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A FINITE ELEMENT COMPUTER PROGRAM FOR PLANE ELASTICITY WITH CONSIDERATION OF THERMAL EFFECTS

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SUMMARY

In the present report, a programming method for the technique of finite elements in plane elastic problems will be developed. The characteristic feature in the programming method is that it approaches closely the treatment in the theory of structures.

The program was written in FORTRAN IV for a 7044 IBM computer. Numerical applications to some examples are added as an illustration.

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1. INTRODUCTION

The technique of finite elements offers one of the most promising fields, for research and application in the solution of complex structural problems. The success of the method may be ascribed to the widespread availability of digital computers and the simple, unsophisticated formulation of boundary conditions.

Another advantage of the method is that it makes use of the well known notions in the theory of structures, so that its peculiar programming technique can be adhered to.

In our case, the program was inspired in the work of Gere and Weaver (3),(4) for structures made up of linear elements.

For the time being, triangular elements were envisaged. It is intended to enlarge the program, in order to include also elements of other shapes.

An interesting feature in the method is that it makes the consideration of thermal effects possible in a simple way, without introduction of additional complications.

In the interest of the program's user, the input data will be supplied by a direct examination of a sketch of the structure. The numbering of points and elements is quite arbitrary. By using a certain discipline in the numbering of elements an economy in storage space can be achieved.

2. BASIC RESULTS FOR TRIANGULAR PLATE ELEMENTS

We shall include a summary of basic results for triangular plate elements subject to in-plane forces. We refer to Przemieniecki (1) for demonstrations.

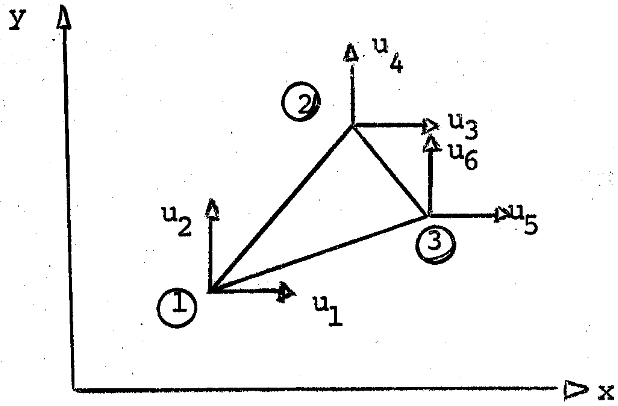


Fig. 1

In figure 1, we depict a triangular element with joints numbered from 1 to 3 with corresponding displacements $u_1, u_2, \dots u_6$.

It can be shown that, if arbitrary displacements are induced at the joints of the element, along with an increase of temperature T in the whole element, the force components $S_1, S_2, S_3 \dots S_6$ in the joints are determined form the matrix equation

$$\{S\}=[K]\{u\} + \{h\}\,\alpha\,T \quad (1)$$

where different quantitiesand symbols are defined below.

$[K]$ is the symmetric stiffness matrix of the element, given by $[K]=[K_n]+[K_s]$

where

$$[K_n]=\frac{Et}{4A_{123}(1-\nu^2)}\begin{bmatrix} y_{32}^2 & & & & & \\ -\nu y_{32}x_{32} & x_{32}^2 & & & & \\ -y_{32}y_{31} & \nu x_{32}y_{31} & y_{31}^2 & & & \\ \nu y_{32}x_{31} & -x_{32}x_{31} & -\nu y_{31}x_{31} & x_{31}^2 & & \\ y_{32}y_{21} & -\nu x_{32}y_{21} & -y_{31}y_{21} & \nu x_{31}y_{21} & y_{21}^2 & \\ -\nu y_{32}y_{21} & x_{32}x_{21} & \nu y_{31}x_{21} & -x_{31}x_{21} & -\nu y_{21}x_{21} & x_{21}^2 \end{bmatrix} \quad (2')$$

and

$$[K_s]=\frac{Et}{8A_{123}(1+\nu)}\begin{bmatrix} x_{32}^2 & & & & & \\ -x_{32}y_{32} & y_{32}^2 & & & & \\ -x_{32}x_{31} & y_{32}x_{31} & x_{31}^2 & & & \\ x_{32}y_{32} & -y_{32}y_{31} & -x_{31}y_{31} & y_{31}^2 & & \\ x_{32}y_{21} & -y_{32}x_{21} & -x_{31}x_{21} & y_{31}x_{21} & x_{21}^2 & \\ -x_{32}y_{21} & y_{32}y_{21} & x_{31}y_{21} & -y_{31}y_{21} & -x_{21}y_{21} & y_{21}^2 \end{bmatrix} \quad (2'')$$

In (2') and (2''), $x_{mn}=x_m-x_n$, $y_{mn}=y_m-y_n$, with m,n=1,2,3. (x_1y_1), (x_2,y_2) are the cartesian coordinantes of the joints in Fig. 1.

Also, $A_{123} = 1/2|x_{32}y_{21} - x_{21}y_{32}|$, the area of the triangle 1,2,3 and E,ν, t are the elastic constants and the thickness of the element respectively.
{h} is a vector related with thermal effect, whose components are

$$\{h\} = \frac{Et}{2(1-\nu)} \begin{Bmatrix} -y_{32} \\ x_{32} \\ y_{31} \\ -x_{31} \\ -y_{21} \\ x_{21} \end{Bmatrix} \quad (3)$$

Finally, {S}, {u} are the joint force and joint displacements vectors, ie,

$$\{S\} = \begin{Bmatrix} S_1 \\ S_2 \\ S_3 \\ S_4 \\ S_5 \\ S_6 \end{Bmatrix}; \quad \{u\} = \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \end{Bmatrix} \quad (4)$$

αt is the coefficient of thermal expansion, times the temperature.
If can be seen from (1), that $\{h\}\alpha$ T is the joint force vector, arising form a temperature change in the element, if no joint displacements are allowed, ie, if {u}=0.
This consideration is important, because it suggests the treatment of thermal effects as a special kind of modal forces, as it was effectively done in the program.
As soon as the displacements are determined, the stresses are evaluated according to the formula

$$\begin{Bmatrix} T_x \\ T_y \\ T_{xy} \end{Bmatrix} = \frac{E}{2A_{123}(1-D^2)} \begin{bmatrix} y_{32} & -\nu x_{32} & -y_{31} & \nu x_{31} & y_{21} & -\nu x_{21} \\ \nu y_{32} & -x_{32} & -\nu y_{31} & x_{31} & \nu y_{21} & x_{21} \\ \frac{-(1-\nu)x_{32}}{2} & \frac{(1-\nu)y_{32}}{2} & \frac{-(1-\nu)y_{31}}{2} & \frac{-(1-\nu)y_{31}}{2} & \frac{-(1-\nu)x_{21}}{2} & \frac{(1-\nu)y_{21}}{2} \end{bmatrix} x$$

$$x \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \end{Bmatrix} - \frac{E\alpha T}{1-\nu} \begin{Bmatrix} 1 \\ 1 \\ 0 \end{Bmatrix} \qquad (5)$$

3. THE MAIN PROGRAM FEATURES

The finite element program must accomplish the following basic tasks:

a) input of data, including cartesian coordinates of joints, information concerning the incidence of elements in the joint network, properties of the structure and degrees of freedom;

b) computation of the stiffness matrix of each element;

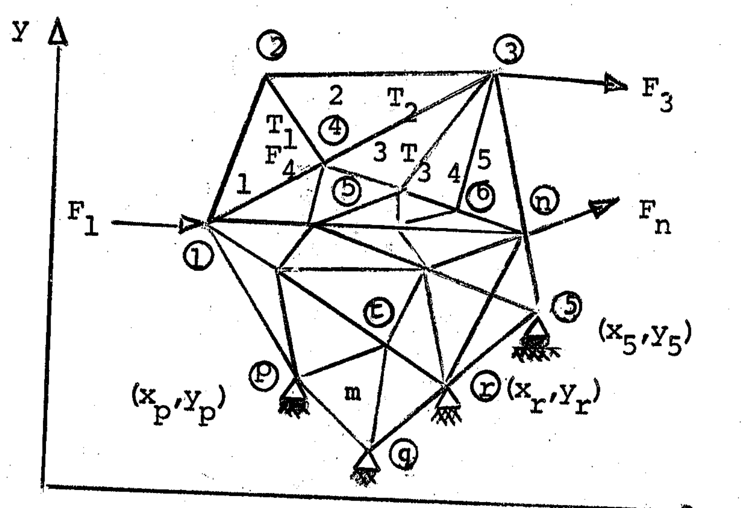
c) construction of the overall stiffness matrix of the structure from the stiffness matrix of the elements;

d) input of loading and temperature data;

e) determination of the free joint displacements;

f) evaluation of stresses.

Let us consider an arbitrary two-dimensional elastic body sketched in Fig.2, where we have stablished an appropriate subdivision into triangular finite elements.



The joints are numbered in an arbitrary way, starting from 1. Joint are enclosed in circles. The elements are also numbered in sucession and the element number is written on each element, without a circle. There is no necessary connection between the element and joint numbers.

We may have arbitrary forces F_1, F_2,... applied at all or some joints.

The elements may also be subject to arbitrary temperature variations T_1, T_2, T_3....
A number of joints, as ⓟ ⓠ, ⓡ and ⓢ must contain restrictions, in order to preclude any rigid body motion of the system. The positions of joints are determined by their cartesian coordinates, refered to the general reference frame x,y.

The computer program must contain information regarding the joint numbers, which are associated to a certain element, so that the overall stiffness matrix is assembled correctly.

For example, the element "m" is associated to ⓟ, ⓠ, ⓣ.

Before we get down to the description of the flow chart, it is interesting to discuss the physical motivation of the way in which the over-all stiffness matrix is assembled form stiffness matrices of the elements.

First, we imagine that all joints in the element assembly are completely restricted against translation. In such a case, all forces can be resolved into joint forces. These are the external applied joint forces and the joint forces necessary to restrict the translation of joints in the case of a temperature variation. In a next stage, we relax the joints, by assuming motions of the joints of each element in the direction of the coordinate axes.

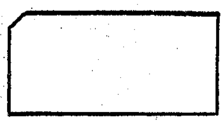
Such a task is performed mathematically, by determining the joint displacements as a result of modal forces.

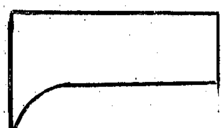
The stresses in all elements are determined in a subsequent stage from the joint displacements.

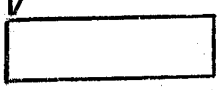
The rules employed to build up the overall stiffness matrix of the finite element assembly, as well the identification of the degrees of freedom, are similar to the analogous rules in the theory of structures (3) (4).

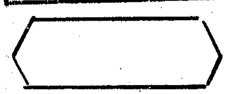
A FLOW CHART

A flow chart for the finite element program is given in the appendix. The following flow chart conventions are observed.


- input


- output


- assignment statement


- iteration


- logical decision

The main symbols used in the program are listed below:

M - number of finite elements

NJ - number of joints

NR - number of restrictions

NRJ - number of restricted joints

E,NI - modulus of elasticity E and Poisson ratio ν

T_1 - thickness of plate

ALFA - coefficient α of thermal expansion

N - number of degrees of freedom

X(I), Y(I) - vectors for storage of cartesian coordinates of the joints

JJ(I),JK(I),JL(I) - vectors used to define the incidence of joints in elements

RL(K) - vector used in order to define degrees of freedom and restrictions ("0" for degree of freedom; "1" for a restriction).

CRL(K) - vector of cumulative restrictions

T(I) - temperature vector

A(K) - joint force vector

SM(I,J) - stiffness matrix of a finite element

S(I,J) - assembled stiffness matrix of the whole structure

D(J) - joint displacement vector

AK(K) - vector of restricted joint reactions.

SUBROUTINE STHMTR

This subroutine computes the stiffness matrix of a finite element with the help of formulas (2'), (2"), along with the temperature vector given by formula (3). The results are stored in SM(...) and H(...) respectively.

Next we shall make some very brief comments on the flow chart. In section I-II of the flow chart, basic data, as number of elements, number of joints, elastic constants, joint coordinates and so on are read in.

Section II-III of the chart identifies degrees of freedom and restrictions and reads in temperature information for processing.

Section III-IV is most important in the program. The stiffness matrix of each element is computed by means of the subroutine STHMTR and located temporarly in SM(...). The stiffness matrices of the elements are transferred to the overall stiffness matrix of the structures S(...), after an appropriate identification of degrees of freedom and restrictions.

In section IV-V, joint loads are read in and combined with joint loads from temperature effects, and the result rearranged, in order to meet the requirements concerning degrees of freedom and restrictions.

In V-VI, the linear system of equations for the displacement is solved in terms of the joint loads. A subroutine based on Choleski algorithm was used.

Section VI-VII refers to a rearrangement of displacement and fixed joint reactions for printing purposes.

Finally, in section VII-VIII, the stresses are computed from the displacements.

The program was written in FORTRAN IV, for a 7044 IBM computer.

5. ILLUSTRATIVE EXAMPLES

As an illustrative, two special examples were calculated by means of the program

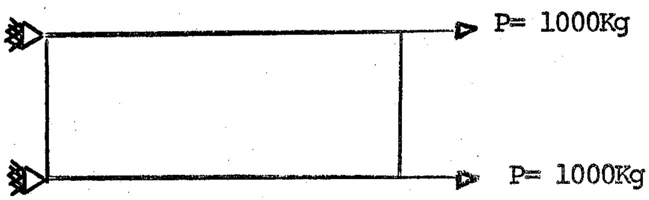


Fig. 2

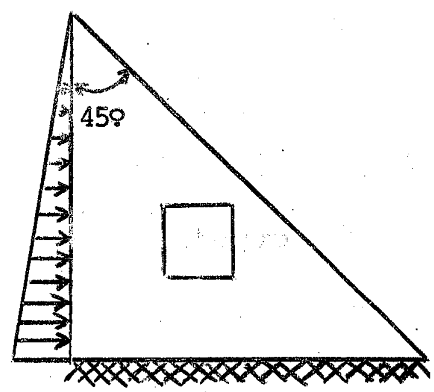


Fig. 3

The examples are shown in Fig. 2 and Fig. 3. The first example is a plate subject to two concentrated equal forces applied at the corners on one side and completely restricted against translation at the opposite corners.

The second example (Fig.3) is a gravity dam with a rectangular hole and subject to a hydrostatic load.

In the first case, different numbers of finite elements were considered in order to appreciate how the solution approaches the continuous model. Also a temperature variation was assumed in all elements.

We shall now list the results in each case.

5.1 Examples of Fig. 2

a)

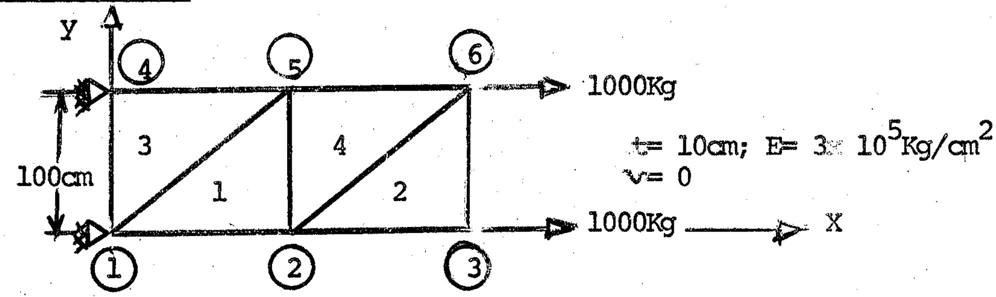


Fig. 4

I) Joint coordinates (cm)

Joint	Joint x	coordinates y
1	0	0
2	100	0
3	200	0
4	0	100
5	100	100
6	200	100

II) Joint Restrictions: 1 and 4, horizontal and vertical

III) Joint Loads: Joints 3 and 6, 1000kg. horizontal

IV) Joint Displacements (cm)

Joint	Horiz.	Vert.
1	0	0
2	0.6667×10^{-3}	$0.2652\ 10^{-10}$
3	0.1333×10^{-2}	$0.5367\ 10^{-10}$
4	0	0
5	0.6667×10^{-3}	$0.1745\ 10^{-10}$
6	0.1333×10^{-2}	$0.5558\ 10^{-10}$

V) Joint Reactions (kg)

Joint	Horiz. Reaction	Vert. Reaction
1	-1000	-0.7629×10^{-5}
4	-1000	-0.1309×10^{-4}

VI) Stress Field (kg/cm^2)

Element	T_x	T_y	Txy
1	2	-0.2722×10^{-7}	-0.2126×10^{-8}
2	2	-0.5733×10^{-8}	-0.3313×10^{-8}
3	2	0.	0.2617×10^{-7}
4	2	-0.2722×10^{-7}	0.1529×10^{-7}

V) Joint Reactions:

Joint	Hor.	Vert.
1	-1000	0.2634×10^3
2	0.	0.6991×10^2
4	0.	-0.2479×10^3
6	0.	-0.1653×10^3
8	0.	-0.2590×10^3
10	0.	0.3389×10^3

VI) Stress field (kg/cm^2)

Element	Tx	Ty	Txy
1	2.9463	0.	-1.0537
2	1.0537	0.3446	-0.2796
3	2.3023	0.3446	-0.3644
4	1.6977	0.1751	0.0227
5	2.3109	0.1751	-0.0141
6	1.6891	0.1014	0.3336
7	2.8332	0.1014	0.1886
8	1.1668	0.1886	0.1167

If, with the above finite element assembly, we repeat the calculations, with zero joint loads and a temperature encrease of 20º in all elements ($\alpha = 10^{-5}$), the results will be:

I) Joint displacements (cm)

Joint	Hor.	Vert.
1	0.	0.
2	0.5335×10^{-2}	0.
3	0.1180×10^{-1}	0.8942×10^{-2}
4	0.1353×10^{-1}	0.
5	0.2243×10^{-1}	0.1005×10^{-1}
6	0.2290×10^{-1}	0.
7	0.3260×10^{-1}	0.1008×10^{-1}
8	0.3273×10^{-1}	0.
9	0.4263×10^{-1}	0.1007×10^{-1}
10	0.4270×10^{-1}	0.

II) Joint Reactions

Joint	Hor.	Vert.
1	0.1062×10^{-1}	-0.1771×10^{5}
2	0.	0.1630×10^{5}
4	0.	0.2232×10^{4}
6	0.	0.3910×10^{3}
8	0.	-0.2774×10^{3}
10	0.	-0.1546×10^{3}

III) Stress field (kg/cm^2)

Element	Tx	Ty	Txy
1	10.821	-60.000	10.821
2	-10.821	-6.347	-5.183
3	3.764	-6.347	-1.873
4	- 3.764	0.273	-1.419
5	1.018	0.273	-1.328
6	-1.018	0.455	-0.400
7	0.200	0.455	-0.418
8	-0.200	0.418	-0.200

5.2 Example of Fig. 3

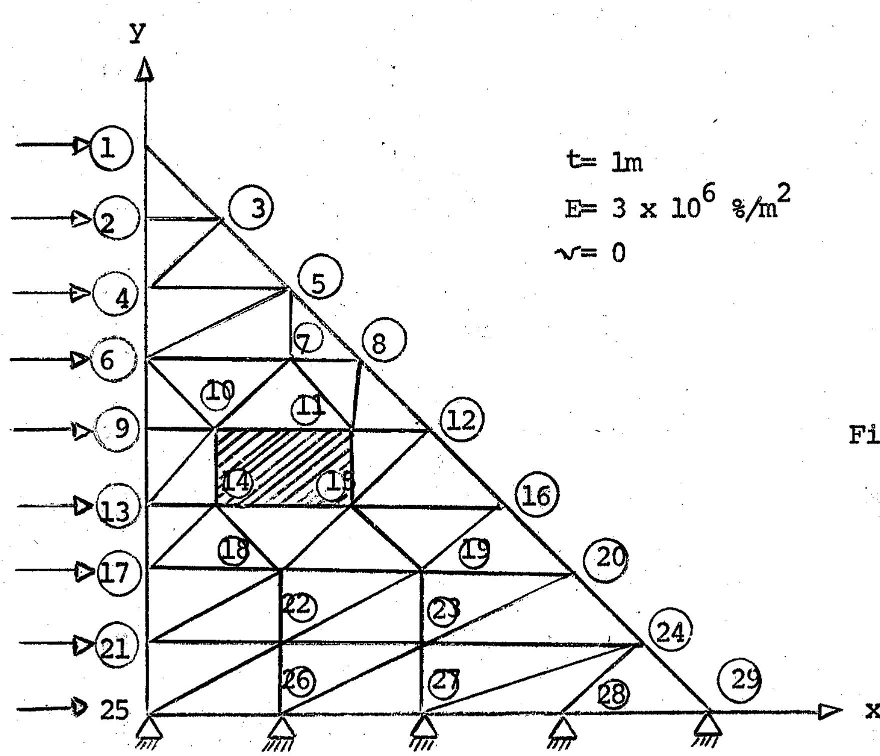


Fig. 6

I) Joint Coordinates (m)

Joint	x	y
1	0.	32
2	0.	28
3	4	28
4	0.	24
5	8	24
6	0	20
7	8	20
8	12	20
9	0.	16
10	6	16
11	11	16
12	16	16
13	0.	12
14	6	12
15	11	12
16	20	12
17	0	8
18	8	8
19	16	8
20	24	8
21	0	4
22	8	4
23	16	4
24	24	4
25	0	0.
26	8	0.
27	16	0.
28	24	0.
29	32	0.

II) Joint Restrictions: Joints 25, 26, 27, 28, 29 Horizontal and Vertical

III) Joint Loads: Horizontal forces are applied as follows:

Joint	Load (t)
1	2
2	16
4	32
6	48
9	164
13	80
17	96
21	112
25	62

IV) Joint displacements (m)

Joint	Hor.	Vert.
1	0.5959×10^{-3}	0.1638×10^{-3}
2	0.5238×10^{-3}	0.1638×10^{-3}
3	0.5182×10^{-3}	0.9428×10^{-4}
4	0.4515×10^{-3}	0.1616×10^{-3}
5	0.4291×10^{-3}	0.2914×10^{-4}
6	0.3820×10^{-3}	0.1572×10^{-3}
7	0.3494×10^{-3}	0.3209×10^{-4}
8	0.3315×10^{-3}	-0.2401×10^{-4}
9	0.3177×10^{-3}	0.1468×10^{-3}
10	0.3909×10^{-3}	0.5565×10^{-4}
11	0.2516×10^{-3}	0.1320×10^{-5}
12	0.2224×10^{-3}	-0.5545×10^{-4}
13	0.2441×10^{-3}	0.1309×10^{-3}
14	0.2046×10^{-3}	0.4148×10^{-4}
15	0.1636×10^{-3}	0.1403×10^{-5}
16	0.1336×10^{-3}	0.6496×10^{-4}
17	0.1710×10^{-3}	0.1016×10^{-3}
18	0.1216×10^{-3}	0.1460×10^{-4}

19	0.9024×10^{-4}	-0.2559×10^{-4}
20	0.7070×10^{-4}	-0.5511×10^{-4}
21	0.9591×10^{-4}	0.6371×10^{-4}
22	0.5684×10^{-4}	0.6254×10^{-5}
23	0.4077×10^{-4}	-0.1357×10^{-4}
24	0.2650×10^{-4}	-0.2454×10^{-4}
25	0.	0.
26	0.	0.
27	0.	0.
28	0.	0.
29	0.	0.

IV) Joint reactions (t)

Joint	Hor.	Vert.
25	-1628	-233.38
26	-1557	- 25.5
27	-114.1	92.1
28	- 79.5	14.7
29	0.	19.9

The stresses will not be given for reasons of space.

6. CONCLUDING REMARKS

Although the above results of the chosen numerical examples are self explaining, we shall drop a few remarks about them.

First, it is interesting to notice that the finite element structure of Fig. 4 gives the same stresses and displacements, as if the loads are applied as a uniform tension in the bar. Such an outcome should have been expected on elementary considerations.

When the numbers of elements is increased, (Fig.5) a stress concentration begins to be felt close to the outer longitudinal edge of the plate. The horizontal displacements are algo larger along the edge, as compared to the inside of the plate. This general tendency should become more evident; as the number of elements is encreased.

The results of the temperature effect can be checked very well by approximate elemetary methods, both for stresses and displacements.

It becomes clear from an analysis of results that the triangular element is not the most favorable shape for the problem of Fig. 2, because it deviates considerably from the natural geometry of the problem. Rectangular elements are better indicated.

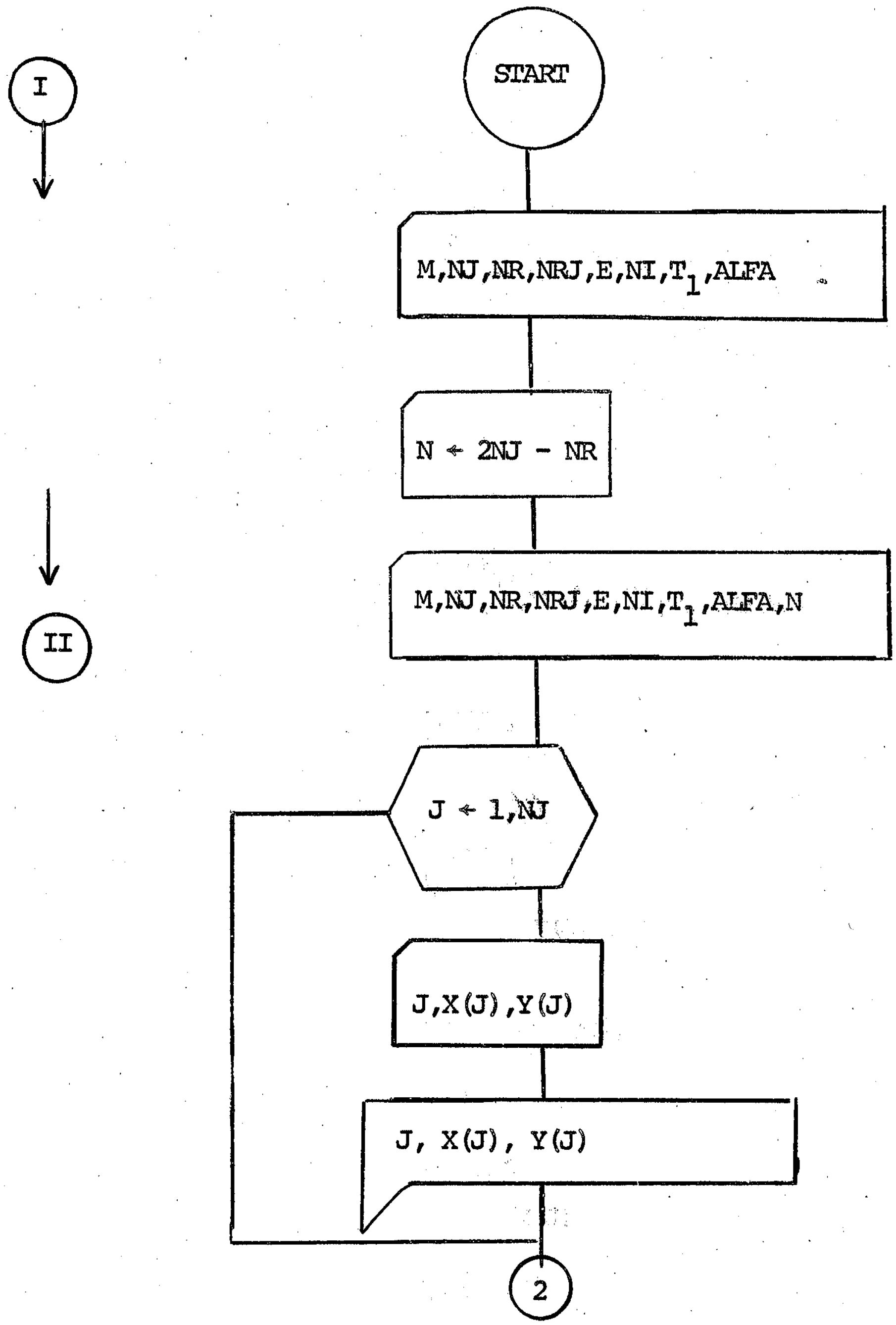
As for the example of the gravity dam with a hole, the displacements obtained from the finite element distribution of Fig, 6 seem quite logical and reasonable. The stresses can not be trusted very much, on account of the very low number of elements.

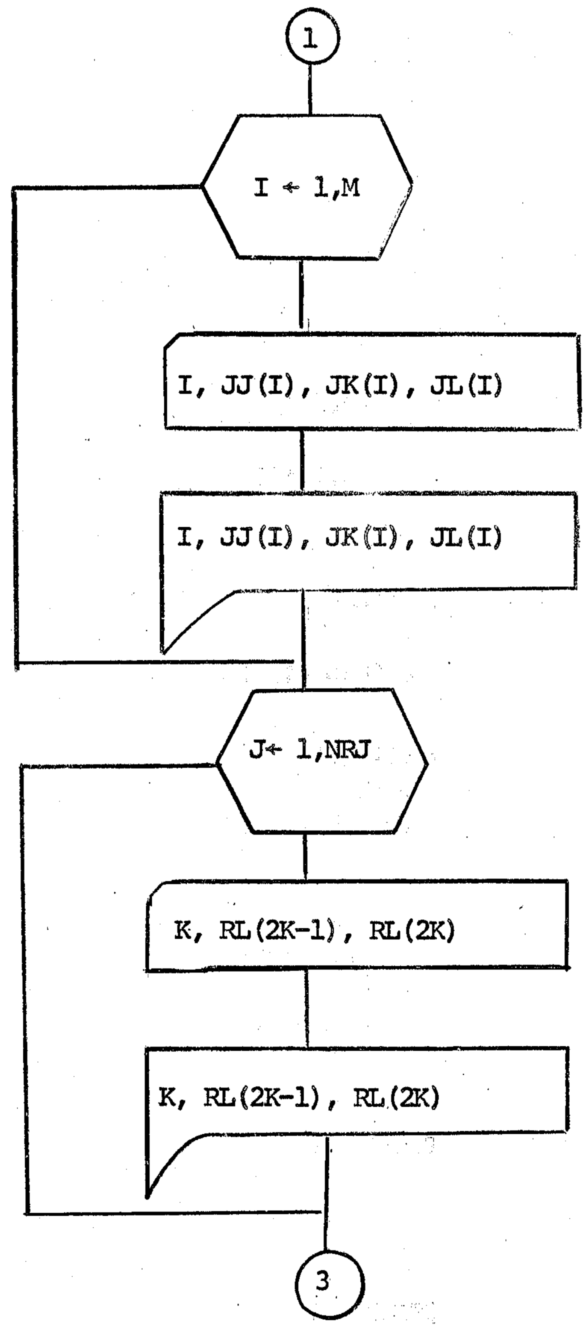
In all examples the equilibrium of the fixed joint reactions with the external loads can be rigoroulsy checked.

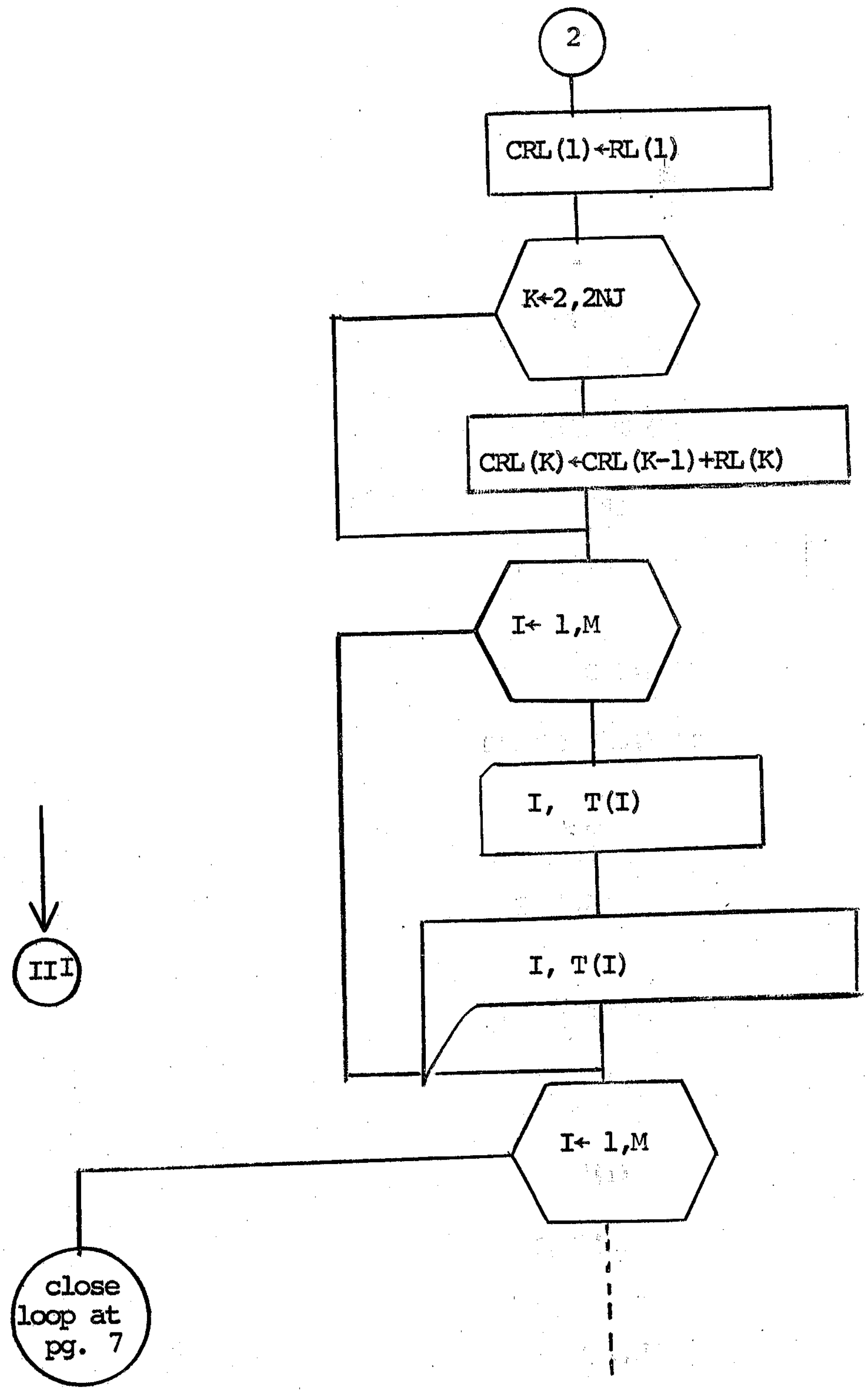
7. REFERENCES

1. J.S. Przeminiecki - "Theory of Matrix Structural Analysis" - Mc Graw Hill.
2. O.C. Zienkiewicz - "The Finite Element Method in Structural and Continuum Mechanics" - Mc Graw Hill.
3. J.M. Gere and Weaver Jr. - "Analysis of Framed Structures" - Van Nostrand.
4. W. Waever Jr. - "Computer Programs for Structural Analysis". Van Nostrand.

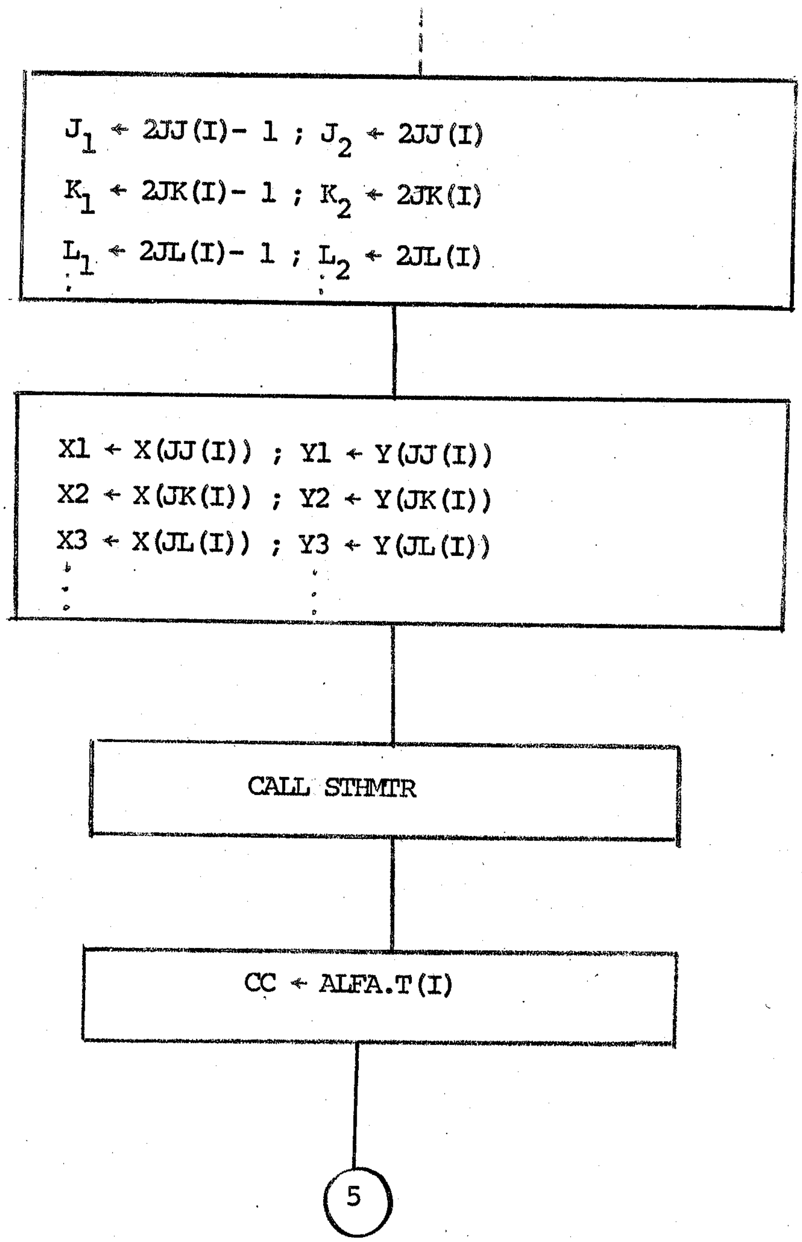
APPENDIX - FLOW CHART

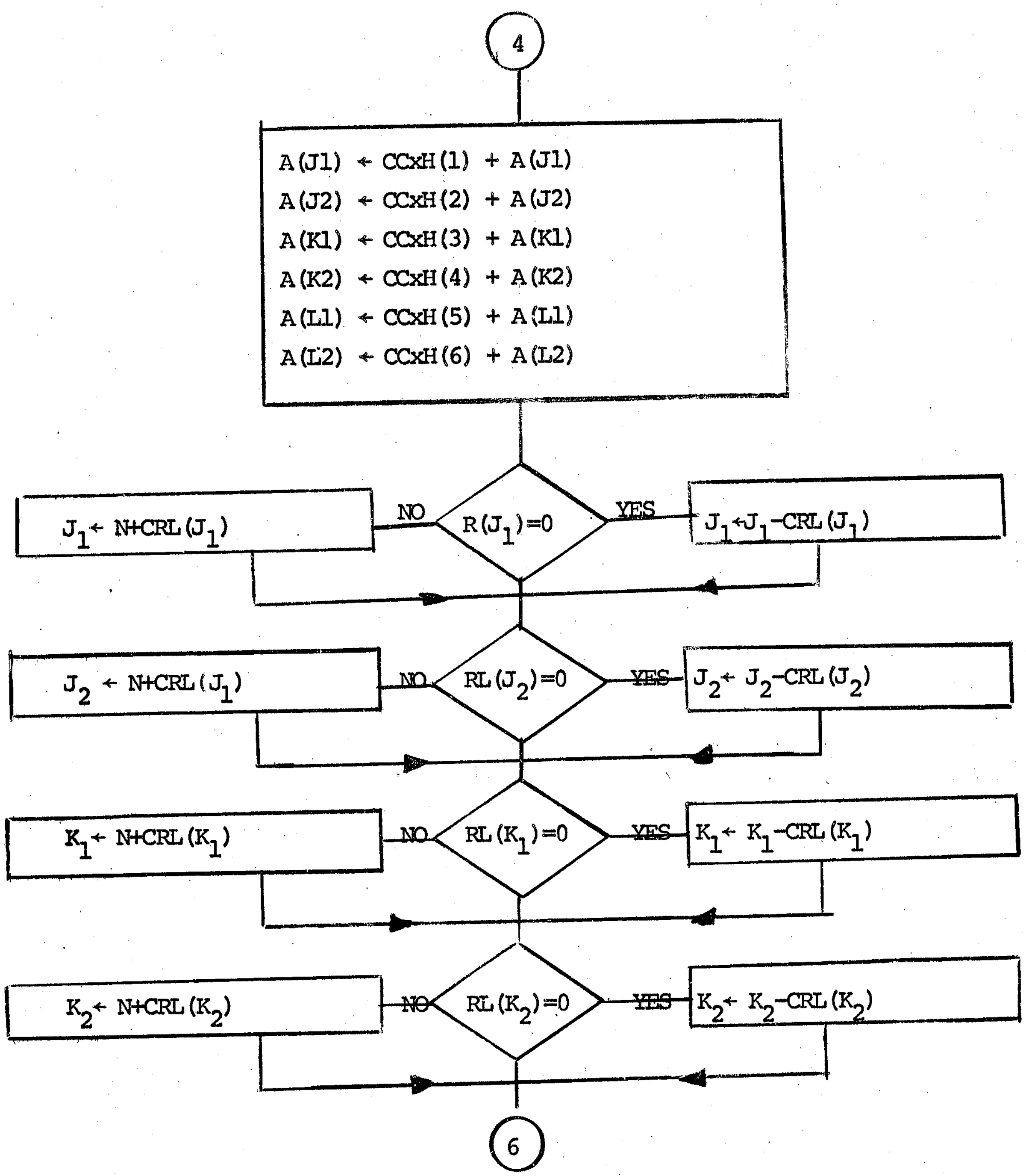



1
I ← 1,M
I, JJ(I), JK(I), JL(I)
I, JJ(I), JK(I), JL(I)
J← 1,NRJ
K, RL(2K-1), RL(2K)
K, RL(2K-1), RL(2K)
3

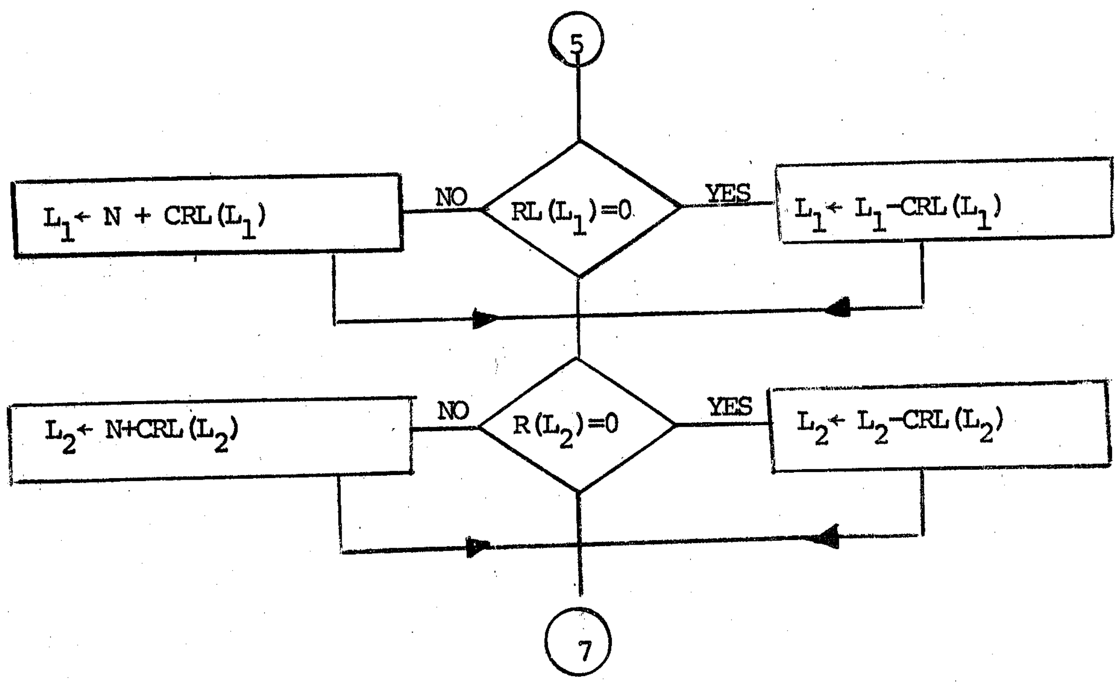


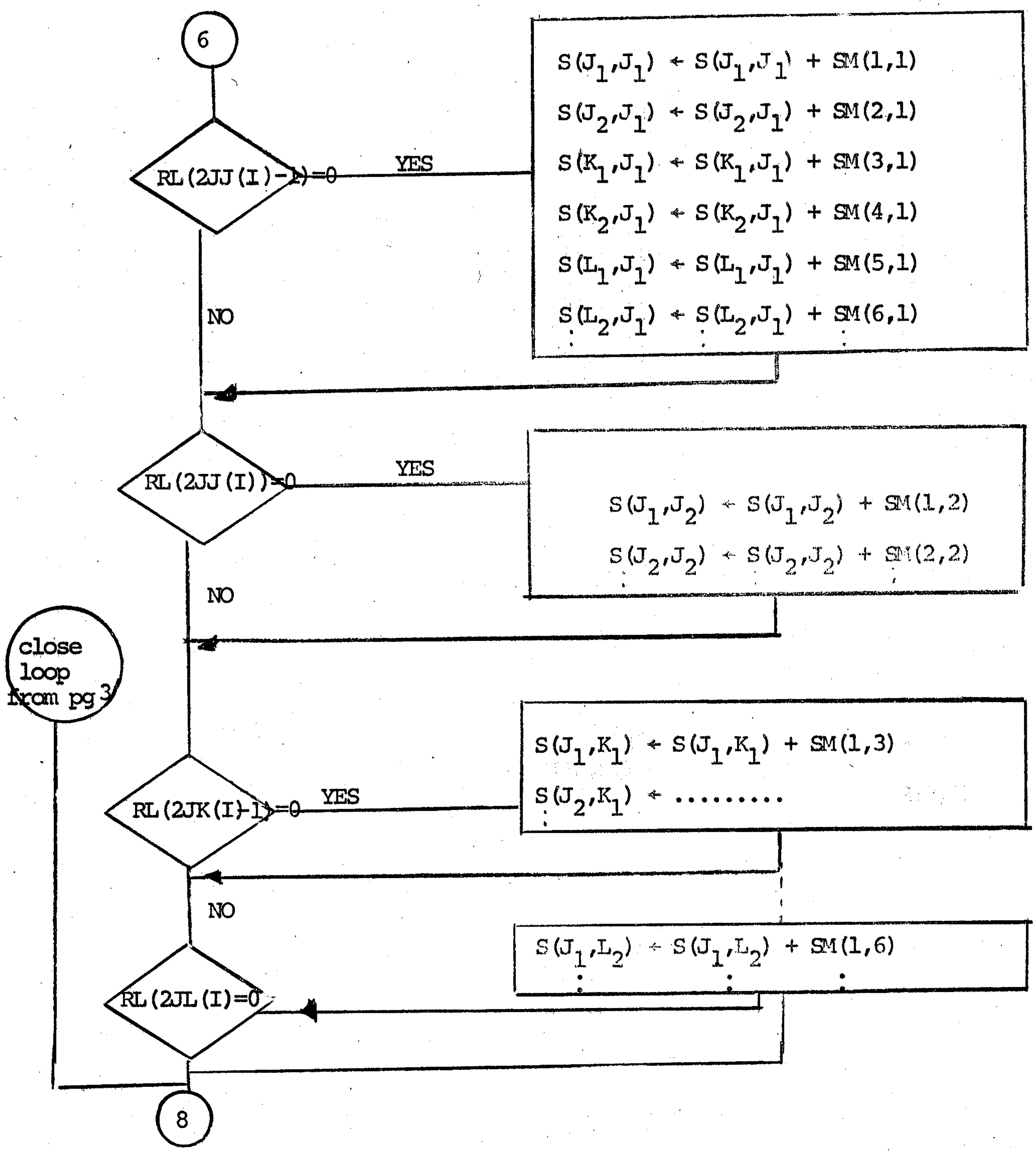
2
CRL(1)←RL(1)
K←2,2NJ
CRL(K)←CRL(K-1)+RL(K)
I← 1,M
I, T(I)
I, T(I)
III
I← 1,M
close loop at pg. 7

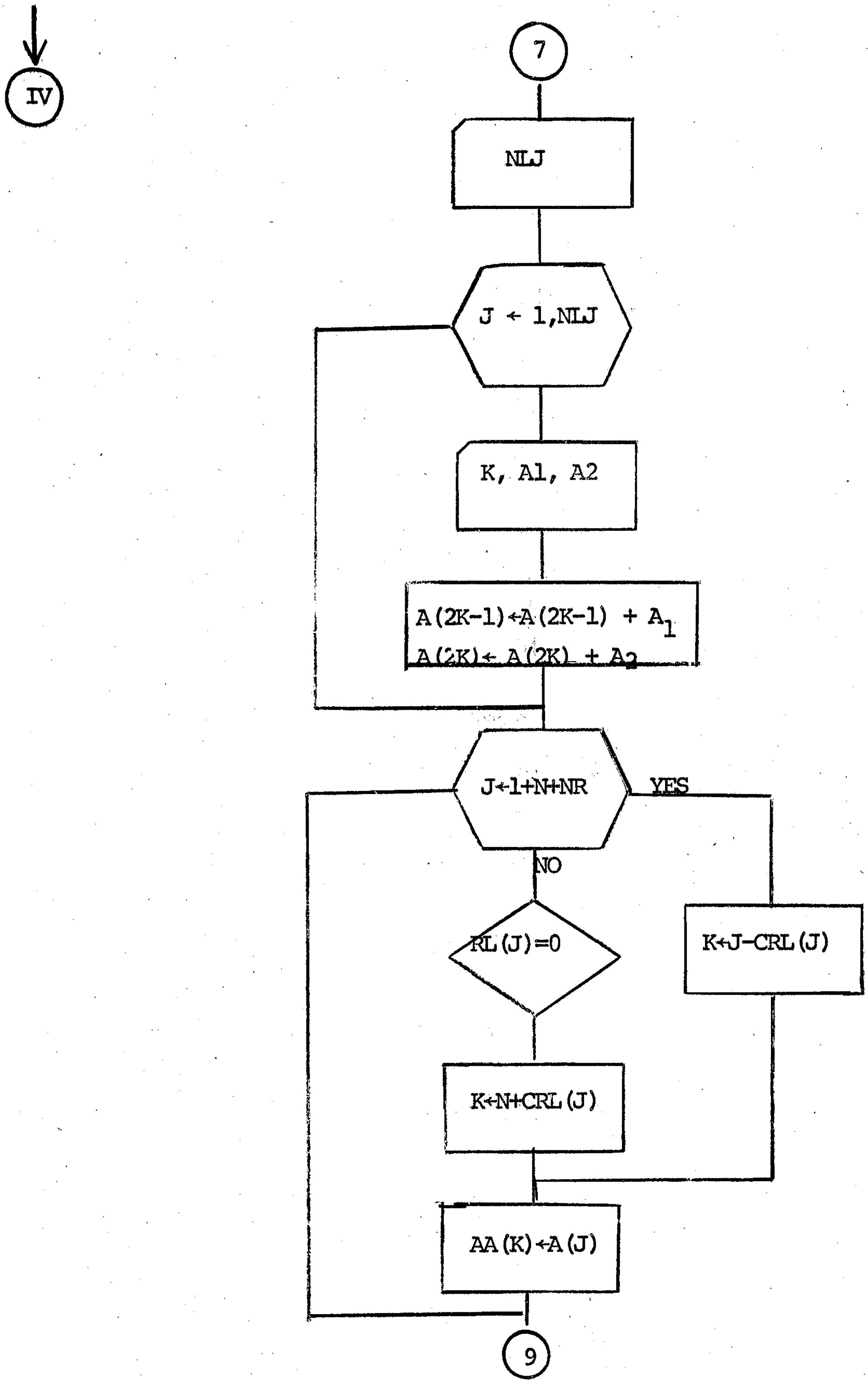

J1 ← 2JJ(I)- 1 ; J2 ← 2JJ(I)
K1 ← 2JK(I)- 1 ; K2 ← 2JK(I)
L1 ← 2JL(I)- 1 ; L2 ← 2JL(I)
X1 ← X(JJ(I)) ; Y1 ← Y(JJ(I))
X2 ← X(JK(I)) ; Y2 ← Y(JK(I))
X3 ← X(JL(I)) ; Y3 ← Y(JL(I))
CALL STHMTR
CC ← ALFA.T(I)
5

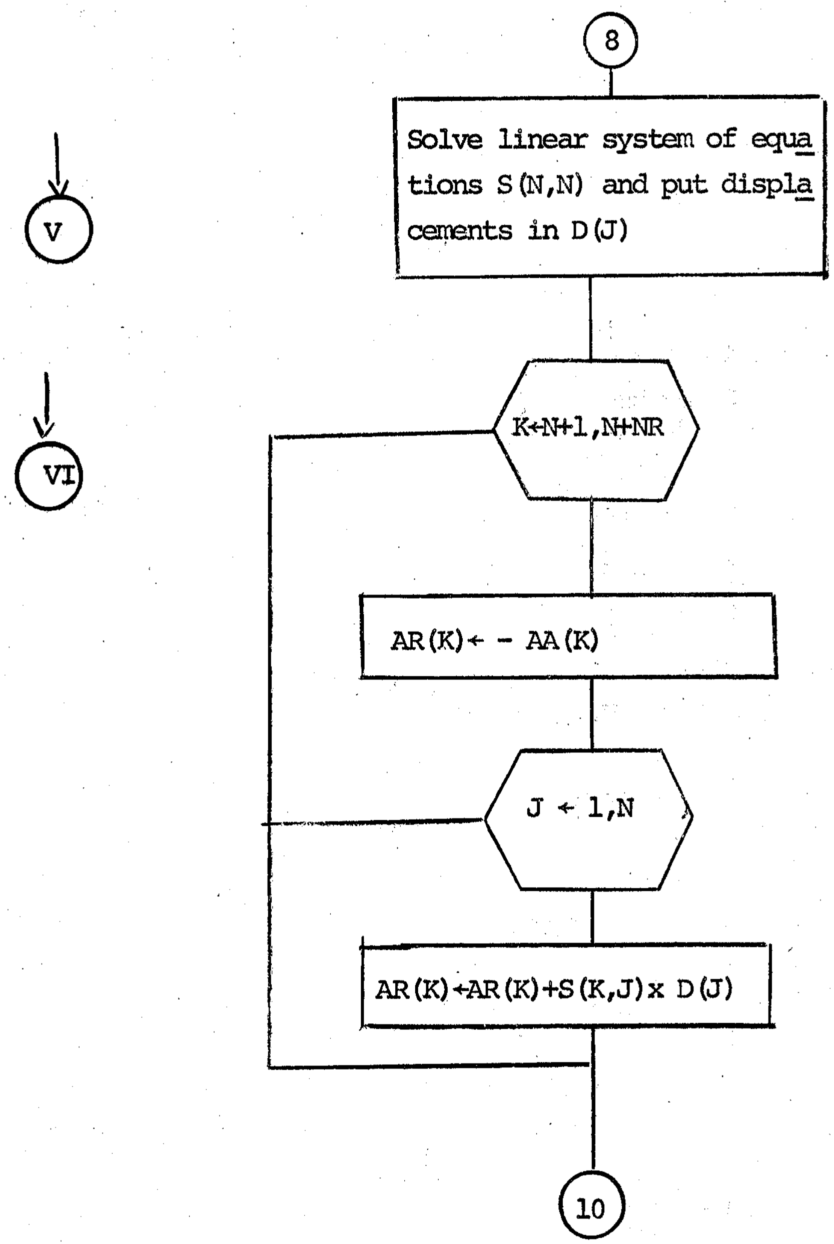


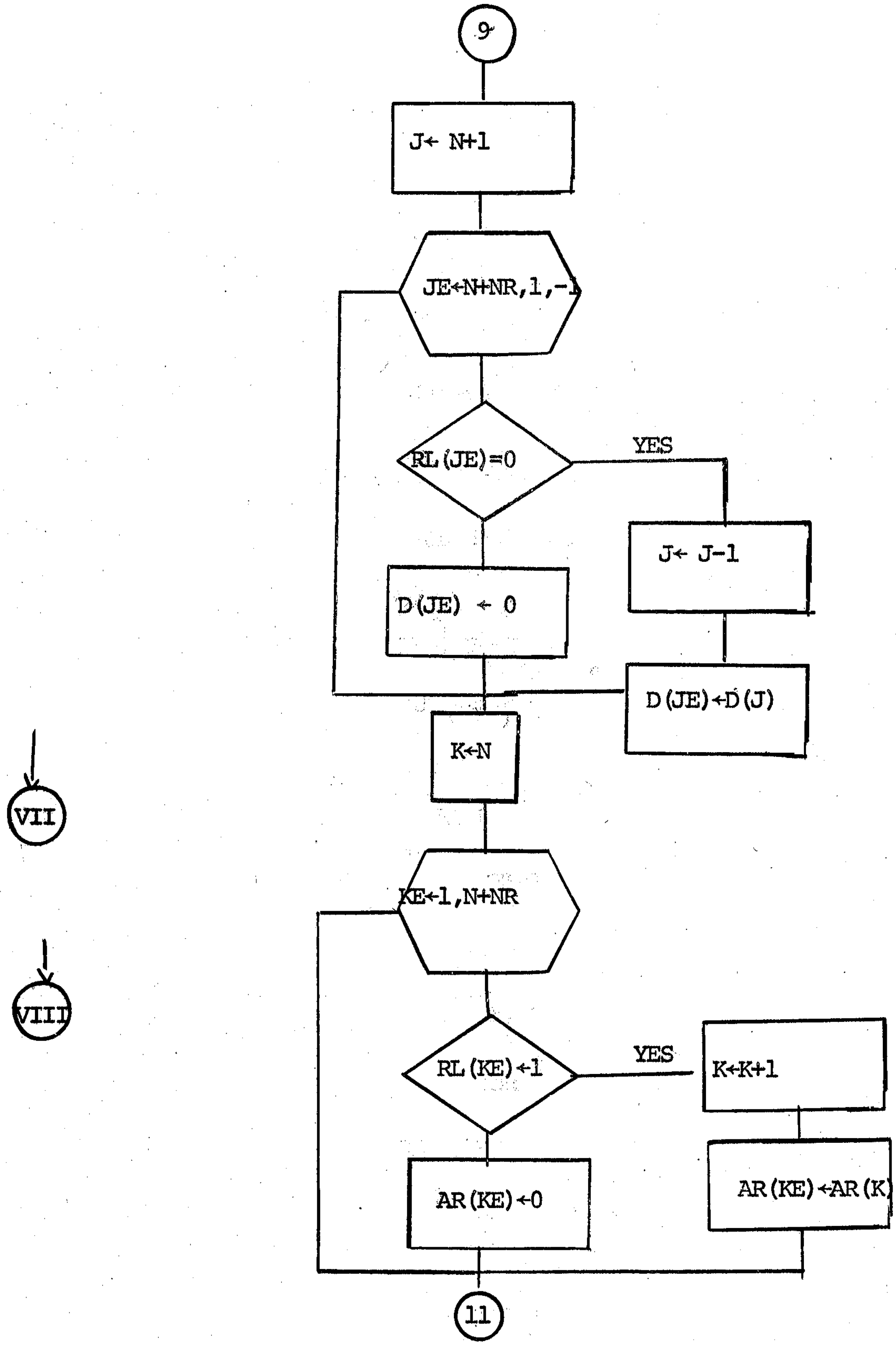
4
A(J1) ← CCxH(1) + A(J1)
A(J2) ← CCxH(2) + A(J2)
A(K1) ← CCxH(3) + A(K1)
A(K2) ← CCxH(4) + A(K2)
A(L1) ← CCxH(5) + A(L1)
A(L2) ← CCxH(6) + A(L2)
R(J1)=0
NO
YES
J1← N+CRL(J1)
J1←J1−CRL(J1)
RL(J2)=0
NO
YES
J2 ← N+CRL(J1)
J2← J2−CRL(J2)
RL(K1)=0
NO
YES
K1← N+CRL(K1)
K1← K1−CRL(K1)
RL(K2)=0
NO
YES
K2← N+CRL(K2)
K2← K2−CRL(K2)
6

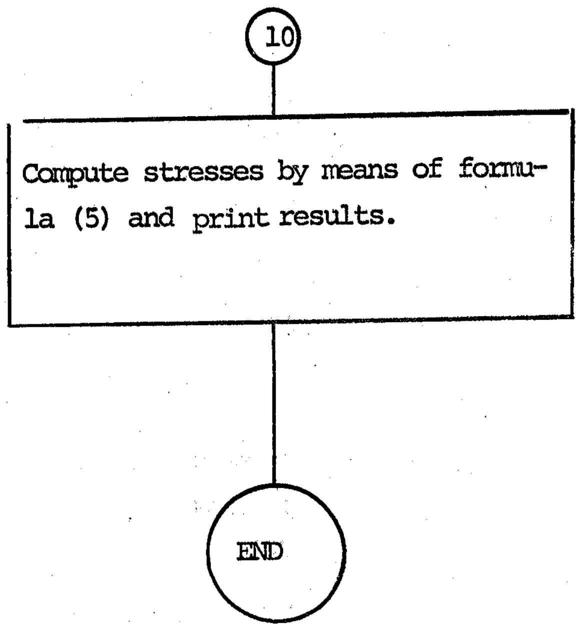

5
NO
RL(L1)=0
YES
L1← N + CRL(L1)
L1← L1-CRL(L1)
NO
R(L2)=0
YES
L2← N+CRL(L2)
L2← L2-CRL(L2)
7


6
RL(2JJ(I)-1)=0
YES
NO
S(J1,J1) ← S(J1,J1) + SM(1,1)
S(J2,J1) ← S(J2,J1) + SM(2,1)
S(K1,J1) ← S(K1,J1) + SM(3,1)
S(K2,J1) ← S(K2,J1) + SM(4,1)
S(L1,J1) ← S(L1,J1) + SM(5,1)
S(L2,J1) ← S(L2,J1) + SM(6,1)
RL(2JJ(I))=0
YES
NO
S(J1,J2) ← S(J1,J2) + SM(1,2)
S(J2,J2) ← S(J2,J2) + SM(2,2)
close loop from pg 3
RL(2JK(I)-1)=0
YES
NO
S(J1,K1) ← S(J1,K1) + SM(1,3)
S(J2,K1) ← ....
S(J1,L2) ← S(J1,L2) + SM(1,6)
RL(2JL(I)=0
8


IV
7
NLJ
J ← 1,NLJ
K, A1, A2
A(2K-1) ← A(2K-1) + A1
A(2K) ← A(2K) + A2
J ← 1+N+NR
YES
NO
RL(J)=0
K ← J-CRL(J)
K ← N+CRL(J)
AA(K) ← A(J)
9


8
V
VI
Solve linear system of equations S(N,N) and put displacements in D(J)
K←N+1,N+NR
AR(K)← - AA(K)
J ← 1,N
AR(K)←AR(K)+S(K,J) x D(J)
10


9
J← N+1
JE←N+NR,1,-1
RL(JE)=0
YES
J← J-1
D(JE) ← 0
D(JE)←D(J)
K←N
VII
VIII
KE←1,N+NR
RL(KE)←1
YES
K←K+1
AR(KE)←0
AR(KE)←AR(K)
11


10
Compute stresses by means of formula (5) and print results.
END