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A Logarithmic Competitive Mechanism for Unit-Demand Combinatorial Auctions

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Abstract. This paper studies the unit-demand combinatorial auction problem in which each of the n consumers can purchase at most one item from a set of k items. We obtain a randomized truthful mechanism that attains an expected competitive ratio of $O(\log(\min\{k, n\}))$ with high probability.

Keywords: Combinatorial Auctions, Approximation Algorithms, Randomized algorithms.

Resumo: Este artigo estuda o problema de leilão combinatório com demanda unitária no qual cada um dos n consumidores podem comprar no máximo um item de um conjunto de k itens. Obtemos um mecanismo truthful aleatório que alcança, com alta probabilidade, uma taxa de competitividade esperada da ordem de $O(\log(\min\{k, n\}))$.

Palavras-chave: Leilões Combinatórios, Algoritmos de Aproximação, Algoritmos Aleatorizados.

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1 Introduction

The unit-demand combinatorial auction problem *(UDCAP)* [1] is the problem of designing a mechanism for an auction in which multiple distinct items are being sold but each consumer can only purchase a single item. Each consumer bids on all items, and the seller uses those bids to decide which consumer receives which item as well as the sale price for each item. Each consumer aims to maximize their own profit, i.e., the difference between the maximum value they would pay for a given item and the actual price paid for that item. On the other hand, the seller intends to maximize the total revenue, that is, the sum of the prices paid for all the items. The auction runs in a single round and it is assumed there is no communication among the consumers (sealed-bid auctions).

An auction is *truthful* if truth-telling is a dominant strategy, that is, the profit of a consumer is maximized when the consumer bids his utility value, regardless of his belief about the other consumers bids. A truthful auction is α-competitive if the revenue of the auction is $\Omega(M/\alpha)$, where M is an upper bound on the revenue of any truthful auction mechanism.

Our Result. We devise a randomized truthful mechanism that attains an expected competitive ratio of $O(\log(\min\{n, k\}))$ with high probability, where n and k are the number of consumers and items, respectively.

2 A Logarithmic Competitive Mechanism for Unit-Demand Combinatorial Auctions

Let n be the number of consumers bidding on k items, numbered 1 through k. Let $u_{i,j}$ be the utility of item j to consumer i.

Let G be the complete bipartite graph among consumers and items where the weight of edge $e_{i,j}$ (associating the i-th consumer to the j-th item) is $u_{i,j}$.

Let M be a matching in G and let l be the weight of the lightest edge of M. We define

$$w(M) = |M|l$$

and

$$c(M) = \sum_{e=ij \in M} u_{i,j}.$$

In addition, let M^* be such that $w(M^*) = \max_{M \in \mathcal{M}}\{w(M)\}$, where $\mathcal{M}$ is the set of all possible matchings in G. On the other hand, let M^{opt} be the matching in G that maximizes $c(\cdot)$ among all the matchings of $\mathcal{M}$. We have that $c(M^{opt})$ is an upper bound on the revenue that any auction can generate.

The next proposition plays a key role in the mechanisms proposed here.

Proposition 1 $w(M^*) = \Omega(c(M^{opt})/\log(\min\{n, k\}))$.

Consider the following mechanism:

Mechanism 1:

1. Flip a fair coin n times to split the consumers into two groups. Let G_1 (G_2) be the bipartite graph induced by the consumers of the first (second) group and the set of all items.

2. Let w^* be the maximum value of function $w(\cdot)$ among all matchings in G_1. In addition, let M^1 be the largest (w.r.t the number of edges) matching in G_1 such that $w(M^1) \geq w^*/3$.

3. Let M^{2^*} be the matching that maximizes $c(\cdot)$ among all the matchings of size $|M^1|/6-1$ in G_2. For every consumer i, let $G_2 \setminus i$ be the graph obtained from G_2 when consumer i is removed. Then, let $M^{2^*}_{-i}$ be the matching which maximizes $c(\cdot)$ among all matchings in $G_2 \setminus i$ of size $|M^1|/6-1$.

4. Use matching M^{2^*} to define which consumer buys each item. Let the sale price of item j being sold to consumer i be $u_{i,j} - c(M^{2^*}) + c(M^{2^*}_{-i})$.

Lemma 1 *Mechanism 1 is truthful.*

Theorem 1 *Mechanism 1 has revenue of* $\Omega(c(M^{opt})/\log(\min\{n,k\}))$ *with high probability.*

Proof: For every j, let M_j be a matching of size j in G which maximizes $w(\cdot)$ (ties are broken arbitrarily). Furthermore, let I be the set of integers such that $i \in I$ if and only if:

- $i \geq |M^*|/3$; and
- $w(M_i) \geq w(M^*)/9$.

For every $i \in I$, let C_i be the set of consumers of matching M_i. With respect to the step (i) of the mechanism above, we define event E_i as the event in which the number of consumers of C_i that lie in the first group is at least $i/3$ and at most $2i/3$. Furthermore, let $E = \bigcup_{i \in I} E_i$.

In what follows we make some observations under the assumption that E occurs. Since at least $|M^*|/3$ consumers of matching M^* lie in the first group there is a matching M in G_1 with $w(M) \geq w(M^*)/3 \geq w^*/3$ and at least $|M^*|/3$ edges. Thus, we conclude that $w^* \geq w(M^*)/3$ and $|M^1| \geq |M^*|/3$.

Since $w(M^1) \geq w^*/3$ it follows that $w(M^1) \geq w(M^*)/9$ and as a consequence $|M^1| \in I$. Therefore, there are at least $|M^1|/3$ consumers of $C_{|M^1|}$ in the second group and, so, there is a matching in G_2, say M^2, of size $|M^1|/3$ where the lowest-priced item costs at least $w(M^*)/(9|M^1|)$.

Now we will argue that the price charged for each item in step (iv) is $\Omega(w(M^*)/|M^1|)$. Note that M^2 has strictly more than twice the number of edges as does M^{2^*}, what means that there is at least one edge $e = uv$ in M^2, such that neither the consumer u nor the item v belong to any edge in M^{2^*}. Hence, $M^{2^*} \cup e$ is a matching in G_2. Because every edge in M^2 costs at least $w(M^*)/(9|M^1|)$, we can conclude that $c(M^{2^*}_{-i}) \geq c(M^{2^*}) - u_{i,j} + w(M^*)/(9|M^1|)$. Thus, the price charged for each item is at least $w(M^*)/(9|M^1|)$. Since the number of items sold is $|M^1|/6-1$, it follows that the total revenue is at least $\Omega(w(M^*)) = \Omega(c(M^{opt})/\log(\min\{n,k\}))$, where the last equality follows from Proposition 1.

Now, it remains to show that event E happens with high probability. By using the Chernoff Bound we have that the probability of event E_i not happenning is at most $2e^{-i/36}$. Applying the union bound we get that the probability of failure of E is at most $\sum_{i \in I} 2e^{-i/36}$. Thus, the event E happens with high probability. ■

Now we show that the mechanism runs in polynomial time. First we argue that we can efficiently calculate w^* and then $|M^1|$. For a real value x, let $K(x)$ be the largest matching in $G_1 = (V_1, E_1)$ where all the edges cost at least x. It is not hard to see that

$$w^* = \max_{e=ij \in E_1} \{u_{i,j} |K(u_{i,j})|\}$$

and that $|M^1| = \max_{e=ij \in E_1}\{|K(u_{i,j})|\}$ constrained to $|K(u_{i,j})| \times u_{i,j} \geq w^*/3$.

Finally, we argue that we can calculate the matching which maximizes $c(\cdot)$ among all matchings of a given size in polynomial time. This is used to calculate M^{2^*} and $M^{2^*}_{-i}$ at step (iii). Given a weighted bipartite graph $H = (V = V^1 \cup V^2, E)$ we construct a weighted unit-capacity network flow $N = (V \cup \{s,t\}, E \cup E')$, where $E' = \{(s,v) | v \in V^1\} \cup \{(v,t) | v \in V^2\}$. All of the edges in E' have weight 0 and the weights of the edges in E are multiplied by -1. The size of the matching which maximizes $c(\cdot)$ in H among all the matchings of a given size, say x, corresponds to the minimum cost flow in N when the the terminal needs to receive x units of flow from s.

References

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