset{\sim}{x} + \frac{R}{2}\int_\Omega (g_o-e(\underset{\sim}{v}))^2 d\underset{\sim}{x} - \end{cases}$$

$$- \int_{\Omega} \mu_o (e(\underset{\sim}{v}) - g_o) d\underset{\sim}{x}.$$

Remark 3.3 : A possible variant of $\mathcal{L}_R$ in (3.15) can be obtained by replacing $e(\underset{\sim}{v})$ by g_o in $\pi(\underset{\sim}{v})$ (leading to a functional $\pi(\underset{\sim}{v}, g_o)$).

4. - ITERATIVE SOLUTION OF THE EQUILIBRIUM PROBLEMS.

We apply to the solution of the equilibrium problem (3.12), the iterative methods described in Sec. 2, keeping in mind that (3.12) is a nonlinear, nonquadratic, non convex problem.

4.1. A first algorithm for solving (3.12).

We follow Sec. 2.4 ; using the notation of Sec. 3.2.3 the algorithm is :

$$(4.1) \qquad \underset{\approx}{\lambda}^o \text{ given in } (L^2(\Omega))^{N\times N},$$

then for $n \geq 0$, $\underset{\approx}{\lambda}^n$ known we obtain $\underset{\approx}{u}^n, \underset{\approx}{F}^n$ and $\underset{\approx}{\lambda}^{n+1}$ $(\underset{\approx}{\lambda}^n \in (L^2(\Omega))^{N\times N})$ by

$$(4.2) \qquad \begin{cases} \{\underset{\sim}{u}^n, \underset{\approx}{F}^n\} \in X\times Y \text{ and } \forall \{\underset{\sim}{v}, \underset{\approx}{G}\} \in X\times Y \text{ we have} \\ \mathcal{L}_R(\underset{\sim}{u}^n, \underset{\approx}{F}^n, \underset{\approx}{\lambda}^n) \leq \mathcal{L}_R(\underset{\sim}{v}, \underset{\approx}{G}, \underset{\approx}{\lambda}^n), \end{cases}$$

$$(4.3) \qquad \underset{\approx}{\lambda}^{n+1} = \underset{\approx}{\lambda}^n - \rho(\underset{\sim}{\nabla}\underset{\sim}{u}^n + \underset{\approx}{I} - \underset{\approx}{F}^n) \quad , \ \rho > 0.$$

Remark 4.1 :Problem (4.2) is equivalent to the nonlinear system

$$(4.4) \qquad \mathcal{L}_R(\underset{\sim}{u}^n, \underset{\approx}{F}^n, \underset{\approx}{\lambda}^n) \leq \mathcal{L}_R(\underset{\sim}{u}^n, \underset{\approx}{G}, \underset{\approx}{\lambda}^n) \quad \forall \underset{\approx}{G} \in Y, \ \underset{\approx}{F}^n \in Y,$$

$$(4.5) \qquad \partial_{\underset{\sim}{v}} \mathcal{L}_R(\underset{\sim}{u}^n, \underset{\approx}{F}^n, \underset{\sim}{\lambda}^n) \cdot \underset{\sim}{v} = 0 \quad \forall \underset{\sim}{v} \in X, \ \underset{\sim}{u}^n \in X,$$

whose block relaxation solution leads to the algorithm described in Sec. 4.2.

4.2. A second algorithm for solving (3.12).

4.2.1. Description of the algorithm.

The second algorithm is given by

$$(4.6) \quad \underset{\sim}{u}^{-1} \text{ given in } X,\ \underset{\approx}{\lambda}^{o} \text{ given in } (L^2(\Omega))^{N\times N}$$

then for $n \geq 0$, $\underset{\sim}{u}^{n-1}, \underset{\approx}{\lambda}^{n}$ given, we obtain $\underset{\approx}{F}^{n}, \underset{\sim}{u}^{n}, \underset{\approx}{\lambda}^{n+1}$ from

$$(4.7) \quad \mathcal{L}_R(\underset{\sim}{u}^{n-1}, \underset{\approx}{F}^{n}, \underset{\approx}{\lambda}^{n}) \leq \mathcal{L}_R(\underset{\sim}{u}^{n-1}, \underset{\approx}{G}, \underset{\approx}{\lambda}^{n}) \quad \forall\ \underset{\approx}{G} \in Y, \underset{\approx}{F}^{n} \in Y,$$

$$(4.8) \quad \partial_{\underset{\sim}{v}}\mathcal{L}_R(\underset{\sim}{u}^{n}, \underset{\approx}{F}^{n}, \underset{\approx}{\lambda}^{n}) \cdot \underset{\sim}{v} = 0 \quad \forall \underset{\sim}{v} \in X,\ \underset{\sim}{u}^{n} \in X,$$

$$(4.9) \quad \underset{\sim}{\lambda}^{n+1} = \underset{\sim}{\lambda}^{n} - \rho(\underset{\sim}{\nabla}\underset{\sim}{u}^{n} + \underset{\approx}{I} - \underset{\approx}{F}^{n}).$$

4.2.2. On the solution of problems (4.7),(4.8).

Problem (4.8) is equivalent to

$$4.10) \quad \begin{cases} \text{Find } \underset{\sim}{u}^{n} \in X \text{ such that} \\ \mathcal{L}_R(\underset{\sim}{u}^{n}, \underset{\approx}{F}^{n}, \underset{\approx}{\lambda}^{n}) \leq \mathcal{L}_R(\underset{\sim}{v}, \underset{\approx}{F}^{n}, \underset{\approx}{\lambda}^{n}) \quad \forall \underset{\sim}{v} \in X \end{cases}$$

which is an unconstrained minimization problem whose solution is quite easy, particularly if R is sufficiently large. If N=2 then the functional to minimize is quadratic, and therefore solving (4.8),(4.10) is equivalent to solving a linear problem related to a second order partial differential operator which is independent of n and whose discrete variants are linear systems associated with positive definite matrices independent of n (we shall use therefore a prefactorization of these matrices). If N=3 or in the axisymmetric case problem (4.8), (4.10) is no longer linear ;it can be, however, efficiently solved by a preconditioned conjugate gradient algorithm. Problem (4.7) is a more delicate one (apparently, at least) ; if N=2, (4.7) is reduced to (omitting indice n) :

$$(4.11) \quad \begin{cases} \text{Find } \underset{\approx}{F} \in Y \text{ and minimizing over } Y \text{ the functional} \\ \underset{\approx}{G} \rightarrow \int_\Omega [RG_{ij}^2 - 2(R(u_{i,j}+\delta_{ij}) - \lambda_{ij})G_{ij}]\,d\underset{\sim}{x} \end{cases}$$

with

$$Y = \{\underset{\approx}{G} \in (L^2(\Omega))^4,\ G_{11}G_{22}-G_{12}G_{21} = 1 \ \underline{\text{a.e. on}}\ \Omega\}.$$

Since derivatives of $\underset{\approx}{G}$ (and $\underset{\approx}{F}$) are not involved in (4.11), we can solve this last problem pointwise. We have therefore to solve an infinity (in theory) of four-dimensional problems of the following class

$$(4.13)\quad \begin{cases} \underline{\text{Find}}\ \{F_{ij}\} \in \mathcal{G}\ \underline{\text{and minimizing over}}\ \mathcal{G}\ \underline{\text{the functional}} \\ \{G_{ij}\} \to RG_{ij}^2 - 2a_{ij}G_{ij}, \end{cases}$$

where

$$(4.14)\quad \mathcal{G} = \{\{G_{ij}\} \in \mathbb{R}^4,\ G_{11}G_{22}-G_{12}G_{21} = 1\}.$$

We diagonalize the above quadratic relation using as new variables

$$(4.15)\quad \begin{cases} b_1 = (F_{11}+F_{22})/\sqrt{2},\ b_2 = (F_{11}-F_{22})/\sqrt{2}, \\ b_3 = (F_{12}+F_{21})/\sqrt{2},\ b_4 = (F_{12}-F_{21})/\sqrt{2}. \end{cases}$$

Using $\underset{\sim}{b}$ defined by (4.15), problem (4.13),(4.14) becomes

$$(4.16)\quad \begin{cases} \underline{\text{Find}}\ \underset{\sim}{b} \in C\ \underline{\text{and minimizing over}}\ C\ \underline{\text{the functional}} \\ \underset{\sim}{c} \to Rc_i^2 - 2z_ic_i, \end{cases}$$

where

$$(4.17)\quad \begin{cases} C = \{\underset{\sim}{c} \mid \underset{\sim}{c} \in \mathbb{R}^4,\ \varepsilon_i c_i^2 = 2\}, \\ \underline{\text{with}}\ \varepsilon_1=\varepsilon_4=1,\ \varepsilon_2=\varepsilon_3=-1. \end{cases}$$

The extremizers of (4.16),(4.17) are given by

$$(4.18)\quad \underset{\sim}{b} \in \mathbb{R}^4,\ b_i=z_i/(R+\varepsilon_i p),\ i=1,2,3,4,$$

where the scalar p (Lagrange multiplier of $\varepsilon_i b_i^2 = 2$) is a solution of

(4.19) $$(z_1^2+z_4^2)/(R+p)^2 = (z_2^2+z_3^2)/(R-p)^2+2 \ .$$

We suppose that $z_1^2+z_4^2 \neq 0$; then it can be shown that (4.19) has a unique solution in]-R,+R[; moreover, using the implicit function theorem (see [24], [25] for more details) one can show that this solution p of (4.19) between -R and R is precisely the one associated with the global minimum of $\underset{\sim}{c} \to Rc_i^2-2z_ic_i$ over $\varepsilon_i c_i^2 = 2$, to which, therefore, corresponds a unique global minimizer given from p in (4.16),(4.17) (actually there is no other minimizer (local or global) than the above global minimizer).

Solving (4.19) on]-R,+R[is a trivial problem ; we have then $\underset{\sim}{b}$ from $\underset{\sim}{p}$ and (4.18), and then $\underset{\approx}{F}$ from $\underset{\sim}{b}$ and (4.15).

Remark 4.2 : In the actual computer experiments that we have done, we never encountered the situation $z_1^2+z_4^2 = 0$; in fact we have the feeling that for R sufficiently large this cannot happen in the context of problem (3.12) if N=2. ■

The solution of problem (4.7), in the axisymmetric case and for genuinely three-dimensional problems, leads to a far more complicated discussion, however the same general ideas still apply . we refer to [24],[25] (resp. [28]) for the axisymmetric (resp. three-dimensional) case.

5. - FINITE ELEMENT APPROXIMATION OF THE EQUILIBRIUM PROBLEMS.

5.1. Synopsis.

A most important step to the computer use of the iterative methods of Sec. 4, for solving the elasticity problems of Sec. 3, is the reduction of these problems to finite dimensional one via convenient approximation methods. As it can be guessed the main difficulty to overcome is the incompressibility condition (3.5), coupled to the other nonlinearities of the problems under consideration ; a guideline to the numerical treatment of the incompressibility condition is given by those results concerning the numerical analysis of incompressible flows governed by Stokes and Navier-Stokes equations (owing to the very abundant litterature concerning this subject we refer to the corresponding bibliographical references in [24], [30] and also to the papers of Malkus and Oden in these proceedings).

In this paper, restricting our attention to two-dimensional problems we shall describe briefly two families of finite element approximations, preserving the decomposition properties of the continuous problems :

a) A fairly classical one, previously used in [24],[25],[29].
b) A very recent one introduced by the third author in [30].

Corresponding numerical results will be shown in Sec. 6.

5.2. Approximation by quadrilateral finite elements.
The numerical solution of the lagrangian problem (3.12) using the algorithms described in Sec. 4 requires the introduction of finite dimensional spaces X_h, Y_h, Z_h approximating X, Y, Z. An approximate gradient operator s_h from X_h to Z_h must also be defined.

In this section 5.2 the spaces X_h, Y_h, Z_h are constructed using quadrilateral finite elements, with the displacements interpolated at the vertices, the pressure and the tensor $\underset{\approx}{F}$ at the center of each element.

5.2.1. Quadrangulation of Ω.
We suppose that our domain Ω is a polygonal open set in $\mathbb{R}^2$ which can be decomposed into quadrilaterals

$$(5.1) \qquad \Omega = \bigcup_{k \in I_h} \Omega_k \ .$$

Here I_h is a finite set of numbers and Ω_k is the image by a mapping $\mathcal{F}_k$ of a reference rectangle $\hat{\Omega}$ such as the one shown in Fig. 5.1 ; this mapping $\mathcal{F}_k$ is defined by

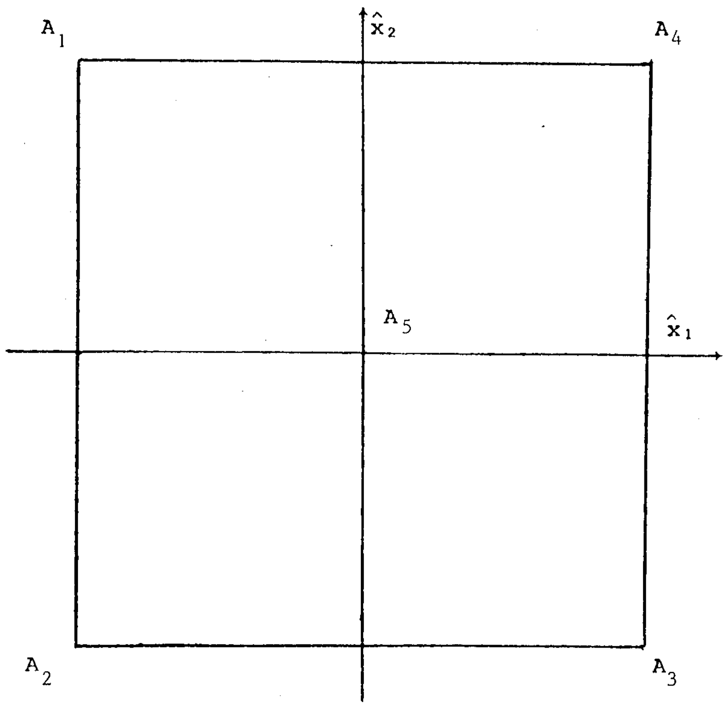

Figure 5.1 : Reference Rectangle

(5.2) $$\underset{\sim}{x} = \mathcal{F}_k(\hat{\underset{\sim}{x}}) = \sum_{\alpha=1}^{4} \underset{\sim}{x}^{\alpha_k} \hat{\phi}(\hat{\underset{\sim}{x}}),$$

where, in (5.2), $\underset{\sim}{x}^{\alpha_k}$ is this vector whose coordinates are those of the node α of the k^{th} element and where $\hat{\phi}^{\alpha}$ is the shape function associated to the node α of the bilinear finite element $\{\hat{\Omega}, \{A_\alpha\}_{\alpha=1}^{4}, Q_1(\hat{\Omega})\}$, $Q_1(\hat{\Omega})$ being the space of the bilinear polynomials defined on $\hat{\Omega}$, i.e.

(5.3) $$\hat{Q}_1(\hat{\Omega}) = \{\hat{q} \mid \hat{q}(\hat{x}) = \hat{a}_{00}+\hat{a}_{10}\hat{x}_1+\hat{a}_{01}\hat{x}_2+\hat{a}_{11}\hat{x}_1\hat{x}_2\}.$$

5.2.2. Discrete spaces and discrete operators.

We can now define the following spaces (with $y_{h|k} = y_{h|\Omega_k}$)

(5.4) $$H_h^1 = \{y_h \in C^0(\bar{\Omega}),\ y_{h|k} = z_h \circ \mathcal{F}_k^{-1},\ z_h \in Q_1(\hat{\Omega})\},$$

(5.5) $P_h = \{q_h \mid q_h|_K = \underline{\text{const.}}\}$,

(5.6) $X_h = H_h^1 \cap X$,

(5.7) $Z_h = (P_h)^4$,

(5.8) $Y_h = Y \cap Z_h$.

Remark 5.1 : It follows from (5.2),(5.4) that the above finite element approximation is of the isoparametric type. ■

Concerning the approximate gradient operator s_h, two quite natural choices can be made :

Either

$$(5.9)_1 \quad s_h(\underset{\sim}{u}_h)|_K = \underset{\sim\sim}{\nabla u}_h(\mathcal{F}_K(A_5)),$$

or

$$(5.9)_2 \quad s_h(\underset{\sim}{u}_h)|_K = L^2\text{-}\underline{\text{projection of}}\ \underset{\sim\sim}{\nabla u}_h\ \underline{\text{on}}\ Z_h.$$

5.2.3. Formulation of the approximate problems.

We consider only the plane strain problem since the extension to the axisymmetric case is quite obvious ; taking into account the results of Sec. 3.2.3, 3.2.4 the discrete approximation of (3.7),(3.8) that we consider is defined by

Find $\{u_h, F_h, \lambda_h\} \in X_h \times Y_h \times Z_h$ such that

$$(5.10)_1 \begin{cases} \partial_{\underset{\sim}{v}} \pi(\underset{\sim}{u}_h)\cdot\underset{\sim}{v}_h + R\displaystyle\int_\Omega (\underset{\approx}{I}+s_h(\underset{\sim}{u}_h)-\underset{\approx}{F}_h)\cdot s_h(\underset{\sim}{v}_h)\,dx\ + \\ -\displaystyle\int_\Omega \underset{\approx}{\lambda}_h\cdot s_h(\underset{\sim}{v}_h)\,d\underset{\sim}{x} = 0 \quad \forall \underset{\sim}{v}_h \in X_h \end{cases}$$

$$(5.10)_2 \begin{cases} \underset{\approx}{F}_h\ \underline{\text{minimizes over}}\ Y_h\ \underline{\text{the functional}} \\ \underset{\approx}{G}_h \to \dfrac{R}{2}\,\|I+s_h(u_h) - \underset{\approx}{G}_h\|^2 + \displaystyle\int_\Omega \underset{\approx}{\lambda}_h \cdot \underset{\approx}{G}_h\,d\underset{\sim}{x}, \end{cases}$$

$$(5.10)_3 \quad \underset{\approx}{I}+s_h(\underset{\sim}{u}_h) = \underset{\approx}{F}_h\ \underline{\text{in}}\ Z_h.$$

It is important to observe that from the construction of Y_h and if we suppose that $\underset{\sim}{u}_h$ and $\underset{\approx}{\lambda}_h$ are known in $(5.10)_2$ then this last minimization problem can be solved elementwise ; therefore the algorithms of Sec. 4 can also be applied to the solution of problem (5.10).

Concerning the convergence of the discrete solutions to the continuous one we refer to LE TALLEC [24].

5.3. Approximation by incomplete quadratic simplicial finite elements.

We shall briefly discuss in this section a new type of finite element approximation, using triangles ; these finite elements have been recently introduced by the third author in [30] (see also [31]) and have been denominated AQL (for Asymmetric Quasi-Linear).

We follow [30] ; this new approximation is defined as follows :

Let $\mathcal{C}_h$ be a triangulation of Ω and let $T \in \mathcal{C}_h$ with vertices $\{\delta_{iT}\}_{i=1}^{3}$; let $P_{4/3}$ be the space of the polynomials in two variables, of degree ≤ 2, defined over T and such that their restriction to two given edges, e_{1T} and e_{2T} for example, are <u>linear functions</u> (e_{iT} is the edge of T, opposite to δ_{iT}) ; we clearly have dim $P_{4/3} = 4$. We introduce now the following set of degrees of freedom $\{a_{iT}\}_{i=1}^{4}$, where :

- a_{iT} is for i=1,2,3 the functional value at δ_{iT},
- a_{4T} is the functional value at the midpoint of edge e_{3T}.

A fundamental property of the AQL element is given by the following

<u>Proposition 5.1</u> : <u>Let</u> $\underset{\sim}{v} = \{v_1, v_2\}$ <u>be a vector valued function defined over a triangle</u> T <u>and whose components belong to</u> $P_{4/3}$, <u>then</u>

$$\underline{\det}\ (\underset{\approx}{I}+\underset{\sim}{\nabla}\underset{\sim}{v})\ \underline{\text{is linear over}}\ T.$$

We refer to [30] for the proof of the above proposition.
It follows then from Prop. 5.1 that

$$\text{(5.11)} \quad \underline{\det}\,(\underset{\approx}{I}+\nabla \underset{\sim}{v})(G_T) = \frac{1}{\text{meas.}(T)} \int_T \underline{\det}(\underset{\approx}{I}+\nabla\underset{\sim}{v}(\underset{\sim}{x}))\,d\underset{\sim}{x}$$

(where G_T = centroid of T), and (5.11) clearly suggests to require the incompressibility condition

$$\det\,(\underset{\approx}{I}+\nabla \underset{\sim}{v}) = 1$$

at G_T only.
Notice that because the nonsymmetric structure of the AQL element we have to make some restriction on the triangulation upon which the space of the approximate displacements is to be defined ; one suggests in [30] the following process that generates nearly as general a mesh as any other finite element triangulation :
We first partition Ω into convex quadrilaterals arbitrarily ; then each quadrilateral is subdivided into two triangles by one of its two diagonals (any of them can be used). The edges over which the restrictions of the second order polynomials of $P_{4/3}$ are allowed to be quadratic are precisely those diagonals.

Using the above AQL finite elements we approximate X,Y,Z by X_h, Y_h, Z_h as follows :

$$\text{(5.12)} \quad H_h^1 = \{y_h \in C^0(\bar{\Omega}),\ y_h|_T \in P_{4/3} \quad \forall T \in \mathcal{C}_h\}\ ,$$

$$\text{(5.13)} \quad P_h = \{q_h | q_h|_T = \underline{\text{const}}.\ \forall T \in \mathcal{C}_h\},$$

$$\text{(5.14)} \quad X_h = X \cap H_h^1\ ,$$

$$\text{(5.15)} \quad Z_h = (P_h)^4,$$

$$\text{(5.16)} \quad Y_h = Y \cap Z_h.$$

Concerning the approximate gradient operator s_h we use

(5.17) $s_h(\underset{\sim}{u}_h)|_T = \underset{\sim}{\nabla}\underset{\sim}{u}_h(G_T)$.

Using the above spaces and operator the approximate problem is then defined exactly as in Sec. 5.2.3 by (5.10), and the local properties of $(5.10)_2$ are preserved by the new element.

Remark 5.2 : We refer to [30], [31] for a detailed discussion of the AQL element, and also for various two and three dimensional generalizations.

Remark 5.3 : It will be interesting to use the AQL element (and its generalizations) for the finite element solution of Stokes and Navier-Stokes problems.

6. - NUMERICAL EXPERIMENTS.

All the numerical experiments displayed in this section deal with Mooney-Rivlin materials and either a plane strain or axisymmetric situation ; moreover the numerical results presented here have been obtained using the quadrilateral finite element approximation discussed in Sec. 5.2. Other numerical experiments are done in [24],[25] (including the numerical treatment of plane stress problems) ; numerical experiments using this Ruas' element of Sec. 5.3 (and some of its generalizations are discussed in [30],[31].

6.1. Stretching of a thick cracked rectangular bar.

We consider a thick rectangular slab of Mooney-Rivlin material, with a non-propagating crack in its middle, submitted to vertical stretching forces applied at its extremities. The initial configuration of the lower part of the bar and of the crack is shown on Fig. 6.1 (a). Under the action of the external forces this bar is stretched and its computed equilibrium position (plane strain assumption, with $\sigma(\underset{\sim}{v}) = E_1(I_1(\underset{\sim}{v})-2))$ is shown on Fig. 6.2(b) (the various data concerning this problem are indicated on Fig. 6.2).
Using $\rho=R=8$ we have convergence of algorithm (4.1)-(4.3) in 20 iterations, corresponding to a computational time of 3.2 seconds on CDC 6400. The computed stresses at the boundary

match the applied tension with a 10^{-4} precision.

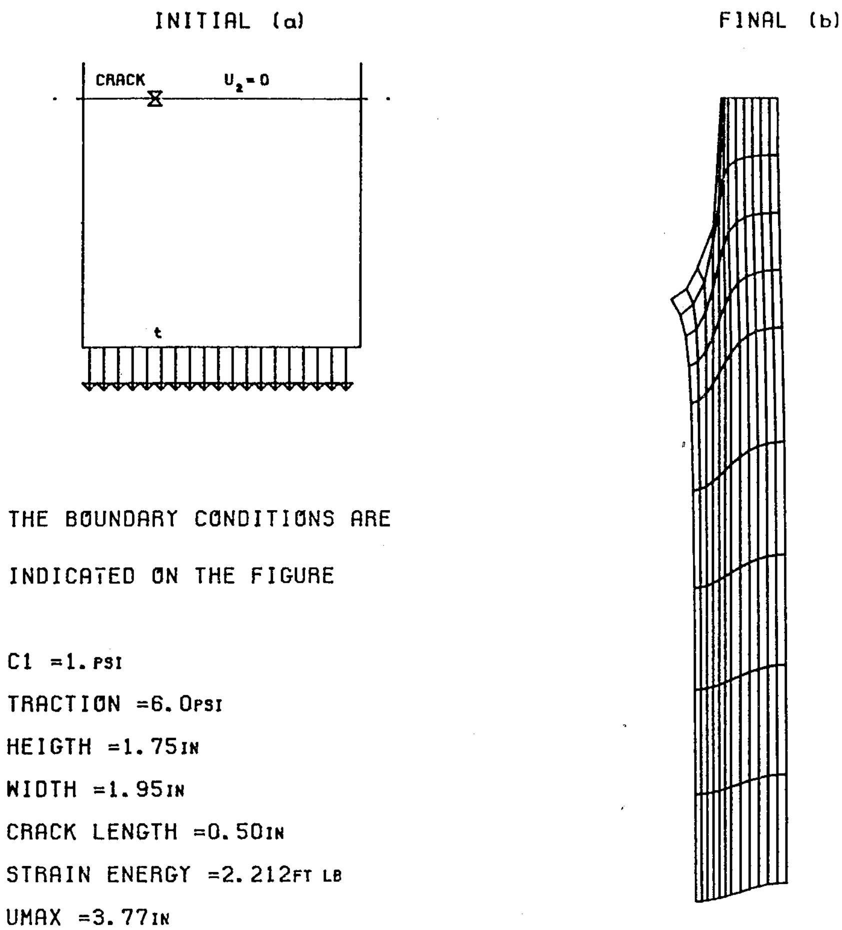

Figure 6.1

6.2. Combined Inflation and Extension of a Circular Cylindrical Tube.

We consider here a circular cylindrical tube, made of an incompressible isotropic elastic material, whose strain function is of Mooney-Rivlin type. This tube is inflated by imposing a fixed radial displacement to the inner surface $\partial\Omega_1$, the outer surface being free of tractions. An analytical solution of this problem is given in CHADWICK-HADDON [32] under the assumption that both extremities are stress free and remain horizontal. We have approximated these conditions by restricting the axial displacement to be zero at the mid cross-section $\partial\Omega_3$ and leaving the upper section traction free. This physical configuration is a close approximation of the case treated in [32] and is described in Fig. 6.2, where we have represented the upper half part of the tube in its initial configuration.

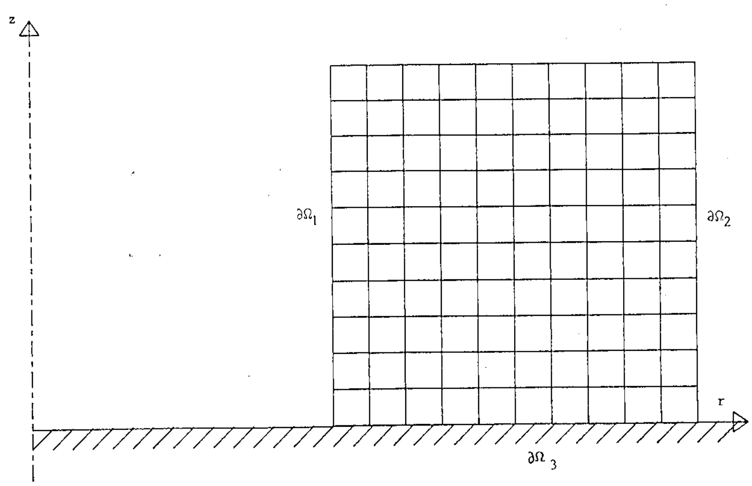

Figure 6.2

Using the notation of [32], the parameters in this problem are

$$2\sigma(\underset{\sim}{v}) = .875(I(\underset{\sim}{v})-3) + .125(I_2(\underset{\sim}{v})-3),$$

N = (outer radius/inner radius) in the reference configuration

Q = final inner radius/initial inner radius.

The numerical values given in Table 6.1, both for the analytical and numerical solution correspond to

EXTV = final height/initial height,

EXTH = final outer radius/initial outer radius.

The computed Cauchy stresses $\sigma_{\theta\theta}$ and σ_{rr} in the mid cross-section are indicated (as functions of r) on Fig. 6.3, for different values of the parameters N and Q ; these computed values (indicated by crosses on Fig. 6.3) are exactly located on the curves obtained analytically in [32] and reproduced on Fig. 6.3 ; this confirms the validity of our computations.

N	1.4	1.4	1.8	1.8	2.2	2.2	2.2
Q	1.2	1.6	1.6	2.0	1.6	2.0	2.2
EXTV analytical	.9460	.8583	.8991	.8432	.9252	.8794	.8578
EXTV computed	.9460	.8582	.8995	.8434	.9261	.8801	.8584
EXTH analytical	1.1191	1.3700	1.2486	1.4334	1.1774	1.3146	1.3879
EXTH computed	1.1192	1.3701	1.2489	1.4339	1.1778	1.3154	1.3882

Table 6.1

Comparison between analytical results (from [32]) and computed results.

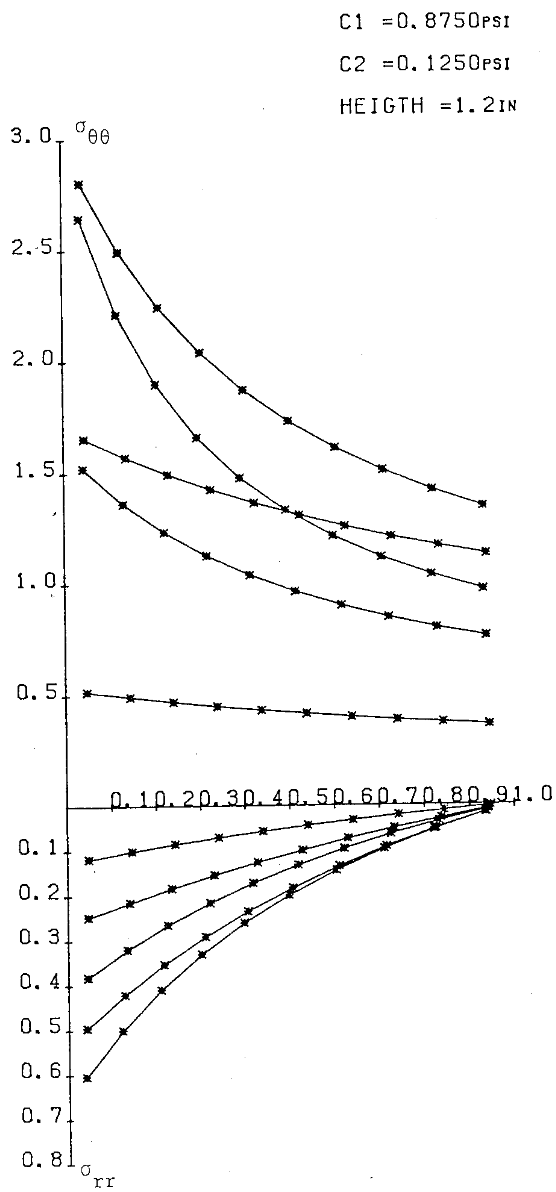
C1 =0.8750PSI
C2 =0.1250PSI
HEIGTH =1.2IN
3.0
2.5
2.0
1.5
1.0
0.5
σθθ
0.10.20.30.40.50.60.70.80.91.0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
σrr

Figure 6.3

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