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by

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A CONVERGENT FINITE ELEMENT METHOD FOR SOLVING DYNAMIC PROBLEMS WITH CONSISTENT DIAGONALIZED MASS MATRIX *

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A B S T R A C T

A finite element method for solving dynamic problems having piecewise quasilinear basis functions is presented. The method leads to a consistent diagonalized mass matrix, whereas the stiffness matrix is essentially the same as in the piecewise linear case. Convergence of the approximate solution to the exact one is guaranteed.

KEY-WORDS: Consistent mass matrix, convergence, diagonalized mass matrix, dynamic problems, finite elements, nodal elimina - tion, quasilinear basis functions.

R E S U M O

Apresenta-se um método de elementos finitos para a re - solução de problemas dinâmicos, com funções de base quasilinea - res por elemento. O método permite a geração de uma matriz de massa consistentemente diagonalizada, ao passo que a matriz de rigidez é essencialmente a mesma que no caso do método linear por elemento clássico. O método produz seqüências de aproximações convergentes à solução do problema contínuo, no sentido habitu - al.

PALAVRAS-CHAVE: Convergência, elementos finitos, eliminação nodal, funções de base quasilineares, matriz de massa consistente, matriz de massa diagonalizada, problemas dinâmicos.

1 - INTRODUCTION

We consider the numerical solution of boundary value problems by Ritz-Galerkin methods. In this framework, many research works have long been devoted to the problem of finding admissible finite dimensional subspaces, with simple orthogonal basis functions with respect to inner products arising from the variational formulation of the differential equation.

As far as the energy inner product is concerned, very few results seem to have been achieved, except for the case of some problems in one dimension space or having particular symmetry or periodicity properties. In the general case however, standard finite element methods are largely in use, eventhough it is a well-known fact that handling the resulting band matrices is often very much time consuming in a computer.

In the case of initial-boundary value problems, the diagonalization of the so-called mass matrix has been sucessfully accomplished with the popular lumped mass scheme (see e.g.[4]). We recall that the basic idea behind it consists of changing the piecewise linear basis functions of the finite element method , into piecewise constant functions having a smaller support, only for the term associated with the derivative with respect to time. Although the global rate of convergence of this finite element approximation is maintained, the above change causes the error itself to increase.

An attempt to generate diagonalized mass matrices avoiding the inconsistency of the lumped mass technique has recently been made by Chien [2]. In his method each piecewise linear basis function is replaced by a quadratic function , and numerical results for a given mesh are shown to be superior in some sense, to those obtained with the standard method. Unfortunately his method fails to generate a convergent sequence of approximations in the energy norm for the dynamic problem he discusses.

Incidentally these problems can be expressed in terms of a system of differential equations of the form:

Given a bounded domain Ω of $\mathbb{R}^N$, $N=1,2$ or 3, with boundary Γ, a vector valued function $\underset{\sim}{u}_0$ defined on Ω, find $\underset{\sim}{u}$ depending on the space variables $x_1, x_2, \ldots x_N$ and on time t such that:

$$(P) \quad \begin{cases} \dfrac{\partial \underset{\sim}{u}}{\partial t} - \mathcal{L}\, \underset{\sim}{u} = f & \text{in } \Omega\times]0,T[\\ \underset{\sim}{u}(\underset{\sim}{x},t) = 0 & \forall \underset{\sim}{x}\in\Gamma\times]0,T[\\ \underset{\sim}{u}(x,0) = \underset{\sim}{u}_0(\underset{\sim}{x}) & \text{in } \Omega \end{cases}$$

where $\mathcal{L}$ is an elliptic second order differential operator and T is a given time.

Without loss of generality for the purpose of our dis - cussion, we take in this work as a model problem the scalar case with:

$$\mathcal{L}\, u = \sum_{i=1}^{N} \frac{\partial}{\partial x_i} \sum_{j=1}^{N} \alpha_{ij} \frac{\partial u}{\partial x_j} \quad ,$$

$\mathcal{A} = \{\alpha_{ij}\}$ being a positive definite matrix independent of $\underset{\sim}{x}$ and t.

We introduce a method giving rise to a consistent dia- gonalized mass matrix, for which convergence is guaranteed.

Clear enough the main purpose of the diagonalization of the mass matrix in the case of such problems, is to allow the straightforward solution of the approximate problem, in case an explicit difference scheme for time discretization is used. Just to have a clear look at this question, we take as an exemple the simplest explicit scheme,

namely:

Let V_h be the N_h-*dimensional* finite element subspace of $H_0^1(\Omega)$ with basis functions ψ_i, $i=1,2,\ldots,N_h$.

Define u_h^0 to be the L^2- projection of u_0 over V_h and let $\vec{v}$ be the vector of components of a function $v_h \in V_h$ with respect to ψ_i, *i.e:*

$$v_h = \sum_{i=1}^{N_h} v_i \psi i$$

Then compute $u_h^1, u_h^2, \ldots, u_h^L \in V_h$ such that for $\Delta t = T/L$ we have:

$$M \frac{\vec{u}^{n+1} - \vec{u}^{n}}{\Delta t} = A\vec{u}^{n} + \vec{f}^{n} \qquad n=0,1,\ldots,L-1$$

where M is the mass matrix, A is the stiffness matrix associa - ted with $\mathcal{L}$ and

$$f_i^n = \int_\Omega f(n\Delta t)\, \psi_i \, d\underset{\sim}{x}$$

Clearly at the $(n+1)$-th time step we must solve for $\vec{u}^{n+1}$ the linear system below (we drop the arrows from now on):

$$M u^{n+1} = M u^{n} + \Delta t (A u^{n} + f^{n}) \tag{1}$$

2 - QUASILINEAR APPROXIMATIONS AND NODAL ELIMINATION

We have shown in previous papers (see e.g. [5] and references therein) that finite element subspaces V_h consisting of quasilinear polynomials defined on simplices of a partition $\mathcal{T}_h$ of Ω, provide a very powerful tool for solving problems related to incompressible media.

We recall that a quasilinear function defined over an N- simplex $K\in\mathcal{T}_h$ is a function of the form

$$p = \sum_{i=1}^{N+1} c_i \lambda_i + c\psi \ ,$$

where ψ is a function of $H^1(K)$ and the λ_i's are the area coordinates of K

In this work we show that this technique can be applied to the space discretization of problem (P), in order to generate consistent diagonalized mass matrices. This will be achieved by taking ψ to be a suitable function of $H_0^1(K)$, ψ being a function of $H_0^1(\Omega)$ as well, for every $K\subset\Omega$.

<u>Remark 1</u>: A function $\psi\notin H_0^1(K)$ will be considered in Section 3 . However it satisfies the same requirements as a function of this space as far as convergence properties are concerned. □

Here the basis functions of the space of test functions will be associated with a node lying in the interior of K, say its centroid, whereas the remaining basis functions will be associated with the vertices of K. We will then eliminate the values at the inner nodes from system(1), in such a way that we obtain a diagonal mass matrix of dimension $N_h - card(\mathcal{C}_h)$ for the remaining unknowns. We procede as follows:

Let the local numbering of the nodes of K be $S_1, S_2, \ldots, S_{N+1}$ for the vertices, and S_{N+2} for the centroid of K. In this way the corresponding basis functions V_h are given by:

$$\begin{cases} p_i = \lambda_i - \dfrac{\psi}{N+1}, & i=1,2,\ldots,N \\ p_{N+2} = \psi & \end{cases} \qquad (2)$$

where $1/\alpha = \psi(S_{N+2})$, a value assumed to be constant over all simplices of $\mathcal{C}_h$.

Let the global numbering of the nodes be such that the vertices of $\mathcal{C}_h$ are nodes from 1 up to I and centroids are nodes from $I+1$ up to $I+J$. We also number the simplices of $\mathcal{C}_h$ from one to J in such a way that the $(j+I)$-th node is the centroid of the j-th simplex.

Clearly the only neighbors of a centroid node are the vertices of the corresponding simplex. Thus it will be easy to compute the value of u^{n+1} at the j-th inner node, $1 \le j \le J$ only in terms of its value at the three vertices of the j-th simplex and of f^n_{I+j}. If we next substitute this value of u^{n+1}_{I+j} in the equations of system (1) corresponding to the vertices of the j-th element for every j, we will only need to solve a system of equations for the unknowns u_i^{n+1}, $1 \le i \le I$.

We will show in Section 4 how to perform all these computations in a simple and unexpensive way. Let us now stress , how the above procedure gives rise to a diagonalized mass matrix.

We can split the matrix M into four matrices P, Q, Q^T and D according to the pattern below:

$$\begin{array}{c|c} & \begin{array}{cc} I & J \end{array} \\ \begin{array}{c} I \\ \\ J \end{array} & \left[\begin{array}{c:c} P & Q \\ \hdashline Q^T & D \end{array}\right] \end{array}$$

Similarly we can split matrix A, u^n and f^n in the fol - lowing way:

$$A = \begin{array}{cc} \begin{array}{cc} I & J \end{array} & \\ \left[\begin{array}{c:c} B & C \\ \hdashline C^T & E \end{array}\right] & \begin{array}{c} I \\ \\ J \end{array} \end{array} \qquad \mathrm{u}^n = \left[\begin{array}{c} \mathrm{u}_I^n \\ \hline \mathrm{u}_j^n \end{array}\right] \qquad f^n = \left[\begin{array}{c} f_I^n \\ \hline f_j^n \end{array}\right]$$

Notice that both D and E are diagonal matrices.

The procedure described above consists then in evaluating:

$$\mathrm{u}_J^{n+1} = D^{-1}Q^T(\mathrm{u}_I^n - \mathrm{u}_I^{n+1}) + \mathrm{u}_J^n + \Delta t(D^{-1}C^T\mathrm{u}_I^n + D^{-1}E\ \mathrm{u}_J^n) + D^{-1}\ f_J^n \qquad (3)$$

and

$$\tilde{P}\mathrm{u}_I^{n+1} = \tilde{P}\ \mathrm{u}_I^n + \Delta t\ (\tilde{B}\ \mathrm{u}_I^n + \tilde{C}\ \mathrm{u}_J^n + f_I^n - QD^{-1}\ f_j^n) \qquad (4)$$

where:

$$\tilde{P} = P-QD^{-1}Q^T$$

$$\tilde{B} = B-QD^{-1}C^T$$

$$\tilde{C} = C-QD^{-1}E$$

We wish to derive practical sufficient conditions under which $\tilde{P}$ is a diagonal matrix.

First, recalling the well-known concept of elementary matrix, one can easily check that the operation $P-QD^{-1}Q^T$ can be performed at the element level. $\tilde{P}$ is precisely the sum of contributions of elementary matrices $\tilde{P}_K$, $K\epsilon\mathcal{C}_h$, where

$$\tilde{P}_K = P_K - q_K\ d_K^{-1}\ q_K^T,$$

P_K, q_K and d_K being obtained by the following splitting of the elementary mass matrix M_K:

$$
M_K = \begin{array}{cc} \begin{array}{cc} \quad N+1 \quad & \quad 1 \quad \end{array} & \\ \left[\begin{array}{c|c} P_K & q_K \\ & \\ \hline q_K^T & d_K \end{array}\right] & \begin{array}{c} N+1 \\ \\ 1 \end{array} \end{array}
$$

Therefore if $\tilde{P}_K$ is a diagonal $(N+1)\times(N+1)$ matrix, $\tilde{P}$ will also be a diagonal matrix.

We then have:

Proposition 1: If function ψ satisfies

$$\int_K \lambda_i\lambda_j - \int_K \lambda_i\psi \int_K \lambda_j\psi \Big/ \int_K \psi^2 = 0 \quad \text{if} \quad i\neq j, \qquad (5)$$

$\forall i,j,\quad 1\leq i,j\leq N+1,\quad \forall K\in\mathcal{T}_h$ then $\tilde{P}$ is a diagonal matrix.

Proof: Recalling the definition (2) of the basis functions we see that

$$(P_K)_{ij} = \int_K [\lambda_i - \alpha\psi/(N+1)][\lambda_j - \alpha\psi/(N+1)]$$

whereas

$$(q_K)_i = \int_K [\lambda_i - \alpha\psi/(N+1)]\alpha\psi$$

and

$$d_K = \int_K \alpha^2\psi^2$$

(5) follows directly from the condition

$$(P_K)_{ij} - (q_K)_i \; d_K (q_K)_j = 0 \qquad \text{for} \quad i\neq j. \qquad \text{q.e.d.} \; \Box$$

Concrete examples of functions ψ satisfying (5) will be given in the next section. Incidentally one can only expect (5) to hold if ψ has symmetry properties with respect to the vertices, in the sense that, if it is expressed in area coordinates, we have:

$$\psi(\lambda_{i_1}, \lambda_{i_2}, \ldots, \lambda_{i_{N+1}}) = \psi(\lambda_1, \lambda_2, \ldots, \lambda_{N+1})$$

$\forall\, \underset{\sim}{i} = (i_1, i_2, \ldots, i_{N+1})$ such that $1 \le i_k \le N+1$ and $i_k \neq i_\ell$ if $k \neq \ell$.

Indeed in this case $\int_K \lambda_i \psi$ is invariant with respect to i, $1 \le i \le N+1$ as is $\int_K \lambda_i \lambda_j$ with respect to i, j if $i \neq j$.

Otherwise stated, it suffices to satisfy (5) for $i=1$ and $j=2$ for instance.

Moreover we will have for diagonal terms

$$\int_K \lambda_i^2 - \left(\int_K \lambda_i \psi\right)^2 \Big/ \int_K \psi^2 = \int_K \lambda_1^2 - \int_K \lambda_1 \lambda_2 \qquad \forall i,\ 1 \le i \le N+1 \tag{6}$$

3 - A SIMPLE CHOICE OF ψ

Clearly the diagonalization process described in the previous section should not imply in intricate and lengthy computations. The choice of ψ described below yields several simplifications.

Let K' be the homotetical reduction of $K \in \mathcal{C}_h$ with center S_{N+2} and ratio $r^{1/N}$. Let $\lambda'_1, \lambda'_2, \ldots, \lambda'_{N+1}$ be the area coordinates of K'

We define:

$$\psi(\underset{\sim}{x}) = \begin{cases} \displaystyle\sum_{i=1}^{N} \sum_{j=i+1}^{N+1} \lambda'_1 \lambda'_j & \text{if} \quad \underset{\sim}{x} \in K' \\ 0 & \text{if} \quad \underset{\sim}{x} \notin K' \end{cases} \tag{7}$$

Notice that ψ is not a function of $H_0^1(\Omega)$, but as far as problem *(P)* in stationary form is concerned, we can prove in the same way as in [6] that its solution is approximated up to an $O(h)$ term in the sense of the discret $H^1(\Omega)$-norm, by using the finite element method equivalent to the one described in the previous section, with ψ given by *(7)*.

Condition (5) then becomes:

$$\int_K \lambda_i \lambda_j - \int_{K'} \lambda_i \psi \int_{K'} \lambda_j \psi / \int_{K'} \psi^2 = 0 \qquad \text{if} \quad i \neq j. \qquad (8)$$

According to well-known quadrature formulae [7] we have:

$$\int_K \lambda_1^{k_1} \lambda_2^{k_2} \ldots \lambda_{N+1}^{k_{N+1}} = \frac{k_1!\ k_2!\ \ldots k_{N+1}!\ N!}{(k_1+k_2+\ \ldots + k_{N+1}+N)!}\ meas(K) \qquad (9)$$

Thus condition *(5)* will be expressed in terms of a simple algebraic equation for κ and we have:

<u>Proposition 2:</u> If ψ is given by *(7)* then condition *(5)* holds if

$$\kappa = \frac{4}{5} \qquad \text{if} \qquad N = 1 \text{ or } 2$$

$$\kappa = \frac{52}{63} \qquad \text{if} \qquad N = 3$$

<u>Proof:</u> It suffices to use formulae *(8)* and *(9)*, and the definition

$\kappa = meas(K')/meas(K)$ q.e.d. □

4- COMPUTATIONAL ASPECTS

The above choice of ψ is convenient, as it allows a simple computation of the matrices appearing in the expressions *(3)* and *(4)* of u_I^{n+1} and u_J^{n+1}, respectively.Nevertheless for any ψ we can compute these values with just a little more effort than it is necessary when using standard piecewise linear functions . More specifically we have:

The computation of u_J^{n+1} is straighforward, since $D^{-1}Q^T$ is a matrix with constant coefficients γ depending on ψ and N but not on the elements themselves. Actually if we let $\hat{K}$ be a reference element such that $K\in\mathcal{C}_h$ is the image of $\hat{K}$ through an affine transformation $\mathcal{F}:\mathbb{R}^N\to\mathbb{R}^N$, and if we define ψ in the usual way (see e.g. [3]), namely:

$$\hat{\psi}[\mathcal{F}^{-1}(\underset{\sim}{x})] = \psi(\underset{\sim}{x}) \qquad \forall x\in K$$

we have:

$$\gamma = \frac{\int_{\hat{K}} \hat{\lambda}_i\hat{\psi}}{\alpha\int_{\hat{K}}\hat{\psi}^2} - \frac{1}{N+1} \qquad \forall i.$$

Notice that the integral in the numerator above is invariant with respect to i, according to the symmetry assumptions for ψ.

On the other hand, both $D^{-1}C^T$ and $D^{-1}E$ can be stored in a J-th dimensional vector. The term of this vector corresponding to the j-th element K_j depends only on ψ and on this element, and are given by $-\beta/(N+1)$ and β respectively, where:

$$\beta = \int_{K_j} \underset{\sim}{\nabla}\psi\underset{\sim}{\nabla}\psi \Big/ \int_{K_j}\psi^2$$

This is because $\displaystyle\int_{K_j} \underset{\sim}{\nabla}\lambda_i\underset{\sim}{\nabla}\psi = \underset{\sim}{\nabla}\lambda_i\int_{K_j}\underset{\sim}{\nabla}\psi = 0 \qquad \forall i,$

according to the assumption $\psi\in H_0^1(K_j)$.

<u>Remark 2</u>: In the case of a function ψ with $\psi\notin H_0^1(K_j)$, such as the one considered in Section 3, we have:

$$\int_{K_j}\underset{\sim}{\nabla}\psi = \sum_{i=1}^{N+1}\int_{K_j}\frac{\partial\psi}{\partial\lambda_i}\underset{\sim}{\nabla}\lambda_i = \sum_{i=1}^{N+1}\underset{\sim}{\nabla}\lambda_i\int_{K_j}\frac{\partial\psi}{\partial\lambda_i} = \int_{K_j}\frac{\partial\phi}{\partial\lambda_1}\sum_{i=1}^{N+1}\underset{\sim}{\nabla}\lambda_i = 0,$$

according to the symmetry assumptions for ψ. Thus the result above also applies to this case. □

Notice that if as usual, one works with the integer array $R(J\times 3)$ which associates with each element of $\mathcal{C}_h$ the numbers of its three vertices, the operation involving the u_I's in (3) can be easily performed.

As for u_I^{n+1}, the essential part of the computational effort is due to the multiplication of u_I^n by $\tilde{B}$, which is the only band matrix to be stored. Incidentally it has the same structure as the piecewise linear finite element matrix associated with $\mathcal{C}_h$. On the other hand, matrix $\tilde{C}$ can be stored in the form a J-th dimensional vector, whose j-th component consists of its equal elementary contributions δ to the coefficients correspondings to the $N+1$ vertices of the element, namely:

$$\delta = -\alpha^2\left(\gamma + \frac{1}{N+1}\right) \int_{K_j} \underset{\sim}{\nabla}\psi\underset{\sim}{\nabla}\psi$$

Using here again the integer array R, this operation becomes very simple as pointed out above.

5 - CONVERGENCE RESULTS

Under suitable assumptions, the convergence of the method presented in this paper can be proved using quite standard estimates. For so doing, we define, as usual, u^n to be the function whose value for every $\underset{\sim}{x}$ is $u(\underset{\sim}{x}, t_n)$ with $t=n\Delta t\le T<\infty$. Then convergence of u_h^n to u^n can be established in several senses, provided

$$\Delta t \le Ch^2 \qquad (10),$$

as we are dealing with an explicit scheme (see e.g. [5]).

For instance if f is smooth in a neighborhood of t, we have:

<u>Theorem 1:</u> Consider the scalar case of problem *(P)* and let *(10)* hold. Then if $f\in L^1[0,t;H^1(\Omega)]$ and $(u,\frac{\partial u}{\partial t}, f)\ (.,\tau\ )\in[H^2(\Omega)]^3$ $\forall\ \tau\in(t-\varepsilon,t)$, $\varepsilon>0$, we have:

$$\| u^n - u_h^n \|_{L^2(\Omega)} \le Ch^2,$$

where C is a bounded constant depending on u, f and T.

<u>**Proof:**</u> According to [1] and [9], the only condition required in this case is the following approximation property of V_h :

$\forall v \in H^2(\Omega)$, the V_h-interpolant v_h of v satisfies:

$$\| v-v_h \|_{L^2(\Omega)} + h \ \| v-v_h \|_{H^1(\Omega)} \leq Ch^2 \ |v|_{H^2(\Omega)}$$

where the H^1-norm is taken in the discrete sense if $\psi \notin H_0^1(K)$, $K \in \mathcal{T}_h$.

Since the space V_h of piecewise quasilinear shape functions contains the space of continuous piecewise linear functions over Ω, the above estimate holds.

q.e.d. □

As for the case of more general systems (P), equivalent estimates can be derived from the results of CROUZEIX [4] for the analogous explicit scheme. Since we are basically interest in systems of two equations corresponding to scalar second order time-dependent problems; i.e., with

$$\underset{\sim}{u} = (u, \frac{\partial u}{\partial t}) \quad \mathcal{L} = \begin{bmatrix} 0 & I \\ \mathcal{L}_1 & 0 \end{bmatrix} \quad \text{and} \quad \underset{\sim}{f} = \begin{bmatrix} 0 \\ f \end{bmatrix} \tag{11}$$

and with
$$\mathcal{L}_1 = \sum_{i=1}^{N} \frac{\partial}{\partial x_i} \sum_{i=1}^{N} \alpha_{ij} \frac{\partial}{\partial x_j} \quad \text{and} \quad f \in L^2(\Omega) \tag{12}$$

we apply his results of Chapter 4.

In order to do so, we need estimates of the error of the Ritz-projection onto V_h of a function $v \in H^2(\Omega) \cap H_0^1(\Omega)$, which we denote by $P_h v$.

Clearly, in the conforming case, i.e., when $V_h \subset H_0^1(\Omega)$, we have:

$$\| P_h v-v \|_{L^2(\Omega)} + h \| P_h v-v \|_{H^1(\Omega)} \leq Ch^2 \ |v|_{H^2(\Omega)} \tag{13}$$

However, in the nonconforming case we have to replace the second term of the left hand side of (13) by the error of $P_h v$ in the sense of the discrete H^1-norm $\|\cdot\|_h$ defined by:

$$\|v\|_h = \{ \sum_{K\in\mathcal{T}_h} \|v\|^2_{H^1(K)} \}^{1/2} \qquad (14),$$

with P_h defined by:

$$a_h(P_h v, v_h) = -(\mathcal{L}_1 v, v_h) \qquad \forall v_h \in V_h \qquad (15),$$

where

$$a_h(u_h, v_h) = \sum_{K\in\mathcal{T}_h} \int_K \sum_{i,j=1}^{N} a_{ij} \frac{\partial u_h}{\partial x_i} \frac{\partial v_h}{\partial x_j} \qquad (16)$$

and (u,v) denotes $\int_\Omega uv$. Notice that this is precisely the case of the method of Section 3. In any case, the key to the problem is the following:

<u>Lemma 1</u>: If P_h and $\|.\|_h$ are defined by (14), (15) and (16) then

$$\forall v\in H^2(\Omega) \qquad \|P_h v - v\|_h \leq Ch|v|_{H^2(\Omega)}.$$

<u>Proof</u>: Let $\pi_h w$ be the interpolant of a function $w\in V_h$ in the space W_h, which consists of continuous piecewise linear functions associated with $\mathcal{T}_h$. We then have:

$$\|P_h v - v\|_h \leq \|\pi_h P_h v - v\|_{H^1(\Omega)} + \|\psi_h\|_h$$

where $\psi_h = \pi_h P_h v - P_h v$.

Now we note that Remark 2 implies that (see also [7]):

$$a_h(P_h v, w_h) = a_h(\pi_h P_h v, w_h) \qquad \forall w_h \in W_h.$$

Thus $\pi_h P_h v$ satisfies:

$$a_h(\pi_h P_h v, w_h) = -(\mathcal{L}_1 v, w_h) \qquad \forall w_h \in W_h \qquad (15),$$

and by standard estimates we have:

$$\|\pi_h P_h v - v\|_{H^1(\Omega)} \leq Ch|v|_{H^2(\Omega)}$$

On the other hand, Remark 2 also implies that

$$\| \psi_h \|_h^2 \le C a_h(\psi_h, \psi_h)^{1/2} = -(\mathcal{L}_1 v, \psi_h) \le |v|_{H^2(\Omega)} \| \psi_h \|_{L^2(\Omega)}.$$

Since we clearly have $\| \psi_h \|_{L^2(\Omega)} \le Ch \| \psi_h \|_h$, the result follows. q.e.d □

Remark 3: Lemma 1 allows us to conclude that the values at the vertices of $\mathcal{C}_h$ of the Ritz-projection of a function of $H_0^1(\Omega)$ onto any associate space of piecewise quasilinear functions with the symmetry properties defined in Section 2, coincide with those of its Ritz-projection onto the space of continuous piecewise linear functions. □

Finally we have:

Theorem 2: Let *(10)* hold and u and $\frac{\partial u}{\partial t}$ belong to $C^\circ[0,t;H^2(\Omega)]$. Then if $\underset{\sim}{u}$, $\mathcal{L}$ and $\underset{\sim}{f}$ are defined by *(11)* and *(12)* the approximate solution of problem *(P)* satisfies:

$$\| u^n - u_h^n \|_{L^2(\Omega)} \le Ch^2$$

where C is a bounded constant depending on T and u.

q.e.d. □

As a conclusion, one can see that the method presented here is explicit and convergent if $\Delta t/h^2$ is bounded by a certain constant. But now we can take Δt small, because the mass matrix inversion is replaced by straightforward and unexpensive operations. Moreover, the local approximation properties of the piecewise linear finite element method can be expected to improve.

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