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EXISTENCE AND STABILITY OF ASYMMETRIC FINITE-ELEMENT APPROXIMATIONS IN NONLINEAR INCOMPRESSIBLE ANALYSIS*

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Abstract. This paper deals with the analysis of existence and numerical stability of a special kind of simplicial finite-element approximation in nonlinear incompressible elasticity. We show that the asymmetric structure of the elements with respect to the centroid of the simplex renders them particularly stable in large strain states, and we prove that, under suitable conditions on the elements, there exists a solution to the corresponding discrete problem. Numerical examples illustrate the efficiency of the method.

Key words. asymmetric, existence, finite elements, incompressible, Mooney-Rivlin material, nonlinear elasticity, quasilinear, simplex, stability

AMS(MOS) subject classifications. 65N30, 73C50, 73K25

1. Introduction. In this work we discuss two mixed finite elements of asymmetric type introduced, respectively, in [13] and [14], for solving finite incompressible elasticity problems.

Let us first define our notation. Let Ω be a bounded set of $\mathbb{R}^n$. Then for every open subset D of Ω, we shall denote by $\|\cdot\|_{m,r,D}$ and $|\cdot|_{m,r,D}$ the usual norm and seminorm, respectively, of the Sobolev space $W^{m,r}(D)$ (see, e.g., [1]), $m, r \in \mathbb{R}$, $m \geqq 0$ and $1 \leqq r \leqq \infty$, with $W^{0,r}(D) \equiv L^r(D)$. Similarly in the case where $r=2$ we denote by $(\cdot,\cdot)_{m,D}$ the usual inner product of $W_0^{m,2}(D) \equiv H_0^m(D)$ and by $|\cdot|_{m,D} = |\cdot|_{m,2,D}$ the corresponding norm, while we will represent the norm of $W^{m,2}(\Omega) \equiv H^m(\Omega)$ by $\|\cdot\|_{m,D}$ instead of $\|\cdot\|_{m,2,D}$. In all cases we shall drop the subscript D whenever D is Ω itself.

For every space of functions V defined on D, $\mathbf{V}$ will represent the space of vector fields whose n components belong to V. In the case where V is $W^{m,r}(D)$ or $W_0^{m,r}(D)$, we define the norm, seminorm, and inner product (if $r=2$) for $\mathbf{V}$, by introducing obvious modifications of the scalar case, and keeping the same notation.

We shall denote by $x \cdot y$ the Euclidian inner product of two vectors x and y of $\mathbb{R}^l$ and by $|\cdot|$ the corresponding norm. l will be either equal to n in the case of vectors of $\mathbb{R}^n$, or equal to n^2 in the case of tensors of $\mathbb{R}^{n\times n}$.

Finally, for every function or vector field y defined over a certain set D, we shall denote by y/S its restriction to a subset S, $S \subset D$.

Now our problem can be described as follows. We are given an elastic body represented by a bounded domain $\Omega \subset \mathbb{R}^n$, $n = 2, 3$, with a boundary Γ. Keeping fixed a part Γ_0 of Γ with meas $(\Gamma_0) \neq 0$, we consider a loading of Ω consisting of body forces having a density $\mathbf{f}$ per unit of measure of Ω, and of surface forces acting on a set $\Gamma^* \subset \Gamma$ (such that meas $(\bar{\Gamma}_0 \cap \bar{\Gamma}^*) = 0$ and $\bar{\Gamma}^* \cup \Gamma_0 = \Gamma$), having a density $\mathbf{g}$ per unit of measure of Γ^*. Although it is physically possible to have $\Gamma^* = \emptyset$, we will not consider this case in this paper.

The effect of $\mathbf{f}$ and $\mathbf{g}$ is to deform Ω into an equilibrium configuration defined by a displacement vector field that we will denote by $\mathbf{u}$. In this way, the new position of every point $\mathbf{x}$ of Ω is given by $\mathbf{x}+\mathbf{u}(\mathbf{x})$. The fact that every element of Ω is measure

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invariant in its deformed state can be expressed mathematically by

(1.1) $$J[\mathbf{x}+\mathbf{u}(\mathbf{x})]=1 \quad \text{for almost every } \mathbf{x}\in\Omega$$

where $J[\mathbf{v}(\mathbf{x})]$ denotes the Jacobian of a vector field $\mathbf{v}$ at point $\mathbf{x}$.

Condition (1.1) is called the incompressibility condition in finite elasticity and we shall often rewrite it as follows:

(1.2) $$\det(\mathbf{I}+\nabla\mathbf{u})=1 \quad \text{a.e. in } \Omega$$

where $\mathbf{I}$ is the $n\times n$ identity tensor and ∇ represents the gradient operator.

Remark. Condition (1.1) is obviously nonlinear but in the case of small strains, that is to say when

$$\max_{\mathbf{x}\in\Omega}|\nabla\mathbf{u}(\mathbf{x})|\ll 1,$$

we can neglect products of derivatives of $\mathbf{u}$ of order higher than 1. Condition (1.1) becomes the well-known linear incompressibility condition arising in infinitesimal elasticity or in fluid dynamics, namely

$$\operatorname{div}\mathbf{u}(\mathbf{x})=0 \quad \text{for a.e. } \mathbf{x}\in\Omega.$$

Although there is a rather large range of incompressible materials, in this work we focus our study on the case of Mooney-Rivlin materials, because they are particularly representative of the class of materials for which (1.1) holds. We note by the way that among Mooney-Rivlin materials rubber is a typical case.

For a Mooney-Rivlin material the elastic energy for a certain admissible displacement vector field $\mathbf{v}$ is given by [13]:

$(1.3)_2$ $$W(\mathbf{v})=\frac{C_1}{2}\int_\Omega[|\mathbf{I}+\nabla\mathbf{v}|^2-2]\,d\mathbf{x}-\int_\Omega\mathbf{f}\cdot\mathbf{v}\,d\mathbf{x}-\int_{\Gamma^*}\mathbf{g}\cdot\mathbf{v}\,ds \quad \text{for } n=2,$$

$(1.3)_3$ $$W(\mathbf{v})=\frac{C_1}{2}\int_\Omega[|\mathbf{I}+\nabla\mathbf{v}|^2-3]\,d\mathbf{x}+\frac{C_2}{2}\int_\Omega[|\operatorname{adj}(\mathbf{I}+\nabla\mathbf{v})|-3]\,d\mathbf{x} -\int_\Omega\mathbf{f}\cdot\mathbf{v}\,d\mathbf{x}-\int_{\Gamma^*}\mathbf{g}\cdot\mathbf{v}\,ds \quad \text{for } n=3$$

where adj A denotes the transpose of the matrix of cofactors of an $n\times n$ matrix A, and C_1 and C_2 are positive physical constants.

Taking into account (1.2) and the fact that W must be finite, it is natural to choose the following set of admissible displacement vector fields:

$$X=\{\mathbf{v}/\mathbf{v}\in\mathbf{W}^{1,r}(\Omega),\ \mathbf{v}/\Gamma_0=\mathbf{0},\ \det[\mathbf{I}+\nabla\mathbf{v}(\mathbf{x})]=1 \text{ a.e. in } \Omega\}$$

with $r\geqq 2(n-1)$, and we shall assume that $\mathbf{f}\in L^2(\Omega)$ and $g\in H^{1/2}(\Gamma^*)$.

The problem we want to solve can now be stated as follows:

(P) Find $\mathbf{u}\in X$ such that $W(\mathbf{u})\leqq W(\mathbf{v})$ for all $\mathbf{v}\in X$.

It is interesting to note that X is a nonconvex set and that it is a subset of the vector space $\mathbf{V}$ defined by

$$\mathbf{V}=\{\mathbf{v}/\mathbf{v}\in\mathbf{W}^{1,r}(\Omega),\ \mathbf{v}/\Gamma_0=\mathbf{0}\},$$

which can be normed by the seminorm $|\cdot|_{1,r}$ (Ω being connected [11]).

Instead of the minimization problem **(P)** itself, here we will work with the following mixed formulation obtained by dualization of (1.2) with the help of a multiplier p, and by differentiation of $W(\mathbf{u})$ along $\mathbf{v}$ over $\mathbf{V}$.

(**P**′) Find $(\mathbf{u}, p) \in \mathbf{V} \times Q$ such that $a(\mathbf{u}, \mathbf{v}) + b'(\mathbf{u}, \mathbf{v}, p) = L(\mathbf{v})$, for all $\mathbf{v} \in \mathbf{V}$ $b(\mathbf{u}, q) = 0$, for all $q \in Q$

where $Q = L^t(\Omega)$, with t such that $n/r + 1/t \leqq 1$, and

$$a(\mathbf{u}, \mathbf{v}) = C_1 \int_\Omega \nabla\mathbf{u} \cdot \nabla\mathbf{v}\, d\mathbf{x} + C_2 \int_\Omega \operatorname{adj}(\mathbf{I} + \nabla\mathbf{u}) \cdot [\operatorname{adj}(\mathbf{I} + \nabla\mathbf{u} + \nabla\mathbf{v}) - \operatorname{adj}\nabla\mathbf{v}] \quad \text{with } C_2 = 0 \quad \text{if } n = 2, \tag{1.4}$$

$$b'(\mathbf{u}, \mathbf{v}, q) = \int_\Omega q[\operatorname{adj}(\mathbf{I} + \nabla\mathbf{u})^{\mathsf{T}} \cdot \nabla\mathbf{v}]\, d\mathbf{x}, \tag{1.5}$$

$$b(\mathbf{v}, q) = \int_\Omega q[\det(\mathbf{I} + \nabla\mathbf{v}) - 1]\, d\mathbf{x}, \tag{1.6}$$

$$L(\mathbf{v}) = \int_\Omega \mathbf{f} \cdot \mathbf{v}\, d\mathbf{x} + \int_{\Gamma^*} \mathbf{g} \cdot \mathbf{v}\, ds - C_1 \int_\Omega \operatorname{div} \mathbf{v}\, d\mathbf{x}. \tag{1.7}$$

According to results of Le Tallec [10], under reasonable assumptions, there exists a hydrostatic pressure p, with $p \in L^t(\Omega)$, associated with every solution $\mathbf{u}$ to problem (**P**), and in this case $(\mathbf{u}, p)$ is a solution to (**P**′).

At this stage we would like to point out that, in practice, it seems unwise to use formulation (**P**′) for numerical computations with associated mixed finite elements. Indeed, there are other mixed formulations of (**P**) much more suitable for such a purpose and in this respect we refer to [6], for instance. However, for the sake of clarity, we prefer to consider (**P**′) in this work, as it appears to be the most natural formulation of all.

Bearing in mind that our mixed finite-element methods apply to other mixed formulations of (**P**) as well, from now on we shall consider approximating problem (**P**′). For this purpose we will define two finite-dimensional spaces V_h and Q_h aimed at approximating **V** and Q, associated with simplicial finite elements, which have an asymmetric structure with respect to the centroid of the simplex. In the three-dimensional case the element can be viewed as a certain generalization of the two-dimensional one first introduced in [14], whereas both have been discussed in [16] for linear problems arising in mechanics of incompressible media.

The structure of this paper is as follows. In § 2 in a general way we define a discrete analogue ($\mathbf{P}'_h$) of (**P**′), based on finite-element approximations. In § 3 we briefly recall the asymmetric elements and we describe the corresponding problem ($\mathbf{P}'_h$), in connection with two kinds of partitions of Ω. In § 4 we study some basic properties of both elements that justify a priori their adequacy for the numerical solution of problem (**P**). In §§ 5 and 6 we consider in detail the well-posedness of ($\mathbf{P}'_h$) in the case of one of the types of partition considered in § 3, for the affine and isoparametric cases, respectively. Finally, in § 7 we conclude with some short remarks concerning the other type of partition; the main assumptions made throughout the paper; and a numerical example.

2. The finite-element approximate problem. Henceforth, except where otherwise specified, we consider Ω to be a domain of $\mathbb{R}^n$, $n = 2, 3$, having a polyhedral boundary Γ. For the case $n = 3$ we also assume that $\bar{\Gamma}^* \cap \bar{\Gamma}_0$ is a set of spatial polygonal lines.

We are given a family $(\tau_h)_h$ of partitions of Ω into n-simplices, satisfying the classical assembling rules for the finite-element method. Some additional compatibility conditions for $(\tau_h)_h$ related to our asymmetric elements will be specified in § 3. We also assume that Γ^* and Γ_0 can be viewed as the union of faces of elements of τ_h and that $(\tau_h)_h$ is regular in the following sense. Denoting by h_K the diameter of the circumscribed sphere and by ρ_K the diameter of the inscribed sphere of element K, $K \in \tau_h$ and setting

$$h = \max_{K \in \tau_h} h_K \quad \text{and} \quad \rho = \min_{K \in \tau_h} \rho_K,$$

there exists a strictly positive constant c such that $\rho h^{-1} > c$, for all h.

With each partition τ_h we associate the finite-dimensional spaces Q_h and $\mathbf{V}_h$, approximations of Q and $\mathbf{V}$, respectively. We assume that $Q_h \subset Q$; however, in general, a similar inclusion will not hold for V_h. Let $|\cdot|$ be the norm of Q_h induced by $L^2(\Omega)$, and let $\|\cdot\|^2_{m,\Omega,h}$ (respectively, $|\ |^2_{m,\Omega,h}$) be obtained by summation over the elements $K \in \tau_h$ of the squares of the $\|\cdot\|_{m,K}$-norms (respectively, $|\cdot|_{m,K}$-seminorm). In particular, we will use the H^1_0-discrete norm for V_h defined by

$$\|v_h\|_h^2 = \sum_{K \in \tau_h} |v_h|^2_{1,K}. \tag{2.1}$$

Now in the discrete analogue of (**P**′), we weaken the requirement that the approximation $\mathbf{u}_h \in \mathbf{V}_h$ of the solution $\mathbf{u}$ to problem (**P**) satisfy exactly (1.1), in the following way. The incompressibility condition is to be satisfied only at those points of Ω to which we attach the degrees of freedom of Q_h. This is equivalent to requiring that $\mathbf{u}_h$ belong to an approximation X_h of X defined by

$$X_h = \{\mathbf{v}_h / \mathbf{v}_h \in \mathbf{V}_h,\ b_h(\mathbf{v}_h, q_h) = 0\ \forall q_h \in Q_h\}$$

where b_h is a suitable approximation of b given by (1.6).

A natural way of defining b_h is to set

$$b_h(\mathbf{v}_h, q_h) = \sum_{K \in \tau_h} b_K(\mathbf{v}_h, q_h) \tag{2.2}$$

where b_K corresponds to an approximation of the integral of (1.6), restricted to element K, whose quadrature points are those associated with the degrees of freedom of Q_h. We consider two possibilities of performing this numerical quadrature, according to the way we define the elements of τ_h.

To be more specific, if the domain Ω is a polygon or a polyhedron, we define b_h as follows.

Case i. Every $K \in \tau_h$ is the reciprocal image of the usual reference simplex $\hat{K}$ (see Fig. 2.1) by an affine transformation $A_K: K \to \hat{K}$.

In this case we define the approximation of

$$\int_K q_h[\det(\mathbf{I} + \nabla \mathbf{v}_h) - 1]\, d\mathbf{x}$$

to be

$$b_K(\mathbf{v}_h, q_h) = \sum_{j=1}^{m} \omega_j q_h(\mathbf{x}_j^K)[\det(\mathbf{I} + \nabla \mathbf{v}_h) - 1]/\mathbf{x}_j^K \text{ meas}(K) \tag{2.3}$$

where $\{\mathbf{x}_j^K\}_{j=1}^m$ is the set of points used to define q_h/K, and the ω_j's are the weights of the numerical quadrature formula.

On the other hand, if Ω has a curved boundary, it may be interesting to partition Ω into curved elements defined in the classical way, namely Case ii.[1]

[1] Now both Γ_0 and Γ^* are approximated by the union of curved faces or edges of elements of τ_h.

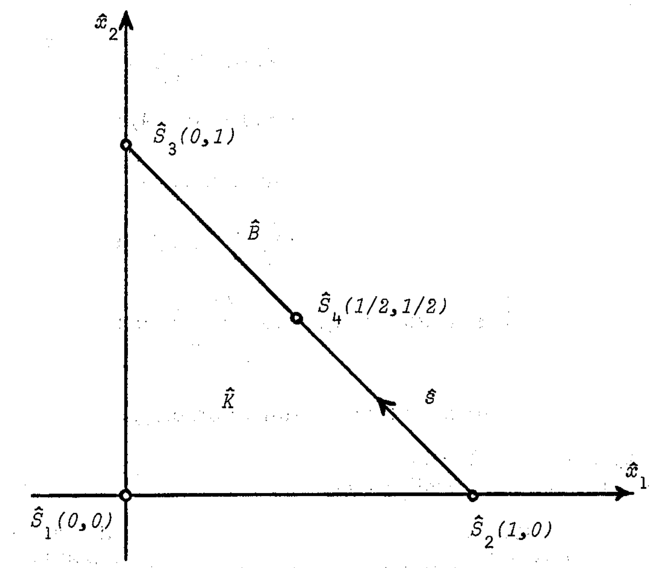

FIG. 2.1. *The reference element $\hat{K}$ for $n=2$.*

Case ii. Every $K \in \tau_h$ is the reciprocal image of $\hat{K}$ by a bijective isoparameteric transformation $\mathscr{A}_K$: $K \to \hat{K}$. This means that $\mathscr{A}_K^{-1}(\hat{\mathbf{x}}) = [a_1^K(\hat{\mathbf{x}}), \cdots, a_n^K(\hat{\mathbf{x}})]$, where $a_i^K \in \hat{P}$, $1 \leqq i \leqq n$, $\hat{P}$ being a space of shape functions defined over $\hat{K}$, such that $\hat{v}_h = v_{h/K} \circ \mathscr{A}_K \in \hat{P}$, for all $v_h \in V_h$ and for all $K \in \tau_h$.

In this case the approximation of $\int_K q_h[\det(\mathbf{I}+\nabla \mathbf{v}_h) - 1]\, d\mathbf{x}$ is given by

$$(2.4) \qquad b_K(\mathbf{v}_h, q_h) = \sum_{j=1}^{m} \omega_j \hat{q}_h(\hat{\mathbf{x}}_j)[\det \hat{\nabla}(\hat{\mathbf{v}}_h + \mathscr{A}_K^{-1}) - \det \hat{\nabla} \mathscr{A}_K^{-1}]/\hat{\mathbf{x}}_j \text{ meas } (\hat{K})$$

where $\{\hat{\mathbf{x}}_j\}_{j=1}^m$ is the set of points of $\hat{K}$ whose reciprocal images through $\mathscr{A}_K$ are the points of K to which we attach the degrees of freedom of Q_h, and $\hat{\nabla}$ denotes the gradient operator for variable $\hat{\mathbf{x}} = \mathscr{A}_K(\mathbf{x})$.

Now, taking into consideration (2.2), we can verify that in both Case i and Case ii we have

$$(2.5) \qquad \forall v_h \in X_h \quad \det(\mathbf{I}+\nabla \mathbf{v}_h)/\mathbf{x}_j^K = 1 \quad \forall j,\ 1 \leqq j \leqq m \text{ and } \forall K \in \tau_h.$$

Indeed, in Case i this is trivial provided meas (K) is nonzero for all $K \in \tau_h$. On the other hand, from the well-known formula of calculus [3] we have

$$(2.6) \qquad J(\mathbf{v}) = \hat{J}(\hat{\mathbf{v}})J(A) \quad \text{where } \hat{\mathbf{x}} = A(\mathbf{x}) \text{ and } \hat{\mathbf{v}} \circ A(\mathbf{x}) = \mathbf{v}(\mathbf{x}).$$

Thus we see that (2.5) also holds for Case ii by setting $\mathbf{v}(\mathbf{x}) = \mathbf{v}_h(\mathbf{x}) + \mathbf{x}$ and $A = \mathscr{A}_K$, and taking into account the identity $J^{-1}(A) = \hat{J}(A^{-1})$.

Remark. If the $\hat{\mathbf{x}}_j$'s are the points of a quadrature formula that integrates exactly functions of form $\hat{J}(\mathbf{v}_h)$ over $\hat{K}$, for all $\mathbf{v}_h \in \hat{P}$, then as in [15], we can draw the following conclusion.

If (2.5) holds and $\sum_{j=1}^m \omega_j = 1$, we have meas $(\tilde{K})$ = meas (K), for all $K \in \tau_h$, $\tilde{K}$ being the deformed state of K induced by $\mathbf{v}_h$.

Furthermore we now set

$$(2.7) \qquad b_h'(\mathbf{u}_h, \mathbf{v}_h, q_h) = \frac{\partial b_h}{\partial \nabla u_h} \cdot \nabla \mathbf{v}_h,$$

and

$$a_h(\mathbf{u}_h, \mathbf{v}_h) = \sum_{K \in \tau_h} a_K(\mathbf{u}_h, \mathbf{v}_h), \qquad L_h(\mathbf{v}_h) = \sum_{K \in \tau_h} L_K(\mathbf{v}_h),$$

a_K and L_K being defined in the same way as a and L in (1.4) and (1.7), respectively, by replacing Ω with K, and Γ^* with $\Gamma^* \cap K$.

We also introduce a "discrete energy" W_h in an analogous way, namely

$$W_h(\mathbf{v}_h) = \sum_{K \in \tau_h} \left[\frac{C_1}{2} \int_K |\mathbf{I} + \nabla \mathbf{v}_h|^2 \, d\mathbf{x} + \frac{C_2}{2} \int_K |\text{adj}\,(\mathbf{I} + \nabla v_h)|^2 \, d\mathbf{x} \right]$$
$$- \int_\Omega \mathbf{f} \cdot \mathbf{v}_h \, d\mathbf{x} - \int_{\Gamma^*} \mathbf{g} \cdot \mathbf{v}_h \, ds - \frac{C_1 + C_2}{2} n \text{ meas}\,(\Omega)$$

with $C_2 = 0$ if $n = 2$.

We now define the discrete mixed formulation of problem (**P**) to be the following:

($\mathbf{P}'_h$) Find $(\mathbf{u}_h, p_h) \in V_h \times Q_h$ such that $a_h(\mathbf{u}_h, \mathbf{v}_h) + b'_h(\mathbf{u}_h, \mathbf{v}_h, q_h) = L_h(\mathbf{v}_h)$, for all $\mathbf{v}_h \in \mathbf{V}_h b_h(\mathbf{u}_h, q_h) = 0$, for all $q_h \in Q_h$.

According to [9], the existence of a solution to problem ($\mathbf{P}'_h$) is directly dependent on the validity of a nonlinear discrete inf-sup type compatibility condition between the spaces $\mathbf{V}_h$ and Q_h. However, this condition must now be expressed in terms of the vector field $\mathbf{u}_h$ itself. Since $\mathbf{u}_h$ is supposed to minimize the energy W_h in some sense, the following result [10, Thm. 4.1] is of crucial importance.

The following problem has a solution:

($\mathbf{P}_h$) Find $\mathbf{u}_h \in X_h$ to minimize $W_h(\mathbf{u}_h)$ over X_h.

Now, let $\mathbf{u}_h$ be a local minimum of W_h. Also let $\|\cdot\|_h$ be the norm of $\mathbf{V}_h$ and $|\cdot|$ be the norm of Q_h induced respectively by $\mathbf{V}$ and $L^2(\Omega)$. The nonlinear compatibility condition can be stated as follows.

There exists $\beta_h > 0$ such that

$$\sup_{\mathbf{v}_h \in \mathbf{V}_h} \frac{b'_h(\mathbf{u}_h, \mathbf{v}_h, q_h)}{\|\mathbf{v}_h\|_h} \geqq \beta_h |q_h| \quad \forall q_h \in Q_h. \tag{2.8}$$

According to Theorem 4.3 of [10], if condition (2.8) is fulfilled, there exists a unique pressure $p_h \in Q_h$ such that $(\mathbf{u}_h, p_h)$ is a solution to ($\mathbf{P}'_h$).

3. The asymmetric finite elements. We first define Q_h to be the space of functions q_h that are constant over each element of τ_h, and we clearly have $Q_h \subset Q$. For convenience we consider the degrees of freedom of Q_h to be function values at the centroid G of the elements. V_h in turn consists of functions whose restriction to each simplex $K \in \tau_h$ belongs to a space P_a defined as follows.

Let S_i denote the vertices of a simplex $K \subset \tau_h$, $i = 1, 2, \cdots, n+1$. We first assign to K a privileged face, say the face opposite to vertex S_{n+1}, that will be called the base B^K of K, and let F_i^K be the face opposite to vertex S_i, $i = 1, 2, \cdots, n$. The F_i^K's will be called the lateral faces of K. Let λ_i denote the barycentric coordinate of K associated with vertex S_i, $i = 1, 2, \cdots, n+1$ and S_{n+2} denote the centroid of B^K.

Now we define P_a to be the $(n+2)$-dimensional space spanned by the functions λ_i, $i = 1, 2, \cdots, n+1$ and ϕ, where

$$\phi = \sum_{\substack{j,k=1 \\ j<k}}^{n} \lambda_j \lambda_k. \tag{3.1}$$

We can easily verify that the set of degrees of freedom $\{a_i\}_{i=1}^{n+2}$, where a_i is the value of the function at point S_i, is P_a-unisolvent and that the associated basis functions are given by

$$p_i = \lambda_i - \frac{2}{n-1}\phi, \quad i = 1, 2, \cdots, n,$$
$$p_{n+1} = \lambda_{n+1}, \tag{3.2}$$
$$p_{n+2} = \frac{2n}{n-1}\phi.$$

In Fig. 3.1 we illustrate these asymmetric finite elements where ○ represents degrees of freedom for $\mathbf{V}_h$ and × represents those for Q_h.

Note that the following inclusions hold: $P_1 \subset P_a \subset P_2$, where P_k denotes the space of polynomials of degree less than or equal to k defined over K.

As has been remarked in [14] and [15], the elements associated with P_a must be used in connection with partitions of Ω into n-simplices constructed in a special way, called compatible partitions. Let us briefly recall two kinds of such partions given in [16] for both elements.

Partition τ_h^1. In the two-dimensional case we first construct a partition of Ω into arbitrary convex quadrilaterals (as in the case of the bilinear Q_1 element). Next, every quadrilateral is subdivided into two triangles by an arbitrarily chosen diagonal. Those diagonals will be the only bases of the elements of the resulting triangulation.

In the three-dimensional case we first construct a partition of Ω into arbitrary convex hexahedrons having quadrilateral faces. Now we refer to Fig. 3.2(b), where we show a classical subdivision of a hexahedron into five tetrahedrons. We next take an arbitrary point in the interior of each central tetrahedron ABCD, say point E, and

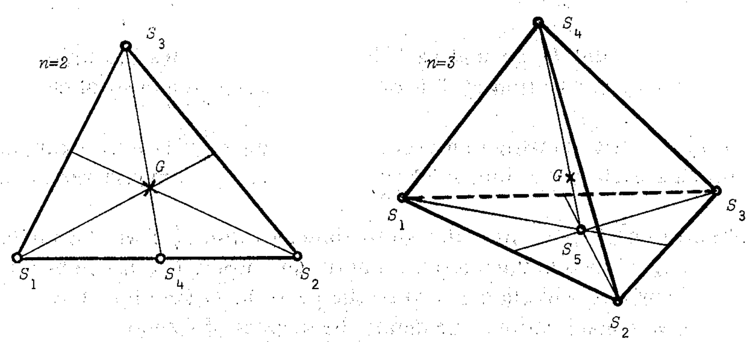

FIG. 3.1. *The asymmetric quasilinear elements.*

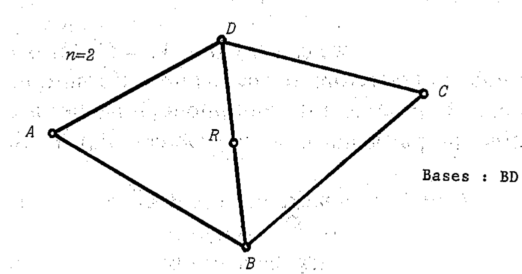

FIG. 3.2(a)

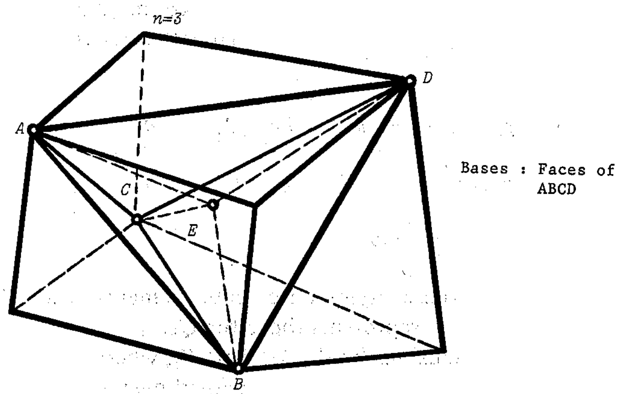

FIG. 3.2(b). *An illustration of compatible partition* τ_h^1.

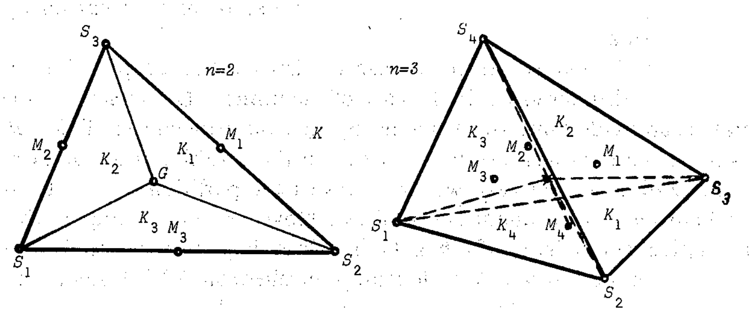

FIG. 3.3. *An illustration of compatible partition* τ_h^2.

we join it to A, B, C, and D, so that each hexahedron becomes the union of eight tetrahedrons. These form partition τ_h^1 if its bases are precisely the faces of tetrahedrons ABCD.

Partition τ_h^2. We first construct an arbitrary partition τ_h of Ω into n-simplices K. Then we subdivide each $K \in \tau_h$ into $n+1$ simplices having a common vertex situated in $\overset{\circ}{K}$.

This subpartition of τ_h becomes the compatible partition τ_h^2 if we choose its bases to be the faces of τ_h. Note that the interior point of the simplex $K \in \tau_h$ can be arbitrary, although in this work we will choose it to be the centroid G (see Fig. 3.3).

With the above considerations, we define the degrees of freedom of V_h to be the function values at the vertices and the centroid of the bases of a compatible partition τ_h of Ω, except the values at those nodes lying on $\bar{\Gamma}_0$, where every function of V_h vanishes necessarily.

With the above definition of V_h we can say that $V_h \subset C^0(\bar{\Omega})$ if $n=2$, but if $n=3$ this inclusion does not hold and therefore we have a nonconforming element. Nevertheless, for $n=3$, a function of V_h is necessarily continuous along the bases of the partition.

Let us now examine the particular case of problem $(\mathbf{P}'_h)$ for the spaces $\mathbf{V}_h$ and Q_h defined above.

We have $m=1$, $\omega_1=1$, and the quadrature point $\mathbf{x}_1^K$ being the centroid of K in Case i, and the image of the centroid of $\hat{K}$ through transformation $\mathcal{A}_K^{-1}$ in Case ii.

It is then possible to verify, using arguments to be developed in § 4, that in both cases i and ii the corresponding numerical quadrature formula calculates exactly the integral of $\det(\mathbf{I}+\nabla\mathbf{u}_j)$ over K; that is to say

$$b_K(\mathbf{u}_h, q_h) = \int_K q_h[\det(\mathbf{I}+\nabla\mathbf{u}_h)-1]\,d\mathbf{x}$$

and

$$\frac{\partial b_K(\mathbf{u}_h, q_h)}{\partial \nabla \mathbf{u}_h}\cdot \nabla \mathbf{v}_h = \int_K q_h[\operatorname{adj}(\mathbf{I}+\nabla\mathbf{u}_h)]^{\mathrm{T}}\cdot\nabla\mathbf{v}_h\,d\mathbf{x}.$$

Remark. In the conforming case ($n=2$), at least when $\Omega=\Omega_h\equiv\bigcup_{K\in\tau_h}K$, we have $b_h=b$ and $b'_h\equiv b'$, for V_h and Q_h defined in this section. However, if the union of the elements K over τ_h is different from Ω, we should redefine problem ($\mathbf{P}'_h$) by replacing a_h and L_h by approximate functionals a_h^* and L_h^* that take into account integration over Ω_h rather than over Ω.

4. Stability properties of the asymmetric elements. In this section we intend to justify our proposal of the elements of asymmetric type for the numerical solution of problem ($\mathbf{P}'$) from the point of view of the simulation of (1.1).

First, let us briefly recall some a priori arguments already considered in [14] and [15].

If a vector field of an approximation space $\mathbf{V}_h$ of $\mathbf{V}$ is such that each component restricted to an element K of τ_h is a polynominal in P_k, its Jacobian is a polynomial in $P_{n(k-1)}$ over K. This implies that we must satisfy constraint (1.1) at a large number of points of K to satisfy the incompressibility condition everywhere, which is the only way of avoiding with absolute certainty that elements "turn inside out" in the deformed state. Note that this question becomes particularly critical in the three-dimensional case, where numerical instabilities are frequently observed whenever the number of these point constraints per element is small, specially under compression loads. Indeed this is precisely the situation where the elements have a tendency to turn inside out.

However, the total number of constraints to be satisfied in the discrete problem associated with ($\mathbf{P}'$), i.e., $\dim Q_h$, must not exceed the total number of displacement degrees of freedom, i.e., $\dim \mathbf{V}_h$; otherwise condition (2.8) fails to hold (see, e.g., [9]). This fact is usually expressed numerically by requiring that the following *asymptotic ratio*:

$$\theta = \lim_{h\to 0}\frac{\dim Q_h}{\dim \mathbf{V}_h}$$

be strictly less than one (actually in practice θ should not be too close to 1).

Just to illustrate some restrictions this fact may impose in practice, we consider an example of $\mathbf{V}_h$ constructed from standard Langrangian elements of degree k in an n-simplex.

If we want to satisfy the incompressibility condition pointwise, the following table indicates the values of θ for different values of k (see [14] for further details).

k	1	2	3	4
$n=2$	1	3/2	5/3	7/4
$n=3$	2	5	56/9	55/8

Remark. Precisely due to the necessity of satisfying the incompressibility condition as exactly as possible for the problem under study, it is not appropriate to choose a space Q_h satisfying continuity requirements at points situated on the interface of the elements. This fact prevents us from reducing the dimension of Q_h significantly as in the case of linear problems solved with the so-called Taylor–Hood elements [7].

Let us also add that $\mathbf{V}_h$ should preferably be conforming. Indeed, even if condition (1.1) is properly satisfied elementwise, the nonconformity may lead to a meaningless representation of the incompressibility phenomenon at the global level, unless we can prove that the resulting interpenetrations of neighboring elements in the deformed state cancel each other, or are negligible.

Summing up, we can say that, except in a very few cases, we cannot expect to approximate problem ($\mathbf{P}'$) by using standard spaces $\mathbf{V}_h$ and Q_h, such as those that work well for viscous flow problems or for linear incompressible elasticity. Therefore, a solution that seems reasonable is to construct $\mathbf{V}_h$ by means of spaces of special polynomials of degree k, for which the Jacobian is of maximal degree significantly less than $n(k-1)$.

Indeed, in so doing, we can expect to come as close as possible to the ideal situation where the incompressibility condition is satisfied pointwise, while still making it possible to satisfy (2.8). As we show hereafter this is precisely the case of P_a.

THEOREM 4.1. *If* $\mathbf{v}=(v_1,\cdots,v_n)$ *defined over* K *is such that* $v_i\in P_a$ *for all* i *then* $J[\mathbf{x}+\mathbf{v}(\mathbf{x})]$ *is a polynomial in* P_1.

Proof. According to (3.2), each component v_i can be written as

$$v_i=\sum_{j=1}^{n+1}\alpha_j^i\lambda_j+\beta^i\phi$$

where the α_j^i's and the β^i's are scalars and ϕ is the quadratic function given in (3.1). We have

$$J[\mathbf{x}+\mathbf{v}(\mathbf{x})]=\begin{vmatrix} c_{11}+\beta^1\dfrac{\partial\phi}{\partial x_1} & c_{12}+\beta^1\dfrac{\partial\phi}{\partial x_2} & \cdots & c_{1n}+\beta^1\dfrac{\partial\phi}{\partial x_n}\\ c_{21}+\beta^2\dfrac{\partial\phi}{\partial x_1} & c_{22}+\beta^2\dfrac{\partial\phi}{\partial x_2} & \cdots & c_{2n}+\beta^2\dfrac{\partial\phi}{\partial x_n}\\ \vdots & \vdots & & \vdots\\ c_{n1}+\beta^n\dfrac{\partial\phi}{\partial x_1} & c_{n2}+\beta^n\dfrac{\partial\phi}{\partial x_2} & \cdots & c_{nn}+\beta^n\dfrac{\partial\phi}{\partial x_n}\end{vmatrix}\tag{4.1}$$

where the constant c_{ij} is the x_j-derivative of the linear part of $x_i+v_i(\mathbf{x})$.

Now we expand the above determinant into a sum of 2^n determinants whose jth column is either $(c_{1j},c_{2j},\cdots,c_{nj})^{\mathsf T}$ or $\partial\phi/\partial x_j\,(\beta^1,\beta^2,\cdots,\beta^n)^{\mathsf T}$. As we can easily see, the only determinants of this expansion that do not vanish identically are those having at most one column with linear entries $(\partial\phi/\partial x_j)\boldsymbol{\beta}^{\mathsf T}$, and the result follows. □

An immediate consequence of Theorem 4.1 is the fact that, to satisfy (1.1) at the centroid G of K, it suffices to have incompressible elements in the following weak sense. The measure of K in the deformed state induced by $\mathbf{u}\in\mathbf{P}_a$ is invariant.

Indeed, if we denote by $\tilde{\mathbf{A}}$ the deformed state induced by $\mathbf{u}$ of any subset A of K, $K\in\tau_h$, according to a well-known numerical quadrature formula, we have

$$\text{meas}\,(\tilde{K})=\int_K J[x+\mathbf{u}(\mathbf{x})]\,d\mathbf{x}=J[G+\mathbf{u}(G)]\text{ meas}\,(K)=\text{meas}\,(K).$$

This shows that the space Q_h defined in § 3 is a proper choice for these asymmetric elements.

Remarks. (i) The relation above also holds in the isoparametric case if G is replaced by the image of $\hat{G}$ under $\mathscr{A}_K$.

(ii) Using the same arguments as in [15], we can conclude that for both $n=2$ and $n=3$, we have $\theta=\frac{1}{2}$, when partition of type τ_h^2 is used. In the case of τ_h^1, the same value of θ applies for $n=2$, while $\theta=\frac{4}{9}$ for $n=3$.

(iii) In the two-dimensional case, the standard $Q_1\times P_0$ element has the same properties as the quasilinear asymmetric element, as far as the degree of the Jacobian and θ are concerned. It can actually give satisfactory numerical results as shown by many examples in [10]. However, in the three-dimensional case, the property of Theorem 4.1 does not hold for the Q_1 element.

Once having proved that incompressibility can be properly treated for each element, we would like to assert that the same is true for Ω.

More precisely, letting A denote any subset of Ω, setting $A_K=A\cap K$, $K\in\tau_h$ and defining

$$\tilde{A}=\bigcup_{K\in\tau_h}\tilde{A}_K\quad\text{with } \tilde{A}_K=\mathbf{u}(A_K)$$

where $\mathbf{u}/K\in\mathbf{P}_a$, we would like to verify that

$$\text{meas}\,(\tilde{K})=\text{meas}\,(K)\quad\forall K\in\tau_h\Rightarrow\text{meas}\,(\tilde{\Omega})=\text{meas}\,(\Omega),$$

or that

$$\text{meas}\,(\tilde{\Omega})=\sum_{K\in\tau_h}\text{meas}\,(\tilde{K}).$$

In fact, denoting by $\hat{\Omega}$ the deformed state of Ω induced by $\mathbf{u}$ to be defined below, we will prove that

$$\text{(4.2)}\qquad \text{meas}\,(\hat{\Omega})=\sum_{K\in\tau_h}\text{meas}\,(\tilde{K})\quad\text{with meas}\,(\tilde{K})=\text{meas}\,(K)\quad\forall K\in\tau_h.$$

In the two-dimensional case we will simply set $\hat{\Omega}=\tilde{\Omega}$ if $J[\mathbf{x}+\mathbf{u}(\mathbf{x})]\geqq 0$, for all $\mathbf{x}\in\Omega$. Indeed in this case (4.2) is trivially satisfied since $\mathbf{V}_h$ is conforming, and therefore the elements in the deformed state do not interpenetrate each other. However, even under the above assumption, this is not necessarily true in the case of a nonconforming $\mathbf{V}_h$. That is why for $n=3$ we will set $\acute{\Omega}=\bigcup_{K\in\tau_h}\acute{K}$, where $\acute{K}$ denotes the deformed state of K induced by the vector field $\pi\mathbf{u}$ that interpolates $\mathbf{u}$ at the vertices of the elements of τ_h. In this way $\acute{\Omega}$ can be viewed as a certain interpolation of $\tilde{\Omega}$ at the points $\tilde{S}$, S being a vertex of an element of τ_h. In so doing we can prove that (4.2) is exactly satisfied for some kind of partitions, whereas in the general case it is satisfied up to an $\mathbf{O}(h^2)$ term.

Again the proofs will be given under the assumption that $\mathbf{u}\in\mathbf{V}_h$ satisfies

$$\text{(4.3)}\qquad J[\mathbf{x}+\mathbf{u}(\mathbf{x})]\geqq 0\quad\text{a.e. for } \mathbf{x}\in\Omega$$

and, of course,

$$\text{(4.4)}\qquad J[G_K+u(G_K)]=1\quad\forall K\in\tau_h$$

where G_K is the centroid of K.

We first note that $(\mathbf{x}+\pi\mathbf{u})/K$ is nothing other than the linear part of $(\mathbf{x}+\mathbf{u})/K$. Therefore, since $\partial\phi/\partial x_j$ vanishes at the vertex S_{n+1}, $j=1,2,\cdots,n$, and, recalling (3.1), we have

$$\text{(4.5)}\qquad J[S_{n+1}+\mathbf{u}(S_{n+1})]=J[\mathbf{x}+\pi\mathbf{u}(\mathbf{x})]\quad\forall\mathbf{x}\in K.$$

Since meas $(\acute{K})=\int_K J[\mathbf{x}+\pi\mathbf{u}(\mathbf{x})]\,d\mathbf{x}$, assumption (4.3) implies that meas $(\acute{K})\geqq 0$, which in this case means that the $\acute{K}$'s are oriented in the same way as the K's, or that the $\acute{K}$'s do not interpenetrate.

We concentrate on the particular case $n=3$, and we further define $\acute{A}$ to be the deformed state induced by $\pi\mathbf{u}$ of every subset A of Ω. Note that we are actually defining $\hat{\Omega}=\acute{\Omega}$.

We first need the following lemma, which has been proved in [16].

LEMMA 4.1. *Let K be a tetrahedron and let $\mathbf{n}_K$ denote the outer unit normal vector with respect to ∂K, the boundary of K. Let $\boldsymbol{\psi}$ be a vector field defined over K such that $\boldsymbol{\psi}=\boldsymbol{\beta}\phi$, with $\boldsymbol{\beta}\in\mathbb{R}^3$ and ϕ be given by* (3.1). *We then have*

$$\int_K \operatorname{div}\boldsymbol{\psi}\,d\mathbf{x}=\frac{2}{3}\int_{B^K}\boldsymbol{\psi}\cdot\mathbf{n}_K\,ds.$$

Now we note that, since $\pi\mathbf{u}$ is conforming, we clearly have

$$\operatorname{vol}(\acute{\Omega})=\sum_{K\in\tau_h}\operatorname{vol}(\acute{K}).$$

In fact, we can prove that, under a reasonable assumption, the above equality also holds if the $\acute{K}$'s are replaced by $\tilde{K}$'s.

THEOREM 4.2. *If τ_h is a compatible partition of Ω that has no base on Γ^* we have*

$$\operatorname{vol}(\acute{\Omega})=\sum_{K\in\tau_h}\operatorname{vol}(\tilde{K}).$$

Remark. Partition τ_h^1 satisfies the assumptions of this theorem.

Proof. A partition satisfying the assumptions of the theorem can be viewed as a subpartition of a first partition χ_h of Ω, consisting of hexahedrons having triangular faces. Each hexahedron H of χ_h generates two tetrahedrons of τ_h, say K_1 and K_2, having a common base lying in the interior of H, and lateral faces coinciding with the faces of the hexahedron (see Fig. 4.1).

Since $\mathbf{u}$ is continuous over B, the common basis of K_1 and K_2, we have

$$\operatorname{vol}(\tilde{H})=\operatorname{vol}(\tilde{K}_1)+\operatorname{vol}(\tilde{K}_2).$$

We now want to prove that in fact we have

$$\operatorname{vol}(\tilde{H})=\operatorname{vol}(\acute{H})\quad\forall H\in\chi_h,$$

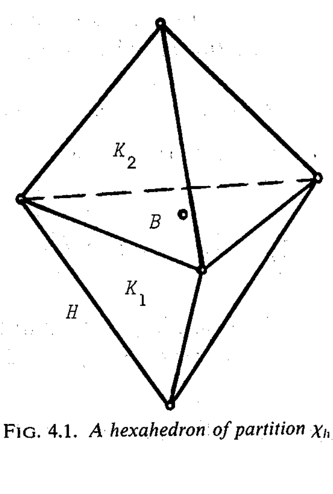

FIG. 4.1. *A hexahedron of partition χ_h.*

which will yield the result we are looking for, since

$$\mathrm{vol}\,(\acute{\Omega}) = \sum_{H \in \chi_h} \mathrm{vol}\,(\acute{H}).$$

For this purpose we introduce a new variable $\acute{\mathbf{x}}$ with the help of the following affine transformation over each K:

$$\mathbf{x} \to \acute{\mathbf{x}} = \mathbf{x} + \pi\mathbf{u}(\mathbf{x}).$$

In this way $\tilde{K}$ can be regarded as the deformed state of $\acute{K}$ obtained by the application of the displacement vector field $\acute{\boldsymbol{\psi}}$ defined by

$$\acute{\boldsymbol{\psi}}(\acute{\mathbf{x}}) = \boldsymbol{\psi}(\mathbf{x}),$$

where $\boldsymbol{\psi} = \boldsymbol{\beta}\phi$, with $\boldsymbol{\beta} = (\mathbf{u})_5 - [\sum_{i=1}^{3} (\mathbf{u})_i]/3$, $(\mathbf{u})_i$ being the value of $\mathbf{u}$ at S_i, $i = 1, 2, \cdots, 5$.

If we denote by $\acute{\lambda}_i(\acute{\mathbf{x}})$ the barycentric coordinates of $\acute{K}$, we have necessarily $\acute{\lambda}_i(\acute{\mathbf{x}}) = \lambda_i(\mathbf{x})$, which means that $\acute{\boldsymbol{\psi}} = \boldsymbol{\beta}\acute{\phi}$, where

$$\acute{\phi} = \sum_{\substack{j,k=1 \\ j<k}}^{n} \acute{\lambda}_j \acute{\lambda}_k.$$

Now we have

$$\mathrm{vol}\,(\tilde{K}) = \int_{\acute{K}} \acute{J}[\acute{\mathbf{x}} + \acute{\boldsymbol{\psi}}(\acute{\mathbf{x}})]\, d\acute{\mathbf{x}}$$

where $\acute{J}$ represents the Jacobian with respect to the new variable $\acute{\mathbf{x}}$.

Expanding the integrand above, we obtain

$$\mathrm{vol}\,(\tilde{K}) = \mathrm{vol}\,(\acute{K}) + \int_{\acute{K}} \acute{\mathrm{div}}\, \acute{\boldsymbol{\psi}}\, d\acute{\mathbf{x}} + \int_{\acute{K}} \left[\sum_{l=1}^{3} [\acute{J}(\acute{\boldsymbol{\psi}}_l) + \acute{J}(\acute{\boldsymbol{\psi}})] \right] d\acute{\mathbf{x}}$$

where $\acute{\boldsymbol{\psi}}_l$ is the vector field obtained by replacing the lth component of $\acute{\boldsymbol{\phi}}$ by $\acute{x}_l$ and $\acute{\mathrm{div}}$ represents the divergence operator associated with $\acute{\mathbf{x}}$.

Since each Jacobian of the second integrand above has at least two columns of form $\boldsymbol{\beta}\acute{\phi}$, they vanish identically.

On the other hand, according to Lemma 4.1 we have

$$\int_{\acute{K}} \acute{\mathrm{div}}\, \acute{\boldsymbol{\psi}}(\acute{\mathbf{x}})\, d\acute{\mathbf{x}} = \frac{2}{3} \int_{\acute{B}} \acute{\boldsymbol{\psi}} \cdot \mathbf{n}_{\acute{K}}\, d\acute{s}.$$

However, since $\pi\mathbf{u}$ is conforming, $\acute{B}$ coincides for both K_1 and K_2 together with $\acute{\psi}/\acute{B}$, whereas $\mathbf{n}/\acute{K}_1/\acute{B} = -\mathbf{n}/\acute{K}_2/\acute{B}$

Therefore we conclude that

$$\mathrm{vol}\,(\tilde{H}) = \mathrm{vol}\,(\tilde{K}_1) + \mathrm{vol}\,(\tilde{K}_2) = \mathrm{vol}\,(\acute{K}_1) + \mathrm{vol}\,(\acute{K}_2) = \mathrm{vol}\,(\acute{H}). \qquad \square$$

Now for the general case the following result applies.

THEOREM 4.3. *For any compatible family $(\tau_h)_h$ of partitions of Ω, we have*

$$\left| \mathrm{vol}\,(\acute{\Omega}) - \sum_{K \in \tau_h} \mathrm{vol}\,(\tilde{K}) \right| \leq Ch^2 |\mathbf{u}|_{2,\infty},$$

where C is a constant independent of h.

Proof. According to Theorem 4.2, all we have to do is to prove that

$$\left| \sum_{K \in \tau_h^*} [\mathrm{vol}\,(\acute{K}) - \mathrm{vol}\,(\tilde{K})] \right| \leq Ch^2 |\mathbf{u}|_{2,\infty}$$

where $\tau_h^* = \{K / K \in \tau_h,\ \mathrm{meas}\,(B^K \cap \Gamma^*) \neq 0\}$.

By a direct computation of the increments of volume of $\acute{K}$ over its faces, due to the quadratic component $\boldsymbol{\beta}\phi$ of $\mathbf{u}$, we obtain

$$\text{vol}(\tilde{K})-\text{vol}(\acute{K})=\int_{\partial\acute{K}}\boldsymbol{\psi}(\acute{\mathbf{x}})\cdot\mathbf{n}_{\acute{K}}\,d\acute{x}.$$

According to Lemma 4.1 we get

$$\text{vol}(\tilde{K})-\text{vol}(\acute{K})=-2\sum_{i=1}^{3}\int_{\acute{F}_i}\boldsymbol{\psi}(\acute{\mathbf{x}})\cdot\mathbf{n}_{\acute{K}}\,d\acute{\mathbf{x}}.$$

Now, F being a lateral face of element K, we define the set Δ_F as follows.

Let E be the edge of F belonging to the base of tetrahedron K and let L_E be the plane surface delimited by $\acute{E}$ and $\tilde{E}$. Δ_F is defined to be the solid delimited by $\tilde{F}$, $\acute{F}$, and L_E, as illustrated in Fig. 4.2 below.

Using classical arguments, if $(\tau_h)_h$ is regular then we can estimate

$$\text{vol}(\Delta_F)\leqq Ch^4|\mathbf{u}|_{2,\infty}\quad\forall F.$$

Noting that

$$|\text{vol}(\Delta_{F_i})|=\left|\int_{\acute{F}_i}\boldsymbol{\psi}(\acute{\mathbf{x}})\cdot\mathbf{n}_{\acute{K}}\,d\acute{\mathbf{x}}\right|,$$

we now have

$$\text{vol}(\tilde{K})-\text{vol}(\acute{K})\leqq 6Ch^4|\mathbf{u}|_{2,\infty}.$$

Since card $\tau_h^*\leqq Ch^{-2}$ the result follows. □

5. Existence results for partition τ_h^2. Case i. Let us now prove that under suitable assumptions on $\mathbf{u}_h$, the compatibility condition (2.8) is satisfied for any partition τ_h^2. We treat Case i in this section, and in the next section we shall consider Case ii for the two-dimensional asymmetric element only.

For the sake of simplicity we will work with the linear manifold $\mathbf{V}_h^x$ of $\mathbf{V}_h$, defined to be $\mathbf{x}+\mathbf{V}_h$. We also define the following subset of $\mathbf{V}_h^x$:

$$\tilde{X}_h^x=\{\mathbf{v}_h^x/\mathbf{v}_h^x-\mathbf{x}\in X_h\}.$$

In both Cases i and ii we shall prove the validity of (2.8) under the following basic assumption on the solution $\mathbf{u}_h$ of the minimization problem (P_h).

Assumption A. Let $\pi\mathbf{u}_h^x$ denote the piecewise linear interpolate of $\mathbf{u}_h^x$ defined in § 3. The triangulation $\acute{\tau}_h^2$ of $\acute{\Omega}_h=\Pi\mathbf{u}_h^x(\Omega_h)$ defined to be

$$\acute{\tau}_h^2=\{\acute{K}/\acute{K}=\Pi\mathbf{u}_h^x(K),\ K\in\tau_h^2\}$$

is such that there exists a constant $\alpha>0$ for which we have

$$\frac{1}{\alpha}\text{area}(K)\geqq\text{area}(\acute{K})\geqq\alpha\ \text{area}(K)\quad\forall K\in\tau_h^2.$$

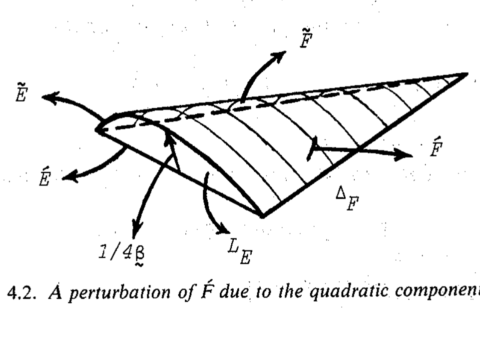

FIG. 4.2. *A perturbation of $\acute{F}$ due to the quadratic components of* $\mathbf{u}$.

Note that Assumption A is equivalent to assuming that $J(\Pi \mathbf{u}_h^x) > 0$ almost everywhere in Ω_h. It also implies that $\acute{\tau}_h^2$ belongs to a regular family of partitions $\{\acute{\tau}_h^2\}_h$, whenever $\mathbf{u}_h$ belongs to a bounded subset of $W^{1,\infty}(\Omega_h)$, for all h.

Indeed, in this case if we set

$$\acute{h} = \max_{K \in \tau_h^2} \{\acute{h}_K = \text{diameter of } \acute{K}\}$$

and

$$\acute{\rho} = \min_{K \in \tau_h^2} \{\acute{\rho}_K = \text{diameter of the inscribed circle in } \acute{K}\}$$

we obtain $\acute{\rho}\acute{h}^{-1} \geqq \acute{c}$, for all $\max_h$, where $\acute{c}$ is given by $(2\alpha/3)(c^2/U^2)$ with $U = \max_h |u_h|_{1,\infty}$, c being the constant such that $\rho h^{-1} \geqq c > 0$, for all $\tau_h^2 \in \{\tau_h^2\}_h$, as we can easily verify.

In Case i both $\Omega = \Omega_h$ and $\acute{\Omega} = \acute{\Omega}_h$ are polygons or polyhedrons. Thus, since $\Pi \mathbf{u}_h^x$ defines an affine transformation over each triangle K onto $\acute{K}$, we can define a space $\acute{V}_h$ over $\acute{\Omega}_h$ associated with $\acute{\tau}_h^2$ in the same way as $\mathbf{V}_h$ is associated with τ_h^2, and $\acute{\mathbf{V}}_h$ will have the same structure as $\mathbf{V}_h$.

We also define $\acute{Q}_h$ to be the space of pressures analogous to Q_h for the triangulation $\acute{\tau}_h^2$.

We shall need the following notation in connection with partition $\acute{\tau}_h^2$. For every function $\acute{v}$ of $L^2(\acute{\Omega})$, whose restriction to every $\acute{K} \in \tau_h^2$ belongs to $H^1(\acute{K})$, we write

$$(\acute{\operatorname{div}}\, \acute{\mathbf{v}}, \acute{q})_{0,\acute{\Omega},h} = \sum_{\acute{K} \in \acute{\tau}_h^2} \int_{\acute{K}} \acute{q}\, \acute{\operatorname{div}}\, \acute{\mathbf{v}}\, d\acute{\mathbf{x}} \quad \forall \acute{q} \in L^2(\acute{\Omega}),$$

and we define

$$|\acute{v}|_{1,\acute{\Omega},h} = \left[\sum_{\acute{K} \in \acute{\tau}_h^2} |\acute{v}|_{1,\acute{K}}^2 \right]^{1/2}.$$

Let us first consider the subspace $\acute{Q}_h^0$ of those pressures that are constant over $\acute{K}$, K being a simplex of τ_h. According to Lemma C2 of [4] if $\acute{V} = \{\acute{\mathbf{v}}/\acute{\mathbf{v}} \in \mathbf{H}^1(\acute{\Omega}),\ \acute{\mathbf{v}} = \mathbf{0}$ on $\acute{\Gamma}_0 \equiv \Gamma_0\}$, for all $\acute{q}_h^0 \in \acute{Q}_h^0$ there exists $\acute{\mathbf{v}} \in \acute{\mathbf{V}}$ such that

$$(5.1) \qquad (\acute{\operatorname{div}}\, \acute{\mathbf{v}}, \acute{q}_h^0)_{0,\acute{\Omega}} \geqq \acute{\beta}_0 |\acute{q}_h^0|_{0,\acute{\Omega}}^2$$

and

$$(5.2) \qquad |\acute{v}|_{1,\acute{\Omega}} \leqq C_0 |\acute{q}_h^0|_{0,\acute{\Omega}}$$

where $\acute{\beta}_0 > 0$ and C_0 are independent of $\acute{q}_h^0$.

LEMMA 5.1. *There exist constants $\acute{\beta}_0 > 0$ and $\acute{C}_0$, such that with every $q_h^0 \in Q_h^0$ we can associate a $\acute{\mathbf{w}}_h \in \acute{\mathbf{v}}_h$ that satisfies the following conditions:*

(5.3) $\acute{\mathbf{w}}_h(\acute{S}) = \mathbf{0}$ *for all vertices $\acute{S}$ of a macrosimplex $\cup_{i=1}^{n+1} \acute{K}_i$, where the K_i's are the simplices of a macrosimplex $K \subset \tau_h$, and τ_h is the first partition of Ω on which τ_h^2 is constructed;*

$$(5.4) \qquad (\acute{\operatorname{div}}\, \acute{\mathbf{w}}_h, \acute{q}_h^0)_{0,\acute{\Omega},h} \geqq \acute{\beta}_0 |\acute{q}_h^0|_{0,\acute{\Omega}}^2;$$

$$(5.5) \qquad |\acute{\mathbf{w}}_h|_{1,\acute{\Omega},h} \leqq \acute{C}_0 |\acute{q}_h^0|_{0,\acute{\Omega}}.$$

Proof. Let $\acute{\mathbf{v}} \in \acute{\mathbf{V}}$ satisfy (5.1) and (5.2). We associate with $\acute{\mathbf{v}}$ a vector field $\acute{\mathbf{w}}_h \in \acute{\mathbf{V}}_h$, such that $\acute{\mathbf{w}}_{h/\acute{K}}$ satisfies for all $K \in \tau_h$ the following:

$$\acute{\mathbf{w}}_h(\acute{S}) = 0 \quad \text{if } \acute{S} \text{ is a vertex of } \acute{K}_i,\ i = 1, 2, \cdots, n+1,$$

$$\acute{\mathbf{w}}_h(\acute{M}_i) = \frac{3}{2} \frac{\int_{\acute{B}_i} \acute{\mathbf{v}}\, ds}{\operatorname{meas}(\acute{B}_i)}$$

where $\acute{B}_i$ is the base of $\acute{K}_i$ and $\acute{M}_i$ is its centroid, thus satisfying (5.3).

Using Lemma 4.1, and letting $\acute{\tau}_h$ be the partition of $\acute{\Omega}$ into macro-simplices $\acute{K}$, $K \in \tau_h$, we obtain

$$\int_{\acute{K}} \acute{\operatorname{div}}\, \acute{\mathbf{w}}_h \, d\mathbf{x} = \int_{\acute{K}} \acute{\operatorname{div}}\, \acute{\mathbf{v}} \, d\mathbf{x} \quad \forall \acute{K} \in \acute{\tau}_h.$$

This yields

$$(\acute{\operatorname{div}}\, \acute{\mathbf{w}}_h, \acute{q}_h^0)_{0,\acute{\Omega},h} = (\acute{\operatorname{div}}\, \acute{\mathbf{v}}, \acute{q}_h^0)_{0,\acute{\Omega}},$$

which in turn gives (5.4), after applying (5.1). To prove (5.5) we first use the Trace Theorem, which gives us

$$|\acute{\mathbf{w}}_h|_{1,\acute{K}_i} = \acute{\mathbf{w}}_h(\acute{M}_i) C(\acute{K}_i) \leqq C'(\acute{K}_i) \|\acute{\mathbf{v}}\|_{1,\acute{K}_i},$$

which, according to Assumption A, yields

$$|\acute{\mathbf{w}}_h|_{1,\acute{\Omega},h} \leqq C(\acute{\Omega}, \acute{\mathbf{u}}_h) |\acute{\mathbf{v}}|_{1,\acute{\Omega}} \quad \text{with } C < \infty.$$

Thus, using (5.2), we get (5.5) with $\acute{C}_0 = C_0 C$. □

Now let $s_i = \operatorname{meas}(\acute{K}_i)$, $1 \leqq i \leqq n+1$. Without loss of generality we can assume that $s_1 \geqq s_2 \geqq \cdots \geqq s_{n+1}$.

Let $\acute{Q}_h^1$ be the subspace of $\acute{Q}_h$ generated by the set of orthogonal functions $\{\eta_2^h, \cdots, \eta_{n+1}^K\}_{K \in \tau_h}$ such that $\operatorname{supp}(\eta_i^K) \subset \acute{K}$, $i = 2, \cdots, n+1$, and

$n = 2$:

$$\begin{cases} \eta_2^K = -1 & \text{if } \acute{\mathbf{x}} \in \acute{K}_1, \\ \eta_2^K = \dfrac{s_1}{s_2 + s_3} & \text{if } \acute{\mathbf{x}} \in \acute{K}_2 \cup \acute{K}_3; \end{cases}$$

$$\begin{cases} \eta_3^K = 0 & \text{if } \acute{\mathbf{x}} \in \acute{K}_1, \\ \eta_3^K = -1 & \text{if } \acute{\mathbf{x}} \in \acute{K}_2, \\ \eta_3^K = \dfrac{s_2}{s_3} & \text{if } \acute{\mathbf{x}} \in \acute{K}_3; \end{cases}$$

$n = 3$:

$$\begin{cases} \eta_2^K = 1 & \text{if } \acute{\mathbf{x}} \in \acute{K}_1, \\ \eta_2^K = \dfrac{-s_1}{s_2} & \text{if } \acute{\mathbf{x}} \in \acute{K}_2, \\ \eta_2^K = 0 & \text{if } \acute{\mathbf{x}} \in \acute{K}_3 \cup \acute{K}_4; \end{cases}$$

$$\begin{cases} \eta_3^K = 0 & \text{if } \acute{\mathbf{x}} \in \acute{K}_1 \cup \acute{K}_2, \\ \eta_3^K = \dfrac{-s_3}{s_4} & \text{if } \acute{\mathbf{x}} \in \acute{K}_3, \\ \eta_3^K = 1 & \text{if } \acute{\mathbf{x}} \in \acute{K}_4; \end{cases}$$

$$\begin{cases} \eta_4^K = 1 & \text{if } \acute{\mathbf{x}} \in \acute{K}_1 \cup \acute{K}_2, \\ \eta_4^K = \dfrac{-s_1 - s_2}{s_3 + s_4} & \text{if } \acute{\mathbf{x}} \in \acute{K}_3 \cup \acute{K}_4. \end{cases}$$

As we can easily verify, we have $\eta_i^K \perp q_h^0$, for all $q_h^0 \in \acute{Q}_h^0$, $i = 2$, and $\acute{Q}_h = \acute{Q}_h^0 \oplus \acute{Q}_h^1$.

Now let q_h^1 be any function of $\acute{Q}_h^1$. We can write

$$q_h^1 = \sum_{K \in \tau_h} \sum_{i=2}^{n+1} q_i^K \eta_i^K \tag{5.6}$$

where the q_i^K's are given scalars.

LEMMA 5.2. *If Assumption* A *holds, then for every* $\acute{q}_h \in \acute{Q}_h$ *there exists a* $\acute{\mathbf{v}}_h \in \acute{\mathbf{V}}_h$ *satisfying* (5.3) *together with*

$$\frac{(\acute{\operatorname{div}}\, \mathbf{v}_h, \acute{q}_h)_{0,\acute{\Omega},h}}{|\acute{\mathbf{v}}_h|_{1,\acute{\Omega},h}} \geqq \acute{\beta}_h |\acute{q}_h|_{0,\acute{\Omega}}, \tag{5.7}$$

for some $\acute{\beta}_h > 0$ *independent of* $\acute{q}_h$.

Proof. Let $\acute{q}_h = q_h^0 + q_h^1$, where $q_h^0 \in \acute{Q}_h^0$ and $q_h^1 \in \acute{Q}_h^1$.

We first construct a vector field $\acute{\mathbf{z}}_h \in \acute{\mathbf{V}}_h$ satisfying (5.3) in the following way. $\acute{\mathbf{z}}_h = \mathbf{0}$ at every vertex or centroid of the bases of $\acute{K}$, $\acute{K} \in \acute{\tau}_h^2$. If $\acute{G}$ is the common vertex of $\acute{K}_i$, $i = 1, 2, \cdots, n+1$, we define $\acute{\mathbf{z}}_h(\acute{G})$ to be of the form

$$\acute{\mathbf{z}}_h(\acute{G}) = \sum_{i=2}^{n+1} \gamma_i^K \mathbf{m}_i$$

where the $\mathbf{m}_i$'s are the oriented edges $\acute{G}\acute{S}_i$ of the $\acute{K}_i$'s, as indicated in Fig. 5.1 below, and the γ_i^K's are given scalars depending on the q_i^K's only (see (5.6)).

First, for $n = 2$ we set

$$\gamma_2^K = -q_2^K \quad \text{and} \quad \gamma_3^K = q_3^K,$$

and using Assumption A we can easily estimate the following:

$$|\acute{\mathbf{z}}_h|_{1,\acute{\Omega}} \leqq C(\mathbf{u}_h)|q_h^1|_{0,\acute{\Omega}}. \tag{5.8}$$

Now, dropping the superscript K, after simple calculations we get:

$$|q_h^1|_{0,\acute{K}}^2 = q_2^2|\eta_2|_{0,\acute{K}}^2 + q_3^2|\eta_3|_{0,\acute{K}}^2 \leqq 2\frac{s_1^3}{s_3^2}(q_2^2 + q_3^2).$$

Since Assumption A implies that $s_3 \geqq \alpha^2 s_1$, we have

$$|q_h^1|_{0,K}^2 \leqq \frac{2s_1}{\alpha^4}(q_2^2 + q_3^2).$$

Now we prove that

$$(\acute{\operatorname{div}}\, \acute{\mathbf{z}}_h, q_h^1)_{0,\acute{\Omega}} \geqq c'|q_h^1|_{0,\acute{\Omega}}^2, \qquad c' > 0. \tag{5.9}$$

A straightforward calculation gives

$$\int_{\acute{K}} \acute{\operatorname{div}}\, \acute{\mathbf{z}}_h\, q_h^1\, d\acute{\mathbf{x}} = \left(s_1 + s_3 + \frac{s_1^2 + s_1 s_3}{s_2 + s_3}\right) q_2^2 + \left(s_1 + \frac{s_1 s_2 + s_2^2}{s_3}\right) q_3^2 + \left(\frac{s_1 s_2}{s_2 + s_3} + 2s_2 - s_1\right) q_2 q_3.$$

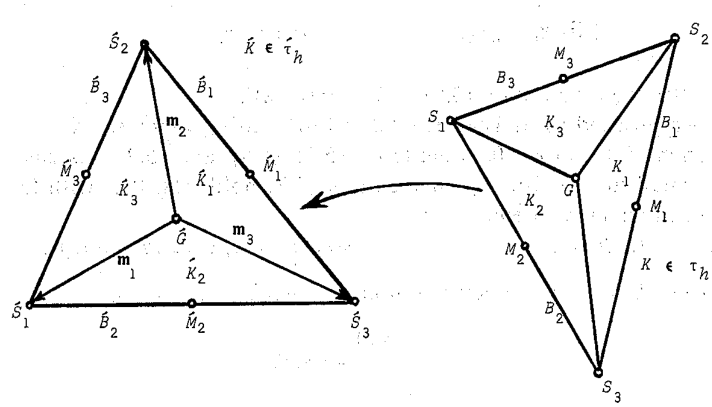

FIG. 5.1. *Macroelements of partitions $\acute{\tau}_h$ and τ_h for $n = 2$.*

Thus we have

$$\int_{\acute{K}} \operatorname{d\acute{i}v} \acute{\mathbf{z}}_h q_h^1 \, d\acute{\mathbf{x}} \geqq (2s_1+s_3)q_2^2+(2s_1+s_2)q_3^2+\left(2s_2-s_1+\frac{s_1s_2}{s_2+s_3}\right)q_2q_3.$$

Now, as we can easily check, for any $\alpha>0$ we have

$$\left|\frac{s_1s_2}{s_2+s_3}+2s_2-s_1\right|^2<4(s_1+s_2)(s_1+s_3),$$

which yields

$$\int_{\acute{K}} \operatorname{d\acute{i}v} \acute{\mathbf{z}}_h q_h^1 \, d\mathbf{x} \geqq s_1(q_2^2+q_3^2).$$

This in turn implies (5.9) with $c'=\alpha^4/2$. In the case $n=3$ we set

$$\gamma_2^K=q_2^K, \quad \gamma_3^K=q_4^K+q_3^K, \quad \gamma_4^K=q_4^K-q_3^K.$$

As in the case $n=2$ it is straightforward to derive the estimate (5.8), and dropping again the superscripts K, we have

$$|q_h^1|_{0,\acute{K}}^2=\sum_{i=2}^{4} q_i^2|\eta_i|_{0,\acute{K}}^2 \leqq 2\frac{s_1^2}{s_4}(q_2^2+q_3^2+2q_4^2).$$

Since Assumption A implies that $s_4 \geqq \alpha^2 s_1$, we have

(5.10) $$|q_h^1|_{0,\acute{K}}^2 \leqq \frac{2 \operatorname{meas}(\acute{K})}{\alpha^2}(q_2^2+q_3^2+2q_4^2).$$

On the other hand, simple calculations yield

$$\frac{1}{\operatorname{meas}(K)}(\operatorname{d\acute{i}v} \acute{\mathbf{z}}_h, q_h^1)_{0,\acute{K}}=\frac{s_1}{s_2}q_2^2-q_2q_4+\left(1+\frac{s_4}{s_3}\right)q_3^2+\left(\frac{s_4}{s_3}-1\right)q_3q_4+\frac{2(s_1+s_2)}{s_3+s_4}q_4^2.$$

Thus we have

$$\frac{1}{\operatorname{meas}(\acute{K})}(\operatorname{d\acute{i}v} \acute{\mathbf{z}}_h, q_h^1)_{0,\acute{K}} \geqq \left(\frac{s_1}{s_2}-\frac{1}{2}\right)q_2^2+\left(\frac{1}{2}+\frac{3s_4}{2s_3}\right)q_3^2+\left(\frac{2s_1+2s_2}{s_3+s_4}-\frac{1}{2}+\frac{s_4}{2s_3}\right)q_4^2,$$

which reduces to

$$(\operatorname{d\acute{i}v} \acute{\mathbf{z}}_h, q_h^1)_{0,\acute{K}} \geqq (q_2^2+q_3^2+2q_4^2)\frac{\operatorname{meas}(\acute{K})}{2}.$$

Using (5.10), we thus obtain (5.9) with $c'=(\alpha/2)^2$.

Finally, we proceed as in Theorem 4.2 of [16], setting $\acute{\mathbf{v}}_h=\theta\acute{\mathbf{w}}_h+\acute{\mathbf{z}}_h$, where $\acute{\mathbf{w}}_h$ is defined in Lemma 5.1 and $\theta>0$. From (5.4), (5.5), (5.8), and (5.9) it is clear that for θ sufficiently small there exists $\acute{\beta}_h>0$ such that (5.7) holds together with (5.3). □

Now we further prove Lemma 5.3.

LEMMA 5.3. *With every $q_h \in Q_h$ we can associate a $\mathbf{v}_h \in \mathbf{V}_h$ that satisfies*

(5.11) $\mathbf{v}_h(S)=\mathbf{0}$ *for every vertex S of a macro-simplex K, $K \in \tau_H$,*

(5.12) $$\frac{\tilde{b}_h'(\Pi\mathbf{u}_h, \mathbf{v}_h, q_h)}{\|\mathbf{v}_h\|_h} \geqq \beta_h\|q_h\|_0$$

where β_h is a strictly positive parameter independent of q_h.

Proof. Using an identity from [9, p. 108] we obtain

$$\tilde{b}_h(\Pi \mathbf{u}_h, \mathbf{v}_h, q_h) = \int_{\acute{\Omega}} \acute{q}_h \,\acute{\operatorname{div}}\, \acute{\mathbf{v}}_h \, d\acute{\mathbf{x}} \tag{5.13}$$

where $\acute{\mathbf{v}}_h(\acute{\mathbf{x}}) = \mathbf{v}_h(\mathbf{x})$.

On the other hand, from Assumption A it is straightforward to establish the existence of a constant $C(\Omega, \mathbf{u}_h)$ such that

$$\|\mathbf{v}_h\|_h \leqq C(\Omega, \mathbf{u}_h)|\acute{\mathbf{v}}_h|_{1,\acute{\Omega},h}.$$

Moreover, we have

$$\|q_h\|_0 \leqq \alpha^{-1/2}|\acute{q}_h|_{0,\acute{\Omega}}.$$

Finally, if $\acute{\mathbf{v}}_h$ is the field defined in Lemma 5.2, we obtain (5.11) and (5.12) with $\beta_h = \alpha^{1/2} C^{-1} \acute{\beta}_h > 0$. □

As a final preparatory result we have Lemma 5.4.

LEMMA 5.4. *Under Assumption A, for any $\mathbf{v}_h \in \mathbf{V}_h$ satisfying* (5.11) *we have*

$$\tilde{b}'_h(\mathbf{u}_h, \mathbf{v}_h, q_h) = \tilde{b}'_h(\Pi \mathbf{u}_h, \mathbf{v}_h, q_h) \quad \forall q_h \in Q_h.$$

Proof. Taking into account the definitions of b'_h and Q_h, if we can show that

$$\int_K \operatorname{adj}^{\mathrm{T}} \nabla \mathbf{u}_h^x \cdot \nabla \mathbf{v}_h \, d\mathbf{x} = \int_K \operatorname{adj}^{\mathrm{T}} \nabla \Pi \mathbf{u}_h^x \cdot \nabla \mathbf{v}_h \, d\mathbf{x} \quad \forall K \in \tau_h^2,$$

we shall prove the lemma. To demonstrate the above equality we rewrite as follows:

$$\int_K \operatorname{adj}^{\mathrm{T}} \nabla u_h^x \cdot \nabla \mathbf{v}_h \, d\mathbf{x}$$
$$= \lim_{\theta \to 0} \frac{1}{\theta} \int_K [J(\mathbf{x} + \Pi \mathbf{u}_h + \boldsymbol{\beta}\phi + \theta \mathbf{v}_h) - J(\mathbf{x} + \Pi \mathbf{u}_h + \boldsymbol{\beta}\phi)] \, d\mathbf{x}$$

where ϕ is given by (3.1) and $\boldsymbol{\beta}$ is a linear combination of $\mathbf{u}_i$, $i = 1, 2, \cdots, n$ and $\mathbf{u}_{n+2}$, where $\mathbf{u}_i$ is the value of $\mathbf{u}_h$ at node S_i of $K \in \tau_h^2$ (see Fig. 3.1).

Passing to element $\acute{K}$, using the affine transformation and notation already encountered in § 4, we get

$$\int_K \operatorname{adj}^{\mathrm{T}} \nabla \mathbf{u}_h^x \cdot \nabla \mathbf{v}_h \, d\mathbf{x} = \lim_{\theta \to 0} \frac{1}{\theta} \int_{\acute{K}} [\acute{J}(\acute{x} + \boldsymbol{\beta}\acute{\phi} + \theta \acute{\mathbf{v}}_h) - \acute{J}(\acute{\mathbf{x}} + \boldsymbol{\beta}\acute{\phi})] \, d\acute{\mathbf{x}}.$$

Expanding the right-hand side above and taking the limit we get

$$\int_K \operatorname{adj}^{\mathrm{T}} \nabla \mathbf{u}_h^x \cdot \nabla \mathbf{v}_h \, d\mathbf{x} = \int_{\acute{K}} \acute{\operatorname{div}}\, \acute{\mathbf{v}}_h \, d\acute{\mathbf{x}} + \sum_{1 \leqq i < j \leqq n} (-1)^{i+j} \int_{\acute{K}} d_{ij}(\boldsymbol{\beta}, \acute{\phi}, \acute{\mathbf{v}}_h) \, d\acute{\mathbf{x}} \tag{5.14}$$

where

$$d_{ij}(\boldsymbol{\beta}, \acute{\phi}, \acute{\mathbf{v}}_h) = \frac{\partial \acute{\phi}}{\partial \acute{x}_i} \begin{vmatrix} \beta_i & \dfrac{\partial \acute{v}_i}{\partial \acute{x}_j} \\ \beta_j & \dfrac{\partial \acute{v}_j}{\partial \acute{x}_j} \end{vmatrix} + \frac{\partial \acute{\phi}}{\partial \acute{x}_j} \begin{vmatrix} \dfrac{\partial \acute{v}_i}{\partial \acute{x}_i} & \beta_i \\ \dfrac{\partial \acute{v}_j}{\partial \acute{x}_i} & \beta_j \end{vmatrix}, \qquad \acute{\mathbf{v}}_h = (\acute{v}_1, \cdots, \acute{v}_n).$$

Now, according to (5.11) we can write $\acute{\mathbf{v}}_h$ as the sum of two components, namely $\acute{\mathbf{v}}_h = \mathbf{a}\acute{\lambda}_{n+1} + \mathbf{b}\acute{\phi}$. Then if we expand the above determinants into sums of two determinants corresponding to these components of $\mathbf{v}_h$, the one associated with $\boldsymbol{\beta}\phi$ is readily seen to vanish identically. Thus we have

$$d_{ij} = \left(\frac{\partial \acute{\lambda}_{n+1}}{\partial \acute{x}_j} \frac{\partial \acute{\phi}}{\partial \acute{x}_i} - \frac{\partial \acute{\lambda}_{n+1}}{\partial \acute{x}_i} \frac{\partial \acute{\phi}}{\partial \acute{x}_j} \right) c_{ij} \quad \text{where } c_{ij} = \begin{vmatrix} \beta_i & a_i \\ \beta_j & a_j \end{vmatrix}.$$

Now we note that

$$\int_{\acute{K}} d_{ij} = \left(\frac{\partial \acute{\lambda}_{n+1}}{\partial \acute{x}_j} \int_{\acute{K}} \frac{\partial \acute{\phi}}{\partial \acute{x}_i} - \frac{\partial \acute{\lambda}_{n+1}}{\partial \acute{x}_i} \int_{\acute{K}} \frac{\partial \acute{\phi}}{\partial \acute{x}_j} \right) c_{ij}. \tag{5.15}$$

Since

$$\frac{\partial \acute{\phi}}{\partial \acute{x}_k} = \sum_{\substack{i,j=1 \\ i \neq j}}^{n} \frac{\partial \acute{\lambda}_j}{\partial \acute{x}_k} \acute{\lambda}_i,$$

we obtain

$$\int_{\acute{K}} \frac{\partial \acute{\phi}}{\partial \acute{x}_k} = \frac{n-1}{n+1} \operatorname{meas}(\acute{K}) \sum_{j=1}^{n} \frac{\partial \acute{\lambda}_j}{\partial \acute{x}_k}.$$

Using the elementary identity $\sum_{j=1}^{n} \acute{\lambda}_j \equiv 1 - \acute{\lambda}_{n+1}$, we finally get

$$\int_{\acute{K}} \frac{\partial \acute{\phi}}{\partial \acute{x}_k} = -\frac{n-1}{n+1} \operatorname{meas}(\acute{K}) \frac{\partial \acute{\lambda}_{n+1}}{\partial \acute{x}_k}.$$

When we use the above relation in (5.15), $\int_{\acute{K}} d_{ij}\, d\acute{\mathbf{x}}$ is readily seen to vanish.

The result then follows taking into account (5.13) and (5.14). □

To conclude, as a consequence of Lemmas 5.2–5.4 we have Theorem 5.1.

THEOREM 5.1. *If* $\mathbf{u}_h$ *satisfies Assumption* A *for some* $\alpha > 0$, (2.8) *holds in Case* i.

6. Existence results for partition τ_h^2; Case ii. Let us now turn to Case ii, which we shall examine for $n = 2$ only.

In this case Ω_h will be the union of triangles with one parabolic edge, such that its boundary Γ_h coincides with Γ at least at the nodes of those triangles that have a parabolic edge (base) on Γ_h. Let Γ_{0_h} be the portion of Γ_h consisting of such parabolic edges that have their three nodes on $\bar{\Gamma}_0$.

Now instead of Assumption A we make a stronger one, namely Assumption B.

Assumption B. $J(\mathbf{u}_h^x) > 0$ almost everywhere in Ω_h.

Taking into account (4.5), the above assumption implies Assumption A. Moreover, it allows us to say that $\mathbf{u}_h^x$ is a bijection between Ω_h and $\tilde{\Omega}_h = \mathbf{u}_h^x(\Omega_h)$. In this case $\tilde{\Omega}_h$ is a domain that has the same structure as Ω_h, in the sense that it can also be viewed as the union of isoparametric elements $\tilde{\mathbf{K}}$, where $\tilde{\mathbf{K}} = \mathbf{u}_h^x(K)$, $K \in \tau_h^2$.

Then let $\tilde{\tau}_h^2$ be the triangulation of $\tilde{\Omega}_h$ consisting of the $\tilde{K}$'s. Similarly, let τ_h be the set of curved macroelements $K = \bigcup_{i=1}^{3} K_i$ on which τ_h^2 is constructed, and let $\tilde{\tau}_h$ be the partition of $\tilde{\Omega}_h$ into curved macroelements $\tilde{K}$, where $\tilde{K} = \mathbf{u}_h^x(K)$, $K \in \tau_h$ (see Fig. 6.1).

For simplicity we consider the case where for all $K \in \tau_h$, area $(K_1) =$ area $(K_2) =$ area (K_3), although the more general case can be treated without major difficulties.

Now if $\acute{K}_i = \Pi \mathbf{u}_h^x(K_i)$, Assumption B, and hence A, implies

$$\frac{1}{\alpha} \operatorname{area}(\tilde{K}_i) \geqq 3 \operatorname{area}(\acute{K}_i) \geqq \alpha \operatorname{area}(\tilde{K}_i), \qquad 1 \leqq i \leqq 3,$$

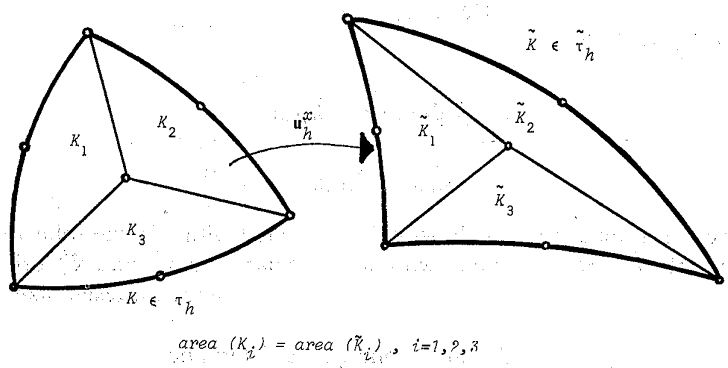

FIG. 6.1. *Macroelements of partitions τ_h and $\tilde{\tau}_h$.*

since

$$\text{area}\,(\tilde{K}_i)=\text{area}\,(K_i)=\tfrac{1}{3}\,\text{area}\,(K)\quad \forall K\in\tau_h.$$

Let us now define the following spaces of functions defined over $\tilde{\Omega}_h$:

$$\tilde{Q}_h=\{\tilde{q}_h/\tilde{q}_h\circ \mathbf{u}_h^x=q_h,\ q_h\in Q_h\},$$

$$\tilde{\mathbf{V}}_h=\{\tilde{\mathbf{v}}_h/\tilde{\mathbf{v}}_h\circ \mathbf{u}_h^x=\mathbf{v}_h,\ \mathbf{v}_h\in \mathbf{V}_h\}.$$

We equip $\tilde{\mathbf{V}}_h$ and $\tilde{Q}_h$ with the norms $\|\cdot\|$ and $|\cdot|$ given, respectively, by $\|\tilde{\mathbf{v}}_h\|=|\tilde{\mathbf{v}}_h|_{1,\tilde{\Omega}_h}$, $\tilde{\mathbf{v}}_h\in\tilde{\mathbf{V}}_h$, and $|\tilde{q}_h|=|\tilde{q}_h|_{0,\tilde{\Omega}_h}$, $\tilde{q}_h\in\tilde{Q}_h$. (Since $\tilde{\mathbf{v}}_h=\mathbf{0}$ on $\Gamma_{0_h}\equiv\tilde{\Gamma}_{0_h}$, $\|\cdot\|$ is actually a norm.)

Let us also denote by $\tilde{\mathbf{x}}$ the new variable $\mathbf{u}_h^x(\mathbf{x})$.

More generally, for every function f defined over Ω_h we denote by $\tilde{f}$ the function defined over $\tilde{\Omega}_h$ such that $\tilde{f}[\mathbf{u}_h^x(\mathbf{x})]=f(\mathbf{x})$, for all $\mathbf{x}\in\Omega_h$.

To prove that (2.8) holds, we use the following theorem given by Le Tallec.

THEOREM 6.1 [10, Thm. 4.5]. *Under Assumption B, (2.8) is equivalent to the following:*

There exists $\beta_h>0$ such that

$$\sup_{\tilde{\mathbf{v}}_h\in\tilde{\mathbf{V}}_h}\frac{\int_{\tilde{\Omega}_h}\tilde{q}_h\,\tilde{\operatorname{div}}\,\tilde{\mathbf{v}}_h\,d\tilde{\mathbf{x}}}{\|\tilde{\mathbf{v}}_h\|}\geqq\tilde{\beta}_h|\tilde{q}_h|\quad\forall\tilde{q}_h\in\tilde{Q}_h \tag{6.1}$$

where $\tilde{\operatorname{div}}$ represents the divergence operator with respect to the $\tilde{\mathbf{x}}$ variable.

The above result states that it suffices to prove the linear discrete compatibility condition between spaces $\tilde{\mathbf{V}}_h$ and $\tilde{Q}_h$ to have existence of a solution to $(\mathbf{P}_h)$ in the isoparametric case.

Now, to prove 6.1 for the asymmetric triangle, we give the following lemmas.

LEMMA 6.1. *Let $\tilde{Q}_h^0$ be the subspace of $\tilde{Q}_h$ of those functions that are constant over $\tilde{K}$, for all $\tilde{K}\in\tilde{\tau}_h$. Then for every $\tilde{q}_h\in\tilde{Q}_h^0$ there exists a vector field $\tilde{\mathbf{w}}_h\in\tilde{\mathbf{V}}_h$ such that*

$$\int_{\tilde{\Omega}_h}\tilde{q}_h\,\tilde{\operatorname{div}}\,\tilde{\mathbf{w}}_h\,d\tilde{\mathbf{x}}\geqq\tilde{\beta}_0|\tilde{q}_h|^2, \tag{6.2}$$

$$\|\tilde{\mathbf{w}}_h\|\leqq\tilde{C}_h|\tilde{q}_h| \tag{6.3}$$

where $\tilde{\beta}_0$ and $\tilde{C}_h$ are strictly positive constants independent of $\tilde{q}_h$.

Proof. According to Lemma C2 of [4] for a given $\tilde{q}_h \in \tilde{Q}_h^0$, there exists a $\mathbf{v} \in \mathbf{H}^1(\tilde{\Omega}_h)$ with $\mathbf{v} = \mathbf{0}$ on Γ_{0_h} such that

$$\int_{\tilde{\Omega}_h} \tilde{q}_h \, \tilde{\operatorname{div}} \, \mathbf{v} \, d\tilde{\mathbf{x}} \geqq \tilde{\beta}_0 |\tilde{q}_h|^2, \tag{6.4}$$

$$|\mathbf{v}|_{1,\tilde{\Omega}_h} \leqq \tilde{C} |\tilde{q}_h|. \tag{6.5}$$

Now we construct a vector field $\tilde{\mathbf{w}}_h \in \tilde{\mathbf{V}}_h$ associated with $\mathbf{v}$ in the following way. For each triangle $\tilde{K} \in \tilde{\tau}_h^2$, we define two perpendicular axes $\vec{x}_3^K$ and $\vec{x}_4^K$ oriented in such a way that they correspond to rotations of the reference Cartesian axes $\vec{x}_1$ and $\vec{x}_2$ of an angle φ.

Dropping the superscript K for simplicity, we determine φ in such a way that the straight line passing through nodes $\tilde{S}_2$ and $\tilde{S}_3$ of $\tilde{K}$ forms an angle of $\Pi/4$ with both $\vec{x}_3$ and $\vec{x}_4$.

Let x_j be the variable with respect to the axes $\vec{x}_j$, $1 \leqq j \leqq 4$. Clearly $\vec{x}_3$ and $\vec{x}_4$ will coincide for any pair of elements of $\tilde{\tau}_h^2$ that have a base $\tilde{B}$ as a common edge. Let the local numbering of the vertices of each element respect the usual permutation convention (in this way, $\tilde{S}_2$ and $\tilde{S}_3$ interchange within each element of such a pair, as shown in Fig. 6.2). Now for each $\tilde{K} \in \tilde{\tau}_h^2$, let s be the curved abscissa along $\tilde{B}$ with origin at S_2, and $\mathbf{n}(s)$ denote the outer unit normal vector along $\tilde{B}$ with respect to $\tilde{K}$. We also denote by $n_j(s)$ the component of $\mathbf{n}$ with respect to $\vec{x}_j$. Let $\mathbf{w} = (w_1, w_2)$, $\mathbf{w} = \mathbf{w}_{h/K}$ and w_3 and w_4 be given by

$$w_3 = w_1 \cos \varphi + w_2 \sin \varphi, \qquad w_4 = -w_1 \sin \varphi + w_2 \cos \varphi.$$

Now we verify that we can uniquely define w_3 and w_4 (and consequently $\mathbf{w}$) in the following way. The values of w_3 and w_4 at the vertices of $\tilde{K}$ are given by

$$w_3(\tilde{S}_i) = w_4(\tilde{S}_i) = 0, \qquad i = 1, 2, 3;$$

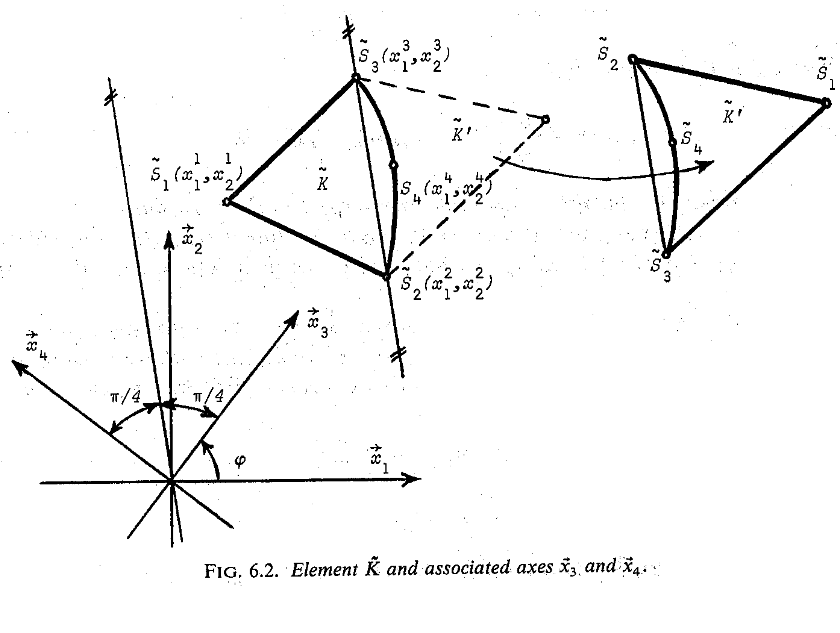

FIG. 6.2. *Element $\tilde{K}$ and associated axes $\vec{x}_3$ and $\vec{x}_4$.*

the value of w_3 and w_4 at node $\tilde{S}_4$ are such that

$$\int_{\tilde{B}} w_j n_j(s)\,ds = \int_{\tilde{B}} v_j n_j(s)\,ds, \qquad j=3,4$$

where v_j is the component of $\mathbf{v}$ with respect to $\tilde{x}_j$, $j=3,4$.

Since $\mathbf{u}_h \in \mathbf{V}_h$, we can compute the coordinates x_3 and x_4 in terms of the reference coordinates $\hat{x}_1$ and $\hat{x}_2$ (see Fig. 2.1) used for defining $\hat{P}_a$ over $\hat{K}$, in the following way:

$$x_3 = [4\xi_1^4 - 2(\xi_1^2+\xi_1^3)]\hat{x}_1\hat{x}_2 + \xi_1^2\hat{x}_1 + \xi_1^3\hat{x}_2 + x_3^1,$$
$$x_4 = [4\xi_2^4 - 2(\xi_2^2+\xi_2^3)]\hat{x}_1\hat{x}_2 + \xi_2^2\hat{x}_1 + \xi_2^3\hat{x}_2 + x_4^1,$$

where $\xi_1^i = x_3^i - x_3^1$ and $\xi_2^i = x_4^i - x_4^1$, $i=2,3,4$ and

$$x_3^i = x_1^i \cos\varphi + x_2^i \sin\varphi,$$
$$x_4^i = -x_1^i \sin\varphi + x_2^i \cos\varphi, \qquad i=1,2,3,4.$$

Using the above relations, we make a change of variables in the integral $\int_{\tilde{B}} w_j n_j(s)\,ds$, $j=3,4$, namely from s to $\hat{s}$, where $\hat{s}$ is the abscissa along the edge $\hat{B}$ of $\hat{K}$ with origin in $\hat{S}_2$ (see Fig. 2.1).

Since we have $n_3(s) = dx_4/ds$ and $n_4(s) = -dx_3/ds$, for a vector field $\mathbf{f}$ defined over $\tilde{B}$, whose components with respect to $\tilde{x}_j$ are f_j, $j=3,4$, we have for the x_3-component

$$\int_{\tilde{B}} f_3 n_3(s)\,ds = \int_{\tilde{S}_2}^{\tilde{S}_3} f_3(s)\frac{dx_4}{ds}\,ds = \int_{\hat{S}_2}^{\hat{S}_3} \hat{f}_3(\hat{s})\left[\frac{\partial x_4}{\partial \hat{x}_1}\frac{d\hat{x}_1}{d\hat{s}} + \frac{\partial x_4}{\partial \hat{x}_2}\frac{d\hat{x}_2}{d\hat{s}}\right]\Big/ \hat{x}_1+\hat{x}_2=1\ d\hat{s}$$

where $\hat{f}_j(\hat{s}) = f_j(s)$. Since $d\hat{x}_i/d\hat{s} = (-1)^i(\sqrt{2}/2)$ we have

$$\int_{\tilde{B}} f_3 n_3(s)\,ds = \int_0^{\sqrt{2}} \frac{1}{\sqrt{2}}\hat{f}_3(\hat{s})\{(\xi_2^3-\xi_2^2) + [4\xi_2^4 - 2(\xi_2^3+\xi_2^2)](1-\hat{s}\sqrt{2})\}\,d\hat{s},$$

whereas an entirely analogous relation holds for the x_4-component.

Now since $\hat{w}_{j/\hat{B}} = 2\hat{w}_j(\hat{S})\hat{s}(\sqrt{2}-\hat{s})$, we have

$$\int_{\tilde{B}} w_3 n_3(s)\,ds = \frac{2}{3}(\xi_2^3-\xi_2^2)w_3(\tilde{S}_4), \tag{6.6}$$

and analogously

$$\int_{\tilde{B}} w_4 n_4(s)\,ds = \frac{2}{3}(\xi_1^2-\xi_1^3)w_4(\tilde{S}_4). \tag{6.7}$$

Since by construction $|\xi_1^3-\xi_1^2| = |\xi_2^3-\xi_2^2| = \sqrt{2}/2$ length $(\hat{B}) \neq 0$, $\mathbf{w}$ can be defined uniquely.

Furthermore, proceeding in the same way for every element, we can define a vector field $\tilde{\mathbf{w}}_h \in \tilde{\mathbf{V}}_h$ such that

$$\int_{\tilde{B}} \tilde{w}_h \cdot \mathbf{n}(s)\,ds = \int_{\tilde{B}} \mathbf{v}\cdot\mathbf{n}(s)\,ds \quad \text{for every base } \tilde{B} \text{ of } \tilde{K} \in \tilde{\tau}_h.$$

This yields

$$\int_{\tilde{\Omega}_h} \tilde{q}_h\, \tilde{\text{div}}\, \tilde{w}_h\, d\tilde{\mathbf{x}} = \int_{\tilde{\Omega}_h} \tilde{q}_h\, \tilde{\text{div}}\, \tilde{\mathbf{v}}\, d\tilde{\mathbf{x}} \quad \forall \tilde{q}_h \in \tilde{Q}_h^0,$$

and consequently (6.2) holds, taking into account (6.4). On the other hand, we have

$$\|\tilde{\mathbf{w}}_h\|^2 = \sum_{\tilde{K}\in\tilde{\tau}_h^2} \int_{\tilde{K}} (|\nabla w_1|^2 + |\nabla w_2|^2)\,d\tilde{\mathbf{x}} = \sum_{\tilde{K}\in\tilde{\tau}_h^2} \int_{\tilde{K}} (|\nabla w_3|^2 + |\nabla w_4|^2)\,d\tilde{\mathbf{x}}.$$

But

$$\int_{\tilde{K}} |\nabla wj|^2 \, d\tilde{\mathbf{x}} = w_j^2(\tilde{S}_4) \int_{\tilde{K}} |\nabla \tilde{p}_4|^2 \, d\tilde{\mathbf{x}}, \qquad j = 3, 4$$

where $\tilde{p}_4(\tilde{\mathbf{x}}) = \hat{p}_4(\hat{\mathbf{x}})$, $\hat{p}_4(\hat{\mathbf{x}}) = 4\hat{x}_1\hat{x}_2$, $\tilde{\mathbf{x}} = \mathbf{u}^x_{h/K}[\mathscr{A}_K^{-1}(\hat{\mathbf{x}})]$. Now, according to Assumption B and standard estimates we have

$$\int_{\tilde{K}} |\nabla \tilde{p}_4|^2 \, d\tilde{\mathbf{x}} \leqq C \frac{h^2 |\mathbf{u}_h|^2_{1,\infty}}{\tilde{\rho}_K^2}$$

where $\tilde{\rho}_K$ denotes the diameter of the inscribed circle in $\tilde{K}$.

Now if area $(\tilde{K}) \geqq$ area $(\acute{K})$ we clearly have

$$\tilde{\rho}_K \geqq \acute{\rho}_K \geqq \frac{2 \text{ area } (\acute{K})}{3\acute{h}_K} \geqq \frac{2\alpha \text{ area } (K)}{3h|\mathbf{u}_h|_{1,\infty}}.$$

If area $(\tilde{K}) \leqq$ area $(\acute{K})$, we use Assumption B together with the geometrical arguments sketched in Fig. 6.3 (we omit the details for the sake of conciseness). It is then possible to prove that $\tilde{\rho}_K$ is greater than the diameter of the inscribed circle in a triangle $\acute{K}'$, which is defined to be the homothetical reduction of $\acute{K}$ with ratio $\frac{1}{2}$.

Hence we have in this case

$$\tilde{\rho}_K \geqq \frac{\frac{1}{4} \text{ area } (\acute{K})}{\frac{3}{4} \acute{h}_K} \geqq \frac{1}{3} \frac{\text{area } (K)}{h|\mathbf{u}_h|_{1,\infty}}.$$

This gives

$$\int_{\tilde{K}} |\nabla \tilde{p}_4|^2 \, d\tilde{\mathbf{x}} \leqq \frac{C}{[\frac{2}{3} \min (\alpha, \frac{1}{2})]^2} \frac{h^4 |\mathbf{u}_h|^4_{1,\infty}}{[\text{area } (K)]^2} \leqq \frac{C'}{c^4} |\mathbf{u}_h|^4_{1,\infty}$$

where c is the constant of regularity of $\{\tau_h^2\}_h$ (see § 2).

On the other hand, using (6.6), (6.7), and the Trace Theorem, we have by construction

$$|w_j(\tilde{S}_4)| \leqq \frac{\int_{\tilde{B}} |v_j| \, ds}{\sqrt{2/3} \, \acute{\rho}_K} \leqq C \frac{\|\mathbf{v}\|_{1,\tilde{K}} |\mathbf{u}_h|_{1,\infty}}{h}.$$

Therefore from the Poincaré inequality we obtain

$$\|\tilde{\mathbf{w}}_h\| \leqq Ch^{-1} |\mathbf{u}_h|^3_{1,\infty} \|\mathbf{v}\|_{1,\tilde{\Omega}_h} \leqq C(\tilde{\Omega}_h) h^{-1} |\mathbf{u}_h|^3_{1,\infty} |\tilde{\mathbf{v}}|_{1,\tilde{\Omega}_h},$$

which from (6.5) proves (6.3) with $\tilde{C}_h = C(\tilde{\Omega}_h)\tilde{C}|\mathbf{u}_h|^3_{1,\infty}/h$. □

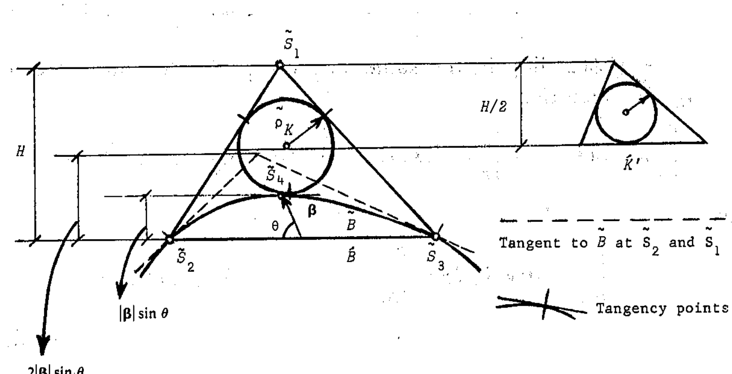

FIG. 6.3. *Triangles $\tilde{K}$ and $\acute{K}$ when* area $(\tilde{K}) \leqq$ area $(\acute{K})$.

Let us now construct a vector field $\tilde{\mathbf{z}}_h \in \tilde{\mathbf{V}}_h$ associated with the subspace $\tilde{Q}_h^1$ of $\tilde{Q}_h$, such that $\tilde{Q}_h = \tilde{Q}_h^0 \oplus \tilde{Q}_h^1$. As is space $\tilde{Q}_h^1$ of Case i, $\tilde{Q}_h^1$ is spanned by a set of orthogonal basis functions $\{\gamma_2^K, \gamma_3^K\}_{\tilde{K} \in \tilde{\tau}_h}$ defined in an entirely analogous way (for $s_1 = s_2 = s_3$). Now we prove Lemma 6.2.

LEMMA 6.2. *Let $\tilde{q}_h^1$ be a function of $\tilde{Q}_h^1$ whose components with respect to γ_2^K and γ_3^K are, respectively, ξ_2^K and ξ_3^K, $K \in \tau_h$. Under Assumption* B, *the vector field $\tilde{\mathbf{z}}_h \in \tilde{\mathbf{V}}_h$, which vanishes at all the vertices of $\tilde{\tau}_h$ and whose value at the common vertex $\tilde{G}$ of $\tilde{K}_i$, $i = 1, 2, 3$, $\tilde{K}_i \subset \tilde{K}$ is given by* (*refer to Fig.* 6.4)

$$\tilde{\mathbf{z}}_h(\tilde{G}) = -\xi_2^K \mathbf{m}_2 + \xi_3^K \mathbf{m}_3,$$

satisfies

$$\|\tilde{\mathbf{z}}_h\| \leqq C(\mathbf{u}_h)|\tilde{q}_h^1|, \qquad C(\mathbf{u}_h) < \infty, \tag{6.8}$$

and

$$\int_{\tilde{\Omega}_h} \tilde{q}_h^1 \, \tilde{\operatorname{div}}\, \tilde{\mathbf{z}}_h \, d\tilde{\mathbf{x}} \geqq \tilde{\beta}_1 |\tilde{q}_h^1|^2 \quad \text{with } \tilde{\beta}_1 > 0. \tag{6.9}$$

Proof. Formula (6.8) is a trivial consequence of the definition of $\tilde{\mathbf{z}}_h$.

On the other hand, a straightforward computation gives

$$\int_{\tilde{K}} \tilde{q}_h^1 \, \tilde{\operatorname{div}}\, \tilde{\mathbf{z}}_h \, d\tilde{\mathbf{x}} = (s_3 + s_1)\frac{3\xi_2^2}{2} + (2s_1 + s_2)\xi_3^2 + (3s_2 + s_3 - s_1)\xi_2\xi_3$$

where $s_i = \text{area}(\acute{K}_i)$, $i = 1, 2, 3$.

Assuming again that the local numbering of the nodes of K is such that $s_1 \geqq s_2 \geqq s_3$, we have

$$\tfrac{1}{4}\left(\tfrac{3}{2}s_2 + s_3 + s_1\right)^2 - \tfrac{1}{2}(s_1 + 3s_3)(s_1 + s_2) < 0,$$

assuming that $s_1 \geqq \alpha$ area $(K)/3 > 0$.

Thus we can write

$$\int_{\tilde{K}} \tilde{q}_h^1 \, \tilde{\operatorname{div}}\, \tilde{\mathbf{z}}_h \, d\tilde{\mathbf{x}} \geqq \frac{\alpha}{3}(\xi_2^2 + \xi_3^2) \text{ area } (K),$$

which yields (6.9) with $\tilde{\beta}_1 = 2\alpha/9$. □

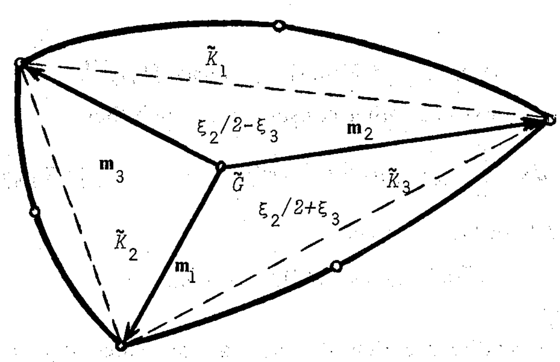

FIG. 6.4. *Superelement* $\tilde{K}$.

Now, defining $\tilde{\mathbf{v}}_h = \theta\, \tilde{\mathbf{w}}_h + \tilde{\mathbf{z}}_h$, from (6.2), (6.3), (6.8), and (6.9), we obtain (6.1) as in Lemma 5.2, for a sufficiently small θ. Hence, as an immediate consequence of Theorem 6.1 and of Lemmas 6.1 and 6.2, we have Theorem 6.2.

THEOREM 6.2. *Under Assumption* B *the compatibility condition* (2.8) *holds for Case* ii.

7. Final remarks and computer tests. Finally, let us consider some relevant aspects related to the problem studied in this paper from both theoretical and computational points of view.

First, Assumption B (hence A), which was crucial to the analysis carried out in §§ 4–6, is far from restrictive in most cases, and fairly reasonable as we can check using different arguments. For example, we will now verify this assumption in a particularly simple case where the loads **f** and **g** are "sufficiently small."

Noting that $\mathbf{0} \in X_h$, we let $n = 2$ and $\mathbf{u}_h$ be a local solution to the minimization problem $(\mathbf{P}_h)$ that satisfies

$$W_h(\mathbf{u}_h) \leqq W_h(\mathbf{0}) = 0.$$

Denoting by Γ_{0_h} and Γ_h^* the portions of the boundary of Ω_h approximating Γ_0 and Γ^*, respectively, we have

$$\begin{aligned} W_h(\mathbf{u}_h) &= C_1\left[\int_{\Omega_h} |\mathbf{I}+\nabla\mathbf{u}_h|^2\, d\mathbf{x} - \int_{\Omega_h} \mathbf{f}\cdot\mathbf{u}_h\, d\mathbf{x} - \int_{\Gamma_h^*} \mathbf{g}_h\cdot\mathbf{u}_h\, ds - 2\,\text{meas}\,(\Omega_h)\right] \\ &= C_1\int_{\Omega_h} (-2\,\text{div}\,\mathbf{u}_h + |\nabla\mathbf{u}_h|^2)\, d\mathbf{x} - \int_{\Omega_h} \mathbf{f}\cdot\mathbf{u}_h\, d\mathbf{x} - \int_{\Gamma_h^*} \mathbf{g}_h\cdot\mathbf{u}_h\, ds \end{aligned}$$

where $\mathbf{g}_h$ denotes the usual approximation of $\mathbf{g}$ over Γ_h^*. Since $\int_{\Omega_h} \text{div}\,\mathbf{u}_h\, d\mathbf{x} = \int_{\Omega_h} \det \nabla\mathbf{u}_h\, d\mathbf{x}$ because $\mathbf{u}_h \in X_h$, after straightforward calculations we obtain

$$C_1\int_{\Omega_h} (|\text{div}\,\mathbf{u}_h|^2 + |\text{curl}\,\mathbf{u}_h|^2)\, d\mathbf{x} \leqq \int_{\Omega_h} \mathbf{f}\cdot\mathbf{u}_h + \int_{\Gamma_h^*} \mathbf{g}\cdot\mathbf{u}_h\, ds. \tag{7.1}$$

Since $\text{div}\,\mathbf{v} = \text{curl}\,\mathbf{v} = 0$ in Ω_h and $\mathbf{v} = \mathbf{0}$ on Γ_{0_h} implies $\mathbf{v} = \mathbf{0}$ almost everywhere in Ω_h,

$$|||\mathbf{v}||| = \left[\int_{\Omega_h} (|\text{div}\,\mathbf{v}|^2 + |\text{curl}\,\mathbf{v}|^2)\, d\mathbf{x}\right]^{1/2} \tag{7.2}$$

is a norm over $\mathbf{V}' = \{\mathbf{v}/\mathbf{v} \in [H^1(\Omega)_h]^2,\ \mathbf{v} = \mathbf{0} \text{ over } \Gamma_{0_h}\}$. Thus, taking into account that $\mathbf{V}_h \subset \mathbf{V}'$ is a finite-dimensional space, there exists a constant C_h^* such that

$$\|\nabla\mathbf{v}_h\|_{0,\infty,\Omega_h} \leqq C_h^* |||\mathbf{v}_h||| \quad \forall \mathbf{v}_h \in \mathbf{V}_h. \tag{7.3}$$

On the other hand, using the Poincaré inequality, the Trace Theorem, and the obvious bound

$$\|\nabla\mathbf{u}_h\|_{0,2,\Omega_h} \leqq \|\nabla\mathbf{u}_h\|_{0,\infty,\Omega_h}[\text{meas}\,(\Omega_h)]^{1/2},$$

we conclude that there exists another constant C_h^{**} such that

$$\int_{\Omega_h} \mathbf{f}\cdot\mathbf{u}_h\, d\mathbf{x} + \int_{\Gamma_h^*} \mathbf{g}_h\cdot\mathbf{u}_h\, ds \leqq C_h^{**}(\|\mathbf{f}\|_{0,2,\Omega_h} + \|\mathbf{g}_h\|_{0,2,\Gamma_h^*})\|\nabla\mathbf{u}_h\|_{0,\infty,\Omega_h}. \tag{7.4}$$

Using (7.2)–(7.4) in (7.1), we get

$$\|\nabla\mathbf{u}_h\|_{0,\infty,\Omega_h} \leqq C_h(\|\mathbf{f}\|_{0,2,\Omega_h} + \|\mathbf{g}_h\|_{0,2,\Gamma_h^*}) \tag{7.5}$$

where $C_h = C_h^{**} C_h^* / C_1$.

From (7.5) we readily see that if **f** and **g** are sufficiently small the relation $\det(\mathbf{I} + \nabla\mathbf{u}_h) > 0$ for all $\mathbf{x} \in \Omega_h$ applies.

For the case $n=3$ a similar argument leads to the same conclusion, but we omit it here for the sake of conciseness.

Second, the analyses given in this paper have been supported by several numerical experiments in both two and three dimensions, using asymmetric elements. Even in tests where strong compression was applied they showed a stable and realistic behavior, whereas standard elements that are expected to satisfy the compatibility condition (2.8), such as the isoparametric bilinear or trilinear rectangular element for displacements with a piecewise constant pressure [10], did not. Therefore, it seems that one of the key factors in the success of these elements for the problem under study is the argument developed in § 4. Indeed, it ensures the "minimum rigidity" to avoid elements turning inside out or hourglassing effects.

We give below an illustration of computer results obtained with our element, using the partition τ_h^2 for a two-dimensional problem. The test consists of applying to the upper base of a cylindrical body, whose lower base is kept fixed, a displacement in the direction of its axis, in such a way that the two bases remain parallel to each other in the deformed state. In Fig. 7.1 we give an illustration of the meridian section of the body in its initial configuration, together with the mesh used in connection with partition τ_h, generated with the MODULEF code.

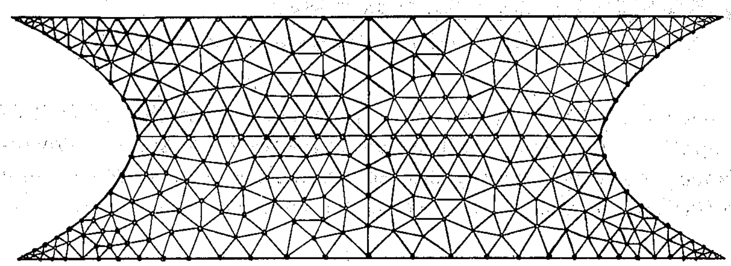

FIG. 7.1. *Meridian section of the cylindrical body.*

Due to symmetry we considered only one quarter of the meridian section in the computations, which were performed for a displacement corresponding to a compression of 25 percent of the body height.

An illustration of this meridian section with the triangles of τ_h^2 in deformed state is given in Fig. 7.2. These computer results obtained with the MODULEF code are in quite good agreement with experimental solutions obtained for similar problems, to be found in the technical literature.

Third, in most of the numerical experiments that we have carried out so far, partitions of type τ_h^1 have been used, because they are less time-consuming. These appeared to be comparable to partitions of type τ_h^2 as far as reliability and accuracy are concerned. Particularly significant examples of large strain simulation for rubber cubes using a partition of type τ_h^1 can be found in [13] and [18] (compression of up to 40 percent).

Fourth, in [13] we give an analysis of existence of solutions applying to partitions of type τ_h^1 in two dimensions, for a particular case. A more detailed analysis for this kind of partition in both two- and three-dimensional cases will be the object of a forthcoming paper.

Acknowledgments. The numerical results given in § 7 were obtained by combining the author's finite-element method with an algorithm of augmented Lagrangian type

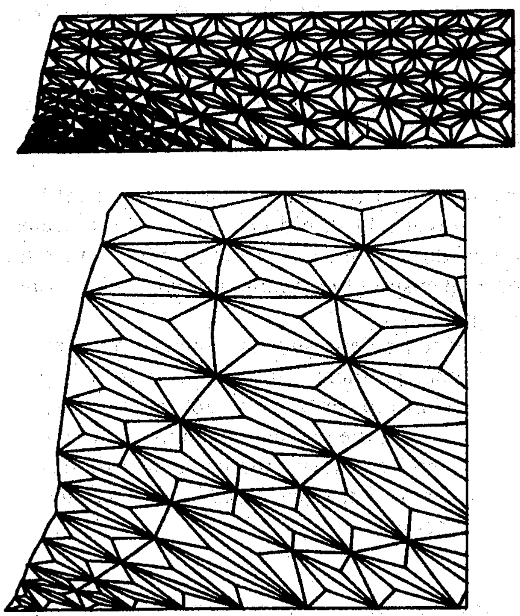

FIG. 7.2. *Quarter meridian section of the body subjected to a compression of 25 percent.*

due to Glowinski and Le Tallec (see, e.g., [6]), implemented in the MODULEF code, for solving nonlinear problem (**P**). The author thanks Professor R. Glowinski and Dr. P. Le Tallec for having so kindly allowed him the use of the code for running this test, at both INRIA and Laboratoire Central des Ponts et Chaussées in Paris.

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