FINITE ELEMENT FLOW ANALYSIS

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ASYMMETRIC FINITE ELEMENTS FOR PROBLEMS ARISING IN MECHANICS OF INCOMPRESSIBLE CONTINUOUS MEDIA

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NOTATION :

- Given a vector space V, $\underset{\sim}{V}$ denotes the space of vector fields $\underset{\sim}{v}$, whose components v_i, $i=1,2,\ldots,n$ belong to V.
- Given vectors $\underset{\sim}{x}$ and $\underset{\sim}{y}$ of $\mathbb{R}^n$, $\underset{\sim}{x}\cdot\underset{\sim}{y}$ denotes the usual inner product given by $\sum_{i=1}^{n} x_i y_i$, and $|x| = \sqrt{\underset{\sim}{x}\cdot\underset{\sim}{x}}$.
- Given a bounded domain Ω of $\mathbb{R}^n$, $L^2(\Omega)$ denotes the space of functions whose square has finite integral over Ω. The inner product and norm of $L^2(\Omega)$ are given respectively by $(f,g) = \int_\Omega fg\, d\underset{\sim}{x}$ and $|f|_o = (f,f)^{1/2}$.
- $L^2_o(\Omega)$ is the subspace of functions f in $L^2(\Omega)$, such that $\int_\Omega f\, d\underset{\sim}{x} = 0$.
- $H^m(\Omega)$ is the space of functions whose derivatives up to the order m belong to $L^2(\Omega)$. The usual semi-norm of $H^m(\Omega)$ is given by $|v|_m = \{\sum_{|\alpha|=m} (\partial^\alpha v, \partial^\alpha v)\}^{1/2}$,

  $\alpha = (\alpha_1,\alpha_2,\ldots,\alpha_n)$ being a multiinteger, $\alpha_i \geq 0$, $|\alpha| = \alpha_1+\alpha_2+\ldots+\alpha_n$, and $\partial^\alpha v = \partial^{|\alpha|} v / \partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \ldots \partial x_n^{\alpha_n}$.
- $H^1_o(\Omega)$ is the subspace of functions in $H^1(\Omega)$ that vanish on the boundary of Ω. $H^1_o(\Omega)$ is normed by the seminorm of $H^1(\Omega)$.
- P_k denotes the space of polynomials in n variables of degree less than or equal to k.

1. - INTRODUCTION.

This paper deals with a new kind of mixed finite element methods for the numerical treatment of incompressible continuous media, such as viscous fluids. In order to avoir non essential difficulties for the description of the methods, we will mainly consider a model problem, namely a stationary Stokes' problem. At the end of the paper we make some important remarks concerning applications to other related problems.

The usual velocity-pressure formulation of our model problem leads to determining a velocity field $\underset{\sim}{u} \in \underset{\sim}{H}^1_o(\Omega)$ and the associated hydrostatic pressure $p \in L^2_o(\Omega)$, Ω being a bounded n-dimensional domain, n=2,3, with boundary $\partial\Omega$, such that :

$$(P) \qquad \begin{cases} (\underset{\sim}{\nabla}\underset{\sim}{u}, \underset{\sim}{\nabla}\underset{\sim}{v}) + (\mathrm{div}\ \underset{\sim}{v}, p) = (\underset{\sim}{f}, \underset{\sim}{v}) & \forall \underset{\sim}{v} \in \underset{\sim}{H}^1_o(\Omega) \\ (\mathrm{div}\ \underset{\sim}{u}, q) = 0 & \forall q \in L^2_o(\Omega) \end{cases}$$

If we approximate $H^1_o(\Omega)$ and $L^2_o(\Omega)$ by finite dimensional spaces V_h and Q_h, whose construction is based upon a partition $\mathcal{C}_h$ of Ω into finite elements having a maximal edge length equal to h, we obtain the following approximation of (P) :

Find $[u_h, p_h] \in V_h \times Q_h$ such that :

$$(P_h) \quad \begin{cases} (\nabla u_h, \nabla v_h) + (\operatorname{div} v_h, p_h) = (f, v_h) & \forall v_h \in V_h \\ (\operatorname{div} u_h, q_h) = 0 & \forall q_h \in Q_h \end{cases}$$

assuming that $V_h \subset H_o^1(\Omega)$ and $Q_h \subset L_o^2(\Omega)$.

Remark : If one of the above inclusions does not hold, the respective inner products in (P_h) must be replaced by inner products $(\cdot,\cdot)_h$ that are obtained by summation of inner products of $L^2(K)$ over $\mathcal{C}_h$, K being the generic element of $\mathcal{C}_h$ (see below).

It is well-known that (P_h) will only be well-posed if the approximation spaces V_h and Q_h satisfy the following Brezzi-type compatibility condition

$$\sup_{v_h \in V_h} \frac{(\operatorname{div} v_h, q_h)_h}{\|v_h\|_h} \geq \beta |q_h|_h \qquad \forall q_h \in Q_h \tag{1}$$

where β must be strictly positive (and possibly independent of h).

Here the norms of V_h and Q_h denoted by $\|\cdot\|_h$ and $|\cdot|_h$ respectively, are discrete $H_o^1(\Omega)$ and $L^2(\Omega)$ norms, obtained by summation over the elements of $\mathcal{C}_h$, namely $\|\cdot\|_h = (\nabla\cdot, \nabla\cdot)_h^{1/2}$ and $|\cdot|_h = (\cdot,\cdot)_h^{1/2}$, where :

$$(\nabla v_h, \nabla v_h)_h = \sum_{K \in \mathcal{C}_h} \int_K \nabla v_h \cdot \nabla v_h \, dx \text{ and } (q_h, q_h)_h = \sum_{K \in \mathcal{C}_h} \int_K q_h^2 \, dx,$$

Notice that these norms will coincide with the norms of $H_o^1(\Omega)$ and $L_o^2(\Omega)$ if $\Omega_h = \bigcup_{K \in \mathcal{C}_h} K = \Omega$ and $V_h \subset H_o^1(\Omega)$, $Q_h \subset L_o^2(\Omega)$.

For the special case of problem (P), a rather significant number of possible choices of spaces V_h and Q_h, not only satisfying condition (1), but also leading to a sequence $\{[u_h, p_h]\}_h$ that converges in $H_o^1(\Omega) \times L_o^2(\Omega)$ to (u,p) as h goes to zero, are known for the two-dimensional case (see e.g. 2, 3, 4 and references therein). As for the three-dimensional case the number of good methods is smaller.

It is important to notice that condition (1) implies in particular that the dimension of Q_h must be suitably exceeded by the dimension of V_h. Therefore, most of the above mentioned methods, are such that dim V_h is too big with respect to dim Q_h. In many respects, this fact can be viewed as a waste, since the velocity is only apparently well-approximated when one uses such a V_h. As a matter of fact, since in this case the approximation of the pressure is relatively poor, the incompressibility condition is badly approximated, which causes the overall discretization error to increase significantly. Therefore, it is advisable to try to reduce to a minimum the discrepancy between the orders of approximation of $H_o^1(\Omega)$ by V_h and of $L_o^2(\Omega)$ by Q_h, while keeping those spaces compatible and, moreover, obtaining a good order of overall approximation of $[u,p]$ by $[u_h, p_h]$. Actually we can say that this order will be optimal if it coincides with the order of approximation of V_h and Q_h considered separately.

Summing up, we can say that the ideal situation is to have :

(i) $\inf_{\underset{\sim}{v}_h \in \underset{\sim}{V}_h} \|\underset{\sim}{v}-\underset{\sim}{v}_h\|_h = 0(h^m) \qquad \forall \underset{\sim}{v} \in \underset{\sim}{H}^1_o(\Omega)$

(ii) $\inf_{q_h \in Q_h} |q-q_h|_h = 0(h^m) \qquad \forall q \in L^2_o(\Omega)$

(iii) $\|\underset{\sim}{u}-\underset{\sim}{u}_h\|_h + |p-p_h|_h = 0(h^m)$

for some real m, $m > 0$, while keeping the asymptotic ratio

$$\theta = \lim_{h\to 0} \frac{\dim Q_h}{\dim \underset{\sim}{V}_h}$$

as close to one as possible.

As an illustration of what happens for what we now call classical methods, we have the following case :

Assume that Q_h is taken to be the simplest, namely the space functions that are constant (P_o) over each triangle of partition $\mathcal{C}_h$. In this case (ii) holds with m=1. Now if for the velocity we take the space of continuous fields, whose restriction to each element is linear (P_1), we satisfy (i) with m=1, but since θ=1 we cannot satisfy (1) and therefore (iii) does not hold.

If on the other hand we keep the same Q_h and we take V_h to be the space of vector fields that are continuous and fully quadratic (P_2) over each triangle, we have (ii) and (iii) with m=1, whereas (i) holds with $m=2^3$. What happens in this case is that V_h was taken too big for a piecewise constant Q_h (indeed we have $\theta = 1/4$).

The above argument compels us to search for a V_h consisting of functions, whose restriction to each triangle is neither P_1 nor P_2, so as to make (i), (ii) and (iii) hold with the same m. We give below the proposed solution.

2. - ASYMMETRIC FINITE ELEMENTS.

Let K be a triangle with vertices S_1, S_2 and S_3 and let B be the edge opposite to vertex S_3, that we call the base of K. We define $P_{4/3}$ to be the space of quadratic functions defined over K, whose restriction to the edges other than B are linear functions. Clearly $\dim P_{4/3} = 4$ and the set of four degrees of freedom $\{a_i\}_{i=1}^4$, where a_i is the value of the function at S_i, S_4 being the mid-point of B, is $P_{4/3}$-unisolvent.

If λ_i denotes the area coordinate of K with respect to S_i, the four associated basis functions are given by :

$$\left.\begin{aligned} p_i &= \lambda_i - 2\lambda_1\lambda_2 \qquad i=1,2 \\ p_3 &= \lambda_3 \\ p_4 &= 4\lambda_1\lambda_2 \end{aligned}\right\} \qquad (2)$$

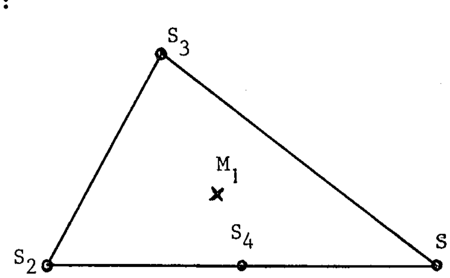


Figure 1

Due to the asymmetric structure of this element, some care is needed when constructing a triangulation, such that the space V_h of functions whose restriction to each triangle belongs to $P_{4/3}$ is conforming. We propose below two such constructions among many other possibilities :

Partition $\mathcal{C}_h^1$: First construct an arbitrary partition $\mathcal{S}_h$ of Ω into convex quadrilaterals J. Then subdivide each J into two triangles K_1 and K_2 by means of anyone of its two diagonals. These diagonals are the only bases of the triangles of $\mathcal{C}_h^1$.

Partition $\mathcal{C}_h^2$: First construct an arbitrary triangulation $\mathcal{S}_h$ of Ω. Then subdivide each triangle J of $\mathcal{S}_h$ into triangles K_1, K_2 and K_3 by taking the centroid M of J and joining it to the vertices of J. The bases of the so-generated triangulation $\mathcal{C}_h^2$ are only the edges of the J's.

Clear enough, the fields of V_h are such that their values at the nodes lying on the boundary of Ω_h vanish. Then for both partitions we have $\underset{\sim}{V}_h \subset \underset{\sim}{H}_0^1(\Omega_h)$ and (i) and (ii) hold with m=1.

Also, as one can easily check, for both $\mathcal{C}_h^1$ and $\mathcal{C}_h^2$ we have $\theta=1/2$. Furthermore, as we will see later on, (1) holds as does (iii) with m=1. For the moment, let us consider a three-dimensional version of the above $P_{4/3}$-element :

Let K be a tetrahedron with vertices S_1, S_2, S_3 and S_4, and B be the face opposite to S_4, called the base of K. Let $P_{7/6}$ be the five-dimensional space of functions defined over K, spanned by the four area coordinates of K, λ_1, $\lambda_2, \lambda_3, \lambda_4$, and $\phi = \lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_2\lambda_3$. The set of degrees of freedom $\{a_i\}_{i=1}^5$, where a_i is the functional value at S_i, S_5 being the centroid of K, is $P_{7/6}$-unisolvent and the associated bases functions are given by :

$$\left.\begin{aligned} p_i &= \lambda_i - \phi \\ p_4 &= \lambda_4 \\ p_5 &= 3\phi \end{aligned}\right\} \qquad (3)$$

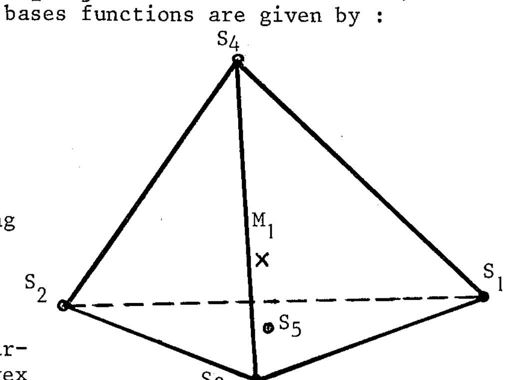


Figure 2

Among many ways of partitioning Ω , suitable for the definition of space $\underset{\sim}{V}_h$ associated with $P_{7/6}$, we consider the following :

Partition $\mathcal{C}_h^1$: First construct an arbitrary partition $\mathcal{S}_h$ of Ω into convex hexahedrons J having quadrilateral faces. Then each J is subdivided into five tetrahedrons in the classical way illustrated in Figure 3 below. Finally each tetrahedron ABCD lying in the interior of J is subdivided into four tetrahedrons by taking its centroid E and joining it to A,B,C and D. The bases of the partition $\mathcal{C}_h^1$ are nothing but the faces of tetrahedron ABCD.

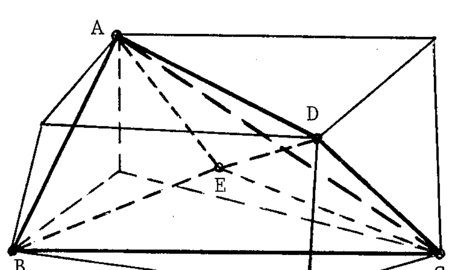


Figure 3

Partition $\mathcal{C}_h^2$: First construct an arbitrary partition $\mathcal{S}_h$ of Ω into tetrahedrons. Then each tetrahedron

J of $\mathcal{S}_h$ is subdivided into four tetrahedrons in the same way as tetrahedron ABCD of partition $\mathcal{C}_h^1$. The bases of the elements of $\mathcal{C}_h^2$ are nothing but the faces of the J's.

Now we define V_h to be the space of functions that are continuous at the vertices and at the centroids of the bases, and that vanish on the nodes lying on the boundary of the polyhedron Ω_h. Notice that $\underset{\sim}{V}_h$ is not a subspace of $H_0^1(\Omega_h)$. Indeed, a field of $\underset{\sim}{V}_h$ is necessarily continuous only over the bases of the partition, since their restriction to a face other than the base depends on the values at the centroid of the latter and at the vertex opposite this face. One can infer this fact by simply examining the basis functions given by (3).

Nevertheless we can prove the following Lemma, that is the key to the convergence analysis to be given later on :

<u>Lemma 1</u> : Let K be a tetrahedron and $\underset{\sim}{v} \in \underset{\sim}{P}_{7/6}$. Let also $\pi\underset{\sim}{v}$ be the linear interpolate of $\underset{\sim}{v}$ at the vertices of K. Then we have :

$$\int_K \operatorname{div} \underset{\sim}{\psi} \, d\underset{\sim}{x} = \frac{2}{3} \int_B \underset{\sim}{\psi} \cdot \underset{\sim}{n} \, ds \tag{4}$$

where $\underset{\sim}{n}$ is the unit outer normal vector to B and $\underset{\sim}{\psi} = \underset{\sim}{v} - \pi\underset{\sim}{v}$.

<u>Proof</u> : From (3) we can conclude that $\underset{\sim}{\psi} = \underset{\sim}{\beta}\psi$, where $\underset{\sim}{\beta} \in \mathbb{R}^3$. From the Stokes' formula we have :

$$\int_K \operatorname{div} \underset{\sim}{\psi} \, d\underset{\sim}{x} = \sum_{i=1}^{3} \int_{F_i} \underset{\sim}{\psi} \cdot \underset{\sim}{n}_i \, ds + \int_B \underset{\sim}{\psi} \cdot \underset{\sim}{n} \, ds,$$

where $\underset{\sim}{n}_i$ is the unit outer normal vector with respect to face F_i opposite to S_i. Exact integration of ψ over the faces of K yields :

$$\int_{F_i} \underset{\sim}{\psi} \cdot \underset{\sim}{n}_i \, dx = \frac{1}{12} \text{ area } (F_i) \underset{\sim}{\beta} \cdot \underset{\sim}{n}_i = \frac{1}{12} \int_{F_i} \underset{\sim}{\beta} \cdot \underset{\sim}{n}_i \, ds$$

and

$$\int_B \underset{\sim}{\psi} \cdot \underset{\sim}{n} \, ds = \frac{1}{4} \text{ area } (B) \underset{\sim}{\beta} \cdot \underset{\sim}{n} = \frac{1}{4} \int_B \underset{\sim}{\psi} \cdot \underset{\sim}{n} \, ds.$$

Finally, from the fact that $\int_K \operatorname{div} \underset{\sim}{\beta} \, d\underset{\sim}{x} = \int_B \underset{\sim}{\beta} \cdot \underset{\sim}{n} \, ds + \sum_{i=1}^{3} \int_{F_i} \underset{\sim}{\beta} \cdot \underset{\sim}{n}_i \, ds = 0$, we obtain (4).

3. - <u>CONVERGENCE RESULTS</u>.

Lemma 1 provides an essential tool for proving convergence results for the three-dimensional $P_{7/6}$-approximation of a Dirichlet problem, for the operator $-\Delta$. Moreover it states that the total flux over the faces of tetrahedron K of the non-conforming component $\underset{\sim}{\psi}$ of $\underset{\sim}{v} \in \underset{\sim}{P}_{7/6}$, corresponds to a fixed fraction (2/3) of the flux along the base B of K, where $\underset{\sim}{v}$ is continuous. This fact is also crucial for deriving inequality (1).

We give below the main lines for achieving this for partition $\mathcal{C}_h^2$ with β independent of h. The case of partition $\mathcal{C}_h^1$ will be discussed more briefly later on.

The key to the proof of the validity of (1) for n=2 and n=3 is the construction of a vector field $\underset{\sim}{v}_h \in \underset{\sim}{V}_h$ associated with an arbitrary $q_h \in Q_h$, such that

$$\sum_{K \in \mathcal{C}_h^2} \int_K q_h \operatorname{div} \underset{\sim}{v}_h \, d\underset{\sim}{x} \geq C_1 |q_h|_h^2 \tag{5}$$

and

$$\|\underset{\sim}{v}_h\|_h \leq C_2 |q_h|_h \,, \tag{6}$$

with C_1 and C_2 independent of h, which gives (1) with $\beta = C_1/C_2$.

This construction of $\underset{\sim}{v}_h$ is very technical and for the details we refer to [5]. Let us simply say here that $\underset{\sim}{v}_h$ is a linear combination of vector fields $\underset{\sim}{v}_h^i \in V_h$, i=0,1,...,n, respectively associated with orthogonal spaces $Q_h^o, Q_h^1, \ldots, Q_h^n$, whose direct sum is precisely Q_h. If

$$J = \bigcup_{i=1}^{n+1} K_i$$

is a macrosimplex of partition $\mathcal{S}_h$, Q is spanned by $\{\gamma_i^J\}_{J \in \mathcal{S}_h}$, where $\gamma_i^J(\underset{\sim}{x})=0$ if $\underset{\sim}{x} \notin J$ and the values of $\gamma_i^J(\underset{\sim}{x})$ for $\underset{\sim}{x} \in J$ are given in [5] for n=2 and 3.

Now, for the case of partition $\mathcal{C}_h^1$ a vector field satisfying both (5) and (6) does not exist in general. However, using an analogous technique, based on a splitting of Q_h, it is possible to prove that (1) holds with a constant β that in general depends on h. This dependence is expressed in exactly the same way as the one of the constant β associated with spaces $\underset{\sim}{V}_h$ and Q_h constructed in the following way :

V_h is the space of functions whose restriction to each quadrilateral (resp. hexahedron) J of $\mathcal{S}_h$ is isoparametric bilinear (i.e. of type Q_1)
Q_h is the space of functions whose restriction to each $J \in \mathcal{S}_h$ is constant (i.e. of type Q_o).

Notice that the above result is a little weaker than the one for partition $\mathcal{C}_h^2$, since it means that (1) holds for partition $\mathcal{C}_h^1$ only if it holds for the classical $Q_1 \times Q_o$ element described above. As a matter of fact, it is known [6] that this element provides very satisfactory results for the velocity, although sometimes it fails to generate a convergent sequence of pressures. Exactly the same may happen to our asymmetric elements if partitions of type $\mathcal{C}_h^1$ are used.

Now assuming that (1) holds with β independent of h, it is possible to use the following bound for the error of $[\underset{\sim}{u}_h, p_h]$ given in [7] :

$$\left.\begin{aligned} \|\underset{\sim}{u}-\underset{\sim}{u}_h\|_h + |p-p_h|_h \leq C \Big(& \inf_{\underset{\sim}{v}_h \in \underset{\sim}{V}_h} \|\underset{\sim}{u}-\underset{\sim}{v}_h\|_h + \inf_{q_h \in Q_h} |p-q_h|_h + \\ & + \sup_{\underset{\sim}{w}_h \in \underset{\sim}{V}_h} \frac{|E_h(\underset{\sim}{u},p,\underset{\sim}{w}_h)|}{\|\underset{\sim}{w}_h\|_h} \Big) \end{aligned}\right\} \tag{7}$$

where

$$E_h(\underset{\sim}{u},p,\underset{\sim}{w}_h) = \sum_{K \in \mathcal{C}_h^i} \int_{\partial K} \frac{\partial \underset{\sim}{u}}{\partial n} \cdot \underset{\sim}{w}_h \, ds + \int_{\partial K} p\underset{\sim}{n} \cdot \underset{\sim}{w}_h \, ds \tag{8}$$

∂K being the boundary of simplex K of $\mathcal{C}_h^i$, i=1 or 2, and $\frac{\partial g}{\partial \underset{\sim}{n}}$ and $\underset{\sim}{n}$ denoting respectively the outer normal derivative and unit vector with respect to ∂K.

The constant C that appears in (7) does not depend on h.

The estimate of the two first terms on the right hand side of (7) is classical, and corresponds to the separate errors of the approximation of $\underset{\sim}{H}_0^1(\Omega)$ by $\underset{\sim}{V}_h$ and of $L_0^2(\Omega)$ by Q_h. Since $P_1 \subset P_a \subset P_2$ with $a=1+1/C_{n+1}^2$, we have the following estimates of type (i) and (ii) with m=1, in which C denotes various constants independent of h :

(i) $$\inf_{\underset{\sim}{v}_h \in \underset{\sim}{V}_h} \|\underset{\sim}{u}-\underset{\sim}{v}_h\|_h \leq Ch|\underset{\sim}{u}|_2$$

(ii) $$\inf_{q_h \in Q_h} |p-q_h|_h \leq Ch|p|_1$$

The last term $|E_h(\underset{\sim}{u},p,\underset{\sim}{w}_h)|$ only needs to be estimated in the case n=3, for it vanishes identically when $\underset{\sim}{V}_h$ is conforming. The key to this estimate is again Lemma 1, together with a classical Lemma due to Ciarlet [8] for non-conforming methods. We get :

$$|E_h(\underset{\sim}{u},p,\underset{\sim}{w}_h)| \leq Ch[|\underset{\sim}{u}|_2 + |p|_1] \|\underset{\sim}{w}_h\|_h ,$$

which applied to (7) leads to :

$$\|\underset{\sim}{u}-\underset{\sim}{u}_h\|_h + |p-p_h|_h \leq Ch[|\underset{\sim}{u}|_2 + |p|_1]$$

that is to say, we have optimal results in (iii) with m=1.

4. - CONCLUDING REMARKS.

1°) The idea of using asymmetry for the definition of functions of V_h and Q_h over each element gives rise to other kind of finite elements. In particular two elements are proposed in [9] for the two and three-dimensional cases respectively. We describe below the two-dimensional one :

Given a triangle K we define $P_{5/3}$ to be the space of quadratic functions defined over K, spanned by $\lambda_1,\lambda_2,\lambda_3$, λ_1,λ_3 and λ_2,λ_3, and $P_{1/2}$ to be the space of linear functions spanned by 1 and λ_3. The set of functional values at points S_i, i=1,2,3,4,5 and M_i=1,2, indicated in Figure 4 below, are respectively $P_{5/3}$ and $P_{1/2}$-unisolvent.

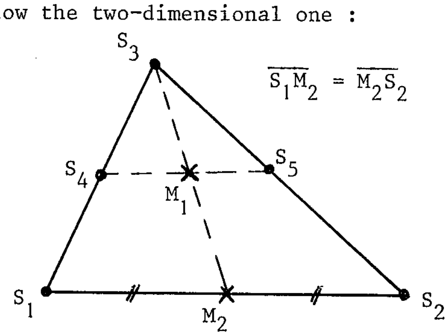
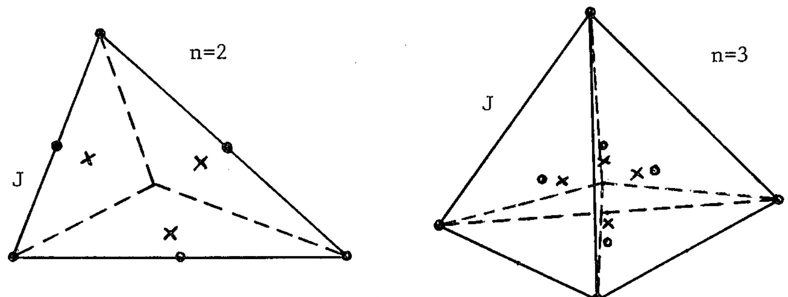


Figure 4

For this element we can construct a partition like $\mathcal{C}_h^1$, where the diagonals of the quadrilaterals are edges S_1S_2, among other possible constructions, including $\mathcal{C}_h^2$ itself. Then we define V_h to be the space of continuous functions, whose restriction to each triangle belongs to $P_{5/3}$, whereas Q_h is the space of discontinuous functions whose restriction to each triangle belongs to $P_{1/2}$. In this way we have $\theta=2/3$.

More details on this element can be found in [9], including numerical examples (these will be shown during the Symposium, together with those related to the two elements treated in this paper).

2°) It is possible to reduce significantly the amount of calculation with the $P_{4/3}$ and $P_{7/6}$ elements introduced above, by eliminating at the matrix level, terms corresponding to nodes lying in the interior of the macro-simplices associated with $\mathcal{C}_h^i$, i=1,2. In particular, in the case of partition $\mathcal{C}_h^2$, if this elimination is performed, the computational effort becomes equivalent to the one that applies to the following elements, illustrated in Figure 5 below for n=1 and n=3.



o degrees of freedom for the velocity
x degrees of freedom for the pressure

Figure 5

Notice that in the case n=2 this simplification means working with the P_2 element associated with macrotriangle J, for the velocity, and with a piecewise constant pressure over $K_1, K_2, \ldots, K_{n+1}$, instead of a constant function over the whole J, like in the case of the $P_2 \times P_0$ element described at the begining of the paper. For n=3, the same remark applies to a 8-node reduced quadratic non-conforming element proposed by the author for the velocity, instead of the P_2 element [7].

3°) The asymmetric structure of the elements considered in this paper has important advantages, besides those that we have considered so far. Among these we have the possibility of treating in a very efficient way the delicate case of the nonlinear incompressibility condition arising in finite elasticity, namely :

$$\det (I+\nabla v) = 1 \text{ a.e. in } \Omega,$$

where I is the identity n×n tensor and v is a displacement vector field. This condition, which becomes div $v = 0$ if $|\nabla v| \ll 1$, is very difficult to be approximated properly, when one uses classical mixed finite element methods for the linear case above. This is explained by the fact that, if the restriction to K of a field of V_h belongs to P_k, then the above determinant over K is a polynomial of degree n(k-1) and not a polynomial of P_{k-1} like div v_h. However, in the case of $P_{4/3}$ and $P_{7/6}$ we can prove [5] that this determinant is not of degree n but only of degree one, i.e., exactly the same as in the case of the linear incompressibility condition. This fact allows us to expect the same approximation properties in the nonlinear case as in the case considered in this paper.

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