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EXTRAPOLATED ITERATIVE METHODS FOR LINEAR SYSTEMS*

P. ALBRECHT† AND M. P. KLEIN‡

Abstract. In this note, we present results on extrapolated iterative methods, especially the extrapolated successive overrelaxation (ESOR). We show that they converge even if the original iteration diverges which increases considerably the scope of application of iterative schemes. The ESOR method is discussed under this aspect for consistently ordered systems and complex eigenvalues of the Jacobi iteration matrix. Comparison theorems are given to show that the ESOR is particularly useful if the SOR diverges or its optimum parameter ω_b cannot be determined; but even if ω_b is known the ESOR may be faster than the SOR method. Further insight into the structure of the method is obtained by relating it to Euler's integration method.

AMS (MOS) subject classifications. primary 65F10

1. Introduction. In a recent paper in this Journal [5], Missirlis and Evans discuss an extrapolated version of the successive overrelaxation (SOR) for the solution of linear algebraic systems. Based upon a different approach to these methods we present further results on this and on related iterative schemes.

Let the system be given by

$$Ax = b \tag{1.1}$$

where $b \in \mathbb{R}^n$ and where the real (n, n) matrix A is nonsingular and, without loss of generality, has all diagonal entries equal to 1.

Any splitting of A, $A = M - N$, M nonsingular, defines an iterative scheme

$$\begin{aligned} Mx^{(j+1)} &= Nx^{(j)} + b, \qquad j = 0, 1, \cdots, \\ x^{(j+1)} &= Sx^{(j)} + c, \qquad S = M^{-1}N, \quad c = M^{-1}b \end{aligned} \tag{1.2}$$

which converges, for arbitrary $x^{(0)} \in R^n$, to the solution u of (1.1) if the spectral radius $\rho(S)$ of the *iteration matrix* S satisfies $\rho(S) < 1$. If $\rho(S_1) < \rho(S_2) < 1$, the iteration with S_1 is called *asymptotically faster* than the iteration with S_2 [6].

With $A = I - L - U$, where I is the identity matrix, and L and U strictly lower and strictly upper triangular matrices, respectively, the iteration matrices of the three classical iterative methods are given by

$$\begin{aligned} &J := L + U, && \text{Jacobi,} \\ &R_1 := (I - L)^{-1}U, && \text{Gauss–Seidel,} \\ &R_\omega := (I - \omega L)^{-1}(\omega U + (1 - \omega)I), && \text{successive overrelaxation.} \end{aligned}$$

To any iterative method of the form (1.2) an *extrapolated method* can be associated by replacing, at each step j, $x^{(j+1)}$ by the extrapolated value $\beta x^{(j+1)} + (1 - \beta)x^{(j)}$:

$$x^{(j+1)} \leftarrow \beta x^{(j+1)} + (1 - \beta)x^{(j)}, \qquad \beta \neq 0, \quad \beta \in \mathbb{R}. \tag{1.3}$$

This requires only a small additional computational effort and corresponds to the iteration

$$x^{(j+1)} = S(\beta)x^{(j)} + \beta c, \qquad j = 0, 1, \cdots \tag{1.4}$$

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† Department of Informatic, Pontifícia Universidade Católica (PUC), R. Marquès de São Vicente, 209, 22453 Rio de Janeiro, Brazil.

‡ Department of Mathematics, Universidade Federal do Ceará, Caixa Postal, 60.000 Fortaleza, Brazil.

with the iteration matrix

$$S(\beta)=\beta S+(1-\beta)I. \tag{1.5}$$

DEFINITION 1.1. The iteration (1.4) is called *β-extrapolation* of the scheme (1.2).

We may generalize the above approach by replacing, at each step of the scheme (1.2), $x^{(j+1)}$ by $Bx^{(j+1)}+(I-B)x^{(j)}$:

$$x^{(j+1)}\leftarrow Bx^{(j+1)}+(I-B)x^{(j)},\qquad B\neq 0 \text{ a real } (n,n) \text{ matrix}.$$

Thus we obtain the iterative scheme

$$x^{(j+1)}=S(B)x^{(j)}+Bc,\qquad j=0,1,\cdots \tag{1.6}$$

with the iteration matrix $S(B)=BS+(I-B)$.

DEFINITION 1.2. The iteration (1.6) is called *B-extrapolation* of the scheme (1.2).

Remarks.

(1) The βI-extrapolation of an iterative method is a β-extrapolation as defined in (1.3).

(2) In (1.3), $x^{(j+1)}$ was replaced *simultaneously* by the extrapolated value; equally well we may replace it *successively* (componentwise) in step j. The *successive β-extrapolation* thus obtained is identical with a $\beta(I-\beta L)^{-1}$-extrapolation and represents a particularly simple B-extrapolation.

(3) Several B-extrapolations may be performed consecutively with different $B=B_i$, $i=1(1)r$, yielding a large variety of methods. In the literature the special case $B_i=\alpha_i I$ is also known as the *semi-iterative method* (e.g. [6], [7]); its basic idea is already discussed in [1].

Examples.

(1) $S=J$, $B=\omega I$ yields $S(\omega)=I-\omega A$. The ω-extrapolation of Jacobi's method thus generates the *Jacobi overrelaxation*[1] (JOR) which has been introduced by Young [7].

(2) $S=J$, $B=\omega(I-\omega L)^{-1}$ yields $S(B)=R_\omega$. Hence, successive ω-extrapolation of Jacobi's method generates the SOR.

(3) $S=R_1$, $B=\beta I$ yields $S(\beta)=(I-L)^{-1}(\beta U+(\beta-1)L+(1-\beta)I)$. This is the *extrapolated Gauss–Seidel method* (EGS).

This paper will be primarily concerned with the case $S=R_\omega$, $B=\beta I$ which generates the *extrapolated successive overrelaxation* (ESOR) method and also with the JOR scheme. These methods are interesting for two reasons:

(1) Their convergence can be analyzed from the eigenvalues of J which are more easily estimated than those of any other iteration matrix (see § 4).

(2) The JOR is a good general purpose method, and ESOR is a good general purpose method in the class of consistently ordered systems, as explained next.

It is not possible to indicate a "best" general purpose iteration method for the solution of linear systems (in the sense of converging faster than any other method). However, we try to find methods that are best in the sense of being at least *convergent* for a maximum set of problems (if necessary, from a restricted class). In order to specify this we define:

DEFINITION 1.3. Let μ_k $(k=1(1)n)$ be the eigenvalues of the Jacobi matrix $J=I-A$. A region G in the complex plane is called region of convergence of an

[1] This name is well established by Young's book; therefore we use it instead of "extrapolated Jacobi method," which would be more consistent with our presentation.

iterative method M_1, if, by adjustment of its parameters, M_1 can be made convergent for any problem with $\mu_k \in G$, $k = 1(1)n$.

This definition enables us to compare two methods by their regions of convergence (which does not say anything about their efficiency for a particular problem). If $G_1 \supset G_2$, we may consider method M_1 the better scheme for general purposes or within a restricted class of problems (e.g. the class of consistently ordered systems).

Young [7] considered the JOR method for irreducible A with weak diagonal dominance and for positive definite A. A convergence theorem for arbitrary A will be given as Corollary 2.2. Dispersed in the literature, the ESOR method can be found under various names [1], [3], [6]. In recent years it has been studied systematically by Hadjidimos [2], Niethammer [8] and Missirlis–Evans [5] and, earlier, in the thesis of Klein [4]. Generalized forms of the latter's results on consistently ordered matrices are presented in § 4.

In [2] and [5] the convergence of the ESOR has been shown for the following cases:

(1) A is irreducible with weak diagonal dominance;

(2) A is positive definite;

(3) A is an L-matrix and $\rho(J) < 1$, $\omega \in (0; 1]$;

(4) A is consistently ordered with real eigenvalues and $\rho(J) < 1$. In all four cases, the classical SOR method converges also (though frequently slower).

As a complement, it will be shown here that the ESOR scheme converges also in cases where the SOR method (as well as Jacobi's method) *diverges* (§ 2); this represents a major advantage of the method. The choice of the parameters ω and β will also be discussed.

Additional insight into the structure of extrapolated methods is gained by relating them to Euler's method for the solution of linear systems of ordinary differential equations (§ 6).

2. Comparison theorem for extrapolated methods. In this section, the convergence of a given method and its β-extrapolation is compared, and the result is applied to the JOR and ESOR methods.

THEOREM 2.1. *Let S be the iteration matrix of the iterative scheme* (1.2) *and $r = \rho(S)$ its spectral radius.*

(a) *Let* (1.2) *converge* ($r < 1$). *Then its β-extrapolation* (1.4) *converges asymptotically faster for some $\beta = \beta_0$, if all eigenvalues λ of S satisfy, exclusively, either* (1) $\operatorname{Re}\lambda < r^2$ *or* (2) $\operatorname{Re}\lambda > r^2$; *in case* (1) *we have $\beta_0 < 1$ and in case* (2) $\beta_0 > 1$.

(b) *Let* (1.2) *diverge* ($r \geqq 1$). *Then its β-extrapolation* (1.4) *converges for some $\beta = \beta_0$ with $|\beta_0| < 1$ if and only if all eigenvalues λ of S satisfy, exclusively, either* (3) $\operatorname{Re}\lambda < 1$ *or* (4) $\operatorname{Re}\lambda > 1$; *in case* (3) *we have $\beta_0 > 0$ and in case* (4) $\beta_0 < 0$.

Proof. Due to (1.5), an eigenvalue λ of S is mapped to an eigenvalue τ of $S(\beta)$ by $\tau = \beta\lambda + (1-\beta)$. This map $\mathbb{C} \to \mathbb{C}$ has the fixed point $\lambda = 1$, and straight lines through $\lambda = 1$ are mapped onto themselves. Consequently, for $r < 1$ the arc $\lambda(t) = re^{it}$, $\operatorname{Re}\lambda(t) \leqq a < r^2$, is mapped into the disc $|\tau| < r$ for some $\beta = \beta_0 < 1$ as well as the arc with $\operatorname{Re}\lambda(t) \geqq a > r^2$ for some $\beta_0 > 1$ (see Fig. 1). For $\beta = 1$ we have $\rho(S(\beta)) = r$. This proves part (a) of the theorem.

For $r > 1$ any set of values λ_k with $\operatorname{Re}\lambda_k < 1$ or $\operatorname{Re}\lambda_k > 1$, exclusively, can be mapped into the unit circle for sufficiently small $|\beta| = |\beta_0|$ and $\beta_0 > 0$ or $\beta_0 < 0$, respectively (see Fig. 2). This proves part (b).

COROLLARY 2.2. *For properly chosen ω, the* JOR *method converges if and only if all eigenvalues of A lie, exclusively, either in the complex left half-plane $\mathbb{C}^-$ or in the complex right half-plane $\mathbb{C}^+$.*

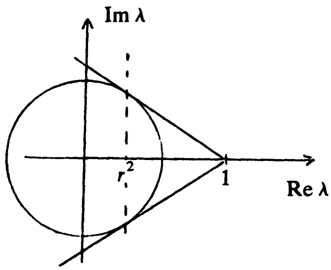

FIG. 1

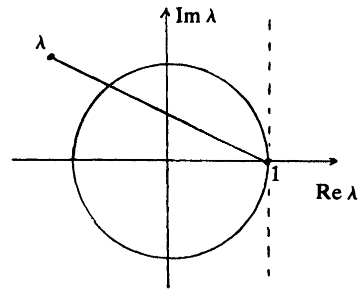

FIG. 2

Remarks.

(1) The JOR converges especially if A is positive or negative definite.

(2) Corollary 2.2 is useful since algorithms exist to decide whether or not all eigenvalues of a matrix belong to $\mathbb{C}^-$.

(3) The half-planes $G_1=\{\mu:\operatorname{Re}\mu<1\}$ and $G_2=\{\mu:\operatorname{Re}\mu>1\}$ are regions of convergence of JOR in the sense of Definition 1.3.

β-extrapolation of the SOR method yields

$$x^{(j+1)}=(\beta R_\omega+(1-\beta)I)x^{(j)}+\beta c,\qquad j=0,1,\cdots$$

with

$$R_\omega=(I-\omega L)^{-1}(\omega U+(1-\omega)I)\quad\text{and}\quad c=\omega(I-\omega L)^{-1}b.$$

For convenience we take $\beta=\gamma/\omega$ and thus obtain the ESOR scheme in the following form

$$x^{(j+1)}=R_\omega(\gamma)x^{(j)}+d,\qquad j=0,1,\cdots, \tag{2.1}$$

$$R_\omega(\gamma)=(I-\omega L)^{-1}[(\gamma-\omega)L+\gamma U+(1-\gamma)I];\qquad d=\gamma(I-\omega L)^{-1}b. \tag{2.2}$$

For $\gamma=\omega$ we have $R_\omega(\omega)=R_\omega$.

Computationally, in the jth step, an ordinary SOR step is performed followed by the replacement

$$x^{(j+1)}\leftarrow\frac{\gamma}{\omega}x^{(j+1)}+\left(1-\frac{\gamma}{\omega}\right)x^{(j)}.$$

For this method we obtain the following corollary to Theorem 2.1.

COROLLARY 2.3. *For properly chosen $\omega\in(0;2)$ and $\gamma=\gamma(\omega)$ with $|\gamma|\leqq\omega$, the* ESOR *method converges if and only if all eigenvalues λ of R_ω satisfy, exclusively, either* $\operatorname{Re}\lambda<1$ *or* $\operatorname{Re}\lambda>1$.

In general, R_ω is not known explicitly, so much the less its eigenvalues; thus it is not easy, at times, to verify the assumptions of Corollary 2.3. Consequently we find ourselves in the same situation as with the SOR method, not being able to decide whether the ESOR converges, except in special cases (see § 1).

However, similar to Young's theory of the SOR, the convergence of the ESOR scheme can be deduced from the eigenvalues of $J=I-A$ in the case of consistently ordered matrices. For this and other purposes we need more results on the eigenvalues λ of R_ω in the case of consistent ordering than those given by the classical theory.

3. On the eigenvalues of R_ω for consistently ordered systems.

DEFINITION 3.1 [6]. The matrix $J=L+U$ (as well as the matrix A) is called *consistently ordered* if the eigenvalues of the matrix $(\alpha L+\alpha^{-1}U)$ are independent of α, $\alpha\neq 0$. We then also say that the linear system $Ax=b$ is consistently ordered. The following theorem of Young is classical [7, p. 142].

THEOREM 3.2. *Let J be consistently ordered. If μ is any eigenvalue of J and if λ satisfies*

$$(\lambda+\omega-1)^2=\omega^2\mu^2\lambda \tag{3.1}$$

then λ is an eigenvalue of R_ω. If λ is an eigenvalue of R_ω then there exists an eigenvalue μ of J such that (3.1) *holds.*

Relation (3.1) can be written as

$$\mu=\omega^{-1}\left(z+\frac{\omega-1}{z}\right),\qquad z=\sqrt{\lambda} \tag{3.2}$$

showing that if $\lambda_1\neq 0$ is an eigenvalue of R_ω so is $\lambda_2=(\omega-1)^2/\lambda_1$. Equation (3.2) establishes a one-to-one mapping between the two-sheeted Riemann surface over the μ-plane (slit between $\mu=c:=2\omega^{-1}\sqrt{\omega-1}$ and $\mu=-c$) and the two-sheeted Riemann surface over the λ-plane (slit along the negative real axis). The first part of (3.2) is known as "Joukowski mapping".

Let C be the circle $\{\lambda:|\lambda|=|\omega-1|\}$. If $\omega\geqq 1$, the segment of the real axis $\{\mu:\mu=t,\ -|c|\leqq t\leqq|c|\}$ is mapped onto C; if $\omega<1$, the segment of the imaginary axis $\{\mu:\mu=it,\ -|c|\leqq t\leqq|c|\}$ is mapped onto C. One sheet of the Riemann μ-surface is mapped onto the inside of C, and the other sheet onto the outside of C.

For $\omega\in(0;2)$, representing the straight line $\operatorname{Re}\lambda=1$ by $\lambda_1(t)=(1+i\operatorname{sh}2t)$, $-\infty<t<\infty$, one can see that the half-plane $\operatorname{Re}\lambda<1$ is mapped onto the open region G_1 in the complex μ-plane (see Fig. 3) that is bounded by $\mu_1(t)$ and $-\mu_1(t)$ with

$$\begin{gathered}\mu(t)=\mu_1(t)=\alpha_1(t)+i\beta_1(t),\\ \alpha_1(t)=\omega^{-1}\left(1+\frac{\omega-1}{\operatorname{ch}2t}\right)\operatorname{ch}t;\ \beta_1(t)=\omega^{-1}\left(1-\frac{\omega-1}{\operatorname{ch}2t}\right)\operatorname{sh}t;\end{gathered} \tag{3.3}$$

for $\omega=1$, (3.3) reduces to the hyperbola $\alpha^2-\beta^2=1$.

For $\omega\in(1;\infty)$, representing the imaginary axis $\operatorname{Re}\lambda=0$ by $\lambda_2(t)=\pm i(\omega-1)\,e^{2t}$, $-\infty<t<\infty$, it can be seen that the half-plane $\operatorname{Re}\lambda<0$ is mapped onto the open region $G_2\subset G_1$ bounded by the hyperbola $\mu(t)=\mu_2(t)=\alpha_2(t)+\beta_2(t)$

$$\alpha_2^2-\beta_2^2=\frac{|c|^2}{2},\qquad c=2\omega^{-1}\sqrt{\omega-1}.$$

For $\omega=1$, $\operatorname{Re}\lambda<0$ is mapped onto the open region $G_2^*\subset G_2$,

$$G_2^*=\{\mu=\alpha+i\beta:|\alpha|<|\beta|\}.$$

For $\omega \in (0; 1)$, $\operatorname{Re} \lambda = 0$ is mapped onto the hyperbola

$$\beta^2 - \alpha^2 = \frac{|c|^2}{2} \tag{3.4}$$

and G_2 is the part of G_1 which is bounded by (3.4) and contains $\mu = \infty$ (see Fig. 4).

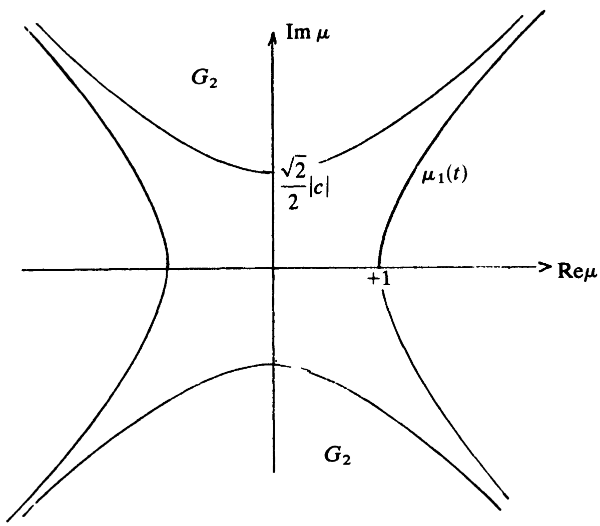

FIG. 3

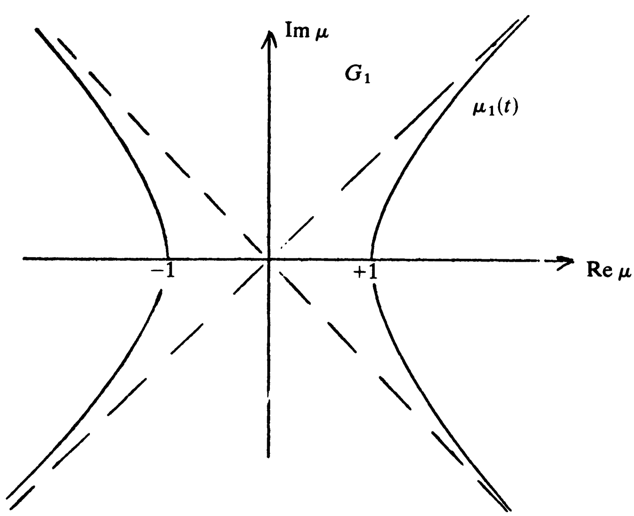

FIG. 4. $\omega < 1$.

4. The convergence of the ESOR for consistently ordered matrices. The convergence of the ESOR method for consistently ordered Jacobi matrices J with *real* eigenvalues and $\rho(J)<1$ has already been treated in [2]; Niethammer [8] also considers eigenvalues $i\mu$ (μ real). In this section, convergence theorems for consistently ordered J with arbitrary (complex) eigenvalues will be given, including the case $\rho(J)\geqq 1$.

THEOREM 4.1 [9], [4]. *Let J be consistently ordered. If μ is any eigenvalue of J and if τ satisfies*

$$(\tau+\gamma-1)^2=\gamma\mu^2(\omega\tau+\gamma-\omega) \tag{4.1}$$

then τ is an eigenvalue of $R_\omega(\gamma)$. If τ is an eigenvalue of $R_\omega(\gamma)$ then there exists an eigenvalue μ of J such that (4.1) *holds.*

Proof. As any eigenvalue τ of $R_\omega(\gamma)$ is related to an eigenvalue λ of R_ω by $\tau=\gamma\lambda/\omega+(1-\gamma/\omega)$ and vice versa, (4.1) follows from (3.1).

Young proved in [7, p. 194] that the SOR method converges for consistently ordered J and properly chosen $\omega\in(0;2)$ if and only if all eigenvalues μ of J satisfy $|\mathrm{Re}\,\mu|<1$. The following theorem shows that the ESOR method converges under a less restrictive assumption:

THEOREM 4.2. *Let J be consistently ordered and G_1 as defined in* § 3. *The* ESOR *method converges for some $\omega\in(0;2)$ and $\gamma=\gamma(\omega)$, $0<\gamma\leqq\omega$, if and only if all eigenvalues μ of J satisfy $\mu\in G_1$.*

Proof. $\mu\in G_1$ implies $\mathrm{Re}\,\lambda<1$ (see § 3); hence it is sufficient for convergence due to Corollary 2.3. $\mu\in\mu_1(t)$, $t\in\mathbb{R}$, implies $\lambda=1$; hence the ESOR diverges (Corollary 2.3). If $\mu\notin G_1$, $\mu\neq\mu_1(t)$, an eigenvalue $\lambda_1=re^{i\phi}$, $r>1$, exists with $\mathrm{Re}\,\lambda_1>1$ (§ 3). As $\lambda_2=(1/r)(\omega-1)^2e^{-i\phi}$ is also an eigenvalue of R_ω and $\mathrm{Re}\,\lambda_2<1$ for $\omega\in(0;2)$, the ESOR diverges due to Corollary 2.3. Thus, $\mu\in G_1$ is necessary for convergence.

The following theorem gives sufficient (but not necessary) conditions for *accelerated* convergence of the ESOR scheme.

THEOREM 4.3. *If, in addition to the assumption of Theorem* 4.2, *the* SOR *method converges for some ω_0 then, for some $\gamma=\gamma(\omega_0)$, the* ESOR *scheme converges asymptotically faster if all eigenvalues μ of J satisfy $\mu\in G_2$.*

Proof. As $\mu\in G_2$ implies $\mathrm{Re}\,\lambda<0\leqq\rho^2(R_\omega)$ (see § 3), the theorem follows from Theorem 2.1 (1).

For $\omega_0\in[1;2)$, $|\alpha|<|\beta|$ implies $\mu=\alpha+i\beta\in G_2$ (see § 3); hence we obtain the following simplification of Theorem 4.3.

COROLLARY 4.4. *Let J be consistently ordered, and let the* SOR *method converge for $\omega_0\in[1;2)$. Then there is a $\gamma=\gamma(\omega_0)$ such that the* ESOR *scheme converges faster if all eigenvalues $\mu=\alpha+i\beta$ of J satisfy $|\alpha|<|\beta|$.*

Remarks.

(1) Comparison of the region of convergence G_1 with that of the JOR method (Corollary 2.2) shows that the JOR scheme may have advantages over the ESOR scheme even for consistently ordered systems.

(2) If $|\alpha|<|\beta|$, the ESOR method also converges for $\omega\in[2,\infty)$, $\gamma=\gamma(\omega)$; this choice, however, does not seem to offer any practical advantages.

(3) Theorem 4.3 and its corollary imply that the ESOR scheme may converge faster than the SOR with *optimum* ω. An example is given in § 5.

The region of convergence G_1 of the ESOR method is largest for $\omega=1$; in this case, its boundaries are given by the hyperbola $\alpha^2-\beta^2=1$. For general problems, we therefore suggest the choice $\omega=1$, i.e. the EGS scheme. If the eigenvalues μ_k of J (or good estimates) are known, the eigenvalues λ_k of R_ω can be obtained (or estimated) from (3.2). For given ω (e.g. $\omega = 1$) the best parameter γ is such that

$$\max_{1\leq k\leq n} \left| \frac{\gamma}{\omega}\lambda_k + \left(1-\frac{\gamma}{\omega}\right)\right| = \text{Min !} \tag{4.2}$$

As is the case for the determination of an optimum relaxation factor ω_b for the SOR method, there is, up to now, no theory that permits to find the optimum *pair* of parameters (ω_0, γ_0) in the general case of complex eigenvalues μ. One can see from examples that $\omega_0 \neq \omega_b$, in general.

If all eigenvalues of J are *real* the ESOR method converges only if the SOR converges that is, if $\rho(J)<1$. However, in certain cases, it is possible to obtain *accelerated* convergence as specified in the following theorem [4]. Niethammer [8] obtains more sophisticated results under the additional assumption of symmetric, positive definite J.

THEOREM 4.5. *Let J be consistently ordered with real eigenvalues μ_k such that $0<m:=|\mu_1|\leq|\mu_2|\cdots\leq|\mu_n|:=M<1$. Let*

$$(1-m^2)<\sqrt{(1-M^2)} \tag{4.3}$$

and

$$\omega_b = \frac{2}{1+\sqrt{1-M^2}} \leq \omega < \frac{2-\omega}{2-m^2}. \tag{4.4}$$

Then

(a) $$\rho(R_\omega(\gamma))<\rho(R_\omega) \quad \textit{for } \omega<\gamma<\frac{2-\omega}{1-m^2},$$

(b) $$\rho(R_\omega(\gamma_0))<\rho(R_\omega(\gamma)), \qquad \gamma\neq\gamma_0, \quad \gamma_0=\frac{1}{2}\frac{2-m^2\omega}{1-m^2}.$$

Proof. Formula (4.3) implies $\omega_b<2(2-m^2)^{-1}$. For $\omega_0\leq\omega$ the eigenvalues λ_k of R_ω are conjugate complex pairs (except for $\omega=\omega_b$ when $\lambda=(\omega_b-1)$ is a double eigenvalue), and lie on the circle $|\lambda|=r$ with $r=(\omega-1)$ (see [7, p. 204]). Let λ_1 and λ_2 be the pair associated to μ_1, then we have from (3.1)

$$\operatorname{Re}\lambda_k \geq \operatorname{Re}\lambda_1 = \tfrac{1}{2}\omega^2 m^2-(\omega-1)>r^2 \quad \text{if } \omega<2(2-m^2)^{-1}.$$

Hence, condition (2) of Theorem 2.1(a) is satisfied, which proves (a). Some calculation shows that $f(\gamma_0)<f(\gamma)$ for $\gamma\neq\gamma_0$ with

$$f(\gamma):=\max_k\left|\frac{\gamma}{\omega}\lambda_k+\left(1-\frac{\gamma}{\omega}\right)\right| = \left|\frac{\gamma}{\omega}\lambda_1+\left(1-\frac{\gamma}{\omega}\right)\right|$$

which proves (b).

5. Example. Let

$$A=\begin{bmatrix} 1 & -1.49 & 0 & 0 \\ -1.49 & 1 & -5.41 & 0 \\ 0 & 1.49 & 1 & -1.12 \\ 0 & 0 & -3.43 & 1 \end{bmatrix}.$$

A is tridiagonal and hence consistently ordered. The eigenvalues of $J=I-A$ are $\mu_{1,2}=0.98+1.40i$; $\mu_{3,4}=-0.98+1.40i$. As $|\operatorname{Re}\mu_k|<1$, the SOR method converges for properly chosen $\omega \in (0; 2)$ (see § 4). The best choice is $\omega_b \doteq 0.15261$ which yields $\rho(R_{\omega_b}) \doteq 0.99779$ (see [7, Table 4.1, p. 199]).

R_{ω_b} has the eigenvalues

$$\lambda_{1,2} \doteq 0.7004 \mp 0.1654i \qquad \lambda_{3,4} \doteq 0.9711 \pm 0.2293i.$$

As $\operatorname{Re} \lambda_k < \rho^2(R_{\omega_b})$, the ESOR scheme converges faster than the SOR for properly chosen γ (see Theorem 2.1). From (4.2) we obtain the best choice $\gamma_1 \doteq 0.5414 \cdot \omega_b \doteq 0.0826$ which yields

$$\rho(R_{\omega_b}(\gamma_1)) \doteq 0.9921.$$

If we choose $\omega = 1$ we have $\lambda_{1,2} \doteq -0.9996 \pm 2.7740i$, $\lambda_{3,4} = 0$. From (4.2) we obtain $\gamma_2 \doteq 0.1899$ which yields

$$\rho(R_1(\gamma_2)) \doteq 0.8101 < \rho(R_{\omega_b}(\gamma_1)).$$

This ESOR method is about 100 times faster than the SOR with optimum $\omega = \omega_b$! This also confirms our previous observation that the best parameter ω for the SOR, in general, is not the best for the ESOR method.

6. JOR and ESOR as integration methods. In this section we complete our study by showing that the above methods can be interpreted as *integration* methods for certain systems of linear differential equations.

Consider the system

$$\dot{x} = b - Ax, \qquad x(t_0) = x_0. \tag{6.1}$$

We assume (6.1) to be stable, i.e., the eigenvalues τ of A to have positive real parts. Applying Euler's integration method to (6.1) yields

$$x_{j+1} = x_j + h(b - Ax_j) = (hJ + (1-h)I)x_j + hb. \tag{6.2}$$

This is the JOR *scheme with* $\omega = h$. Hence, the JOR method may be used for the solution of the linear algebraic system $Ax = b$ or the system of differential equations (6.1), the only difference being the choice of ω. In the first case, ω is chosen such that $\rho(J(\omega)) = \text{Min}!$, $J(\omega) := \omega J + (1-\omega)I$, and in the latter according to the required precision.

The ESOR method with $\gamma = \omega h$ has the form

$$x_{j+1} = x_j + h\omega(b - Ax_j) + \omega L(x_{j+1} - x_j), \tag{6.3}$$

which may be interpreted as a modification of (6.2). However, it is seen from the representation

$$x_{j+1} = x_j + h\omega(I - \omega L)^{-1}(b - Ax_j)$$

that (6.3) is identical with the application of Euler's method to the problem

$$\begin{aligned} \dot{x} &= \omega(I - \omega L)^{-1}(b - Ax), \qquad & x(t_0) &= x_0 \\ &= (R_\omega - I)x + c, & c &:= \omega(I - \omega L)^{-1}b. \end{aligned} \tag{6.4}$$

Hence, for small $\gamma = \omega h$, the ESOR method yields approximations x_j of the solution $x(t)$ of (6.4) at $t = t_j$. If all eigenvalues of $(R_\omega - I)$ have negative real parts, (6.4) has a steady state solution u (with $Au = b$) that can be obtained with (6.3) using the *optimum stepsize* $h = \gamma/\omega$ defined by $\rho(R_\omega(\gamma)) = \text{Min}!$ The same is true for the differential equation

$$\begin{aligned} \dot{x} &= \omega(I-\omega L)^{-1}(Ax-b), \qquad x(t_0)=x_0 \\ &=(I-R_\omega)x+c \end{aligned} \tag{6.5}$$

if all eigenvalues of $(R_\omega - I)$ have positive real parts. (6.3) then corresponds to the ESOR method with $\gamma = -\omega h$.

All eigenvalues of $(R_\omega - I)$ have, exclusively, negative or positive real parts if $\mu \in G_1$ (see § 3). This shows the relation between the region of convergence of the ESOR scheme and the stability region of the Euler method.

The interesting point is the fact that Euler's method makes sense for large stepsizes, yielding then the steady state solution. The interpretation of iterative methods as integration methods for linear differential equations is not new, however, and can be found in Varga [6, § 8.4]. It seems promising to investigate this aspect further for nonlinear problems or multistep methods.

Conclusion. Taking into account the small additional computational effort involved, it seems advisable to add to any iterative scheme its extrapolated form as an option. Not only does this provide the possibility of accelerating the original iteration, but convergence is achieved for a considerably larger class of problems. The question of further enlarging the area of convergence has not been considered here and needs more extensive investigation.

In the important special case of consistently ordered systems, the ESOR method is particularly useful if the SOR diverges or its optimum parameter ω_b cannot be determined; but even if ω_b is known the ESOR may be faster than the SOR method.

For given parameter ω it is simple to determine the associated optimum $\gamma_0 = \gamma(\omega)$ if good estimates of the eigenvalues μ of J are available. So far, there is no theory that permits one to find an optimum pair of parameters (ω_0, γ_0) in the general case of complex eigenvalues μ. We conjecture that such a pair is not unique. In many cases, good results are obtained with $\omega = 1$ and the associated optimum γ_0.

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