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# ABSTRACT SEMANTICAL SYSTEMS

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**Abstract:**

In this work we introduce a new approach to the study of semantical systems. Traditionally the study of these systems have covered only those semantical systems whose languages are formal and whose models are relational structures. Here, both languages and models are classes, moreover there is no dependency on foundational considerations. The concepts of "theory", "worlds","semantical consequences", "negation" and "completeness" are presented from strictly semantical point of view. Some of the results obtained subsume similar results found in the Theory of Models.

# Abstract Semantical Systems

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**Abstract**

In this work we introduce a new approach to the study of Semantical Systems. Traditionally the study of these systems have covered only those semantical systems whose languages are formal and whose models are relational structures. Here, both languages and models are classes, moreover there is no dependency on foundational considerations. The concepts of **theory, worlds, semantical consequence, negation** and **completeness** are presented from a strictly semantical point of view. Some of the results obtained subsume similar results found in the Theory of Models.

Understanding something means to own a system of criteria to evaluate perceived facts. An evaluation system works as a *filter* through which **concepts** may be generated. Different individuals possess different filters and so perception of reality is something of a subjective nature. Not only do sensorial observations form an individual's knowledge, but concepts may be apprehended by other kinds of perception and deductive processes performed by the human mind. **Knowledge** is, therefore, the mental appropriation of sensorial and extra-sensorial perceptions that can be deepened by the exercise of thought and experimentation. Concepts, in their turn, will influence the evaluation process changing the cultural filter and so our perception of reality.

We can say that for a fixed instant of an individual's development an evaluation system is acting for the immediately following perceptions and such perceptions change the system itself re-elaborating the evaluation criteria. The *steady* existence and functioning of this evaluation system enables the *knowledge acquisition* process to continue, generating *configurations* or *models* of reality.

The impossibility of directly communicating different models by the corresponding *sensation* causes the necessity for the creation of *codes* or *systems of meaning*. The systems of meaning also result from social acculturation and their most obvious manifestation is the usage of various forms of language -

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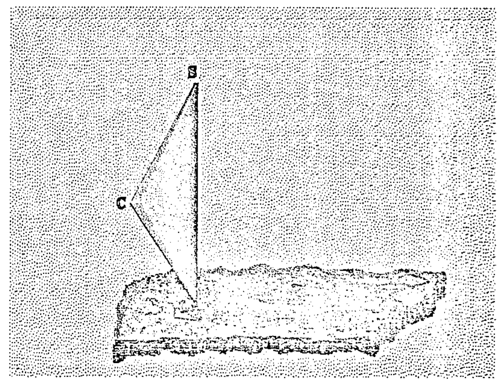
spoken, gestual, written, etc. Such languages emerge from social conventions which associate objects to particular *signs.*

We can say that a signal **S** stands for an object **O** as if it indicates the object itself. This relation is indirect, because its interpretation is **mediated** by the concept **C** that we have of the object. The properties of the objects, the relations among them and the transformations to which they may be subjected are also coded by signals.

Ogden and Richards proposed the **Meaning Triangle** as a representation of the relations between the objects and their corresponding conceptions in our minds and the associated signals (symbols).

The triangle expresses that a referenced object
O (Referent)
impresses our senses producing an image in our mind
C (Concept)
that can be represented by a signal
S (Symbol)
that allow us to establish references, under a certain convention.




# 1 Semantical Systems

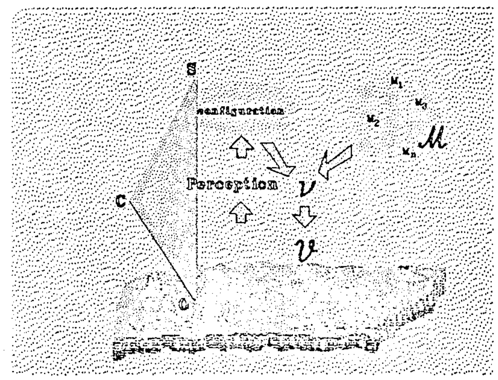
Knowledge about a given *world*, actual or artificial, is formed by the continuous process of perception and evaluation. This knowleldge can be understood as a class of all the conceivable configurations of the considered *world.* Each of these configurations is called a **model**. These models can be fuzzy or fragmented, as our compreehension is limited. The class of models, also fuzzy and fragmented since we cannot perceive all of its elements, is what we usually call *system of beliefs* of an individual regarding a given **world**. The class of models, because of its conceptual nature, is situated at vertex C of the meaning triangle.

Regarding a given state of affairs we may perceive and even conceive different configurations. Each of them has a valuation with respect to a possible world. This valuation comes from a set of values and a process of evaluation. By evaluation we mean a judgement given to a configuration (perceived, communicated or conceived) related to the world. We may think that the new or recent configuration is compared to each of the configurations that are already considered as models or real world configurations. The set of abstract values can be quite extensive and imprecise, varying with the nature of knowledge or the state of affairs considered. Therefore we may have a definite set of values, such as: $\{fair, unfair\}$ or $\{defficient, insufficient, regular, good, excellent\}$, which however not very precise, are well determinate. The most usual ones

$\{true, false\}$ are taken for granted, despite the extreme difficulties of their understanding. Nevertheless other sets may be fuzzy or even indeterminate.

We may note that even for definite sets like those mentioned we may still question the meaning of each element. The degree of indetermination as well as the complexity of the elements are directly related to the extent to which reality is understood. Those values are situated at vertex C of the triangle.

After establishing a set of values, we can proceed to the definition of the judgement criteria, i.e., the forms of relating a given configuration to a value. The evaluation process, that so far we assume to be carried out by human agents, is located at vertex C. Although eventually, and more frequently, such process is an abstraction, similar to a *partial function*, it can also have a symbolic representation whose connotation, instead of being descriptive intentional, is *operational* or *algorithmic*.




Knowledge is coded, classified, structured, transmitted, learned, and improved by its use. Systems of meaning are built - *naturally* or *artificially* - and, in the process, evaluation systems become more elaborated and complex but it is mainly by the use of language that they can be studied and better understood.

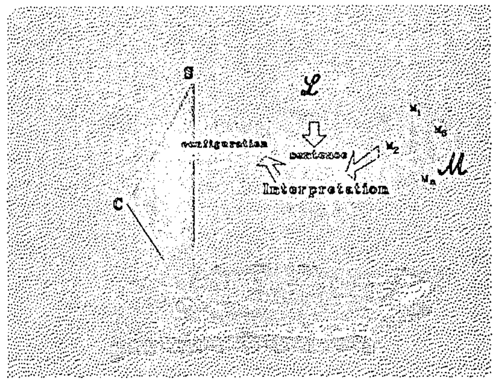
In the following, unless otherwise stated, when we use the word *language* we mean the symbolic portion of the systems of meaning. Languages[1] are composed of: **atoms** - that can be sonorous or visual signs individually identifiable, **words** - which are temporal or spacial sequences of atoms and **sentences** - which are certain temporal or spacial sequences of words.

Thus a language is situated at vertex S of the triangle and frequently it is learned by a gradual process that never ends. Therefore languages are generally fragmented and imcomplete.

Although a language has a symbolic nature it can be fuzzy with no clear boundary, but still coherent for the social group that uses it. The coherence we expect is obtained by communicative interaction among the elements of the social group, and each *significant* unit is a code, sometimes ambiguous, that is interpreted generating configurations in the same way as the direct perception of reality. It is then, something symbolic that by interpretation enables us to *apprehend* new configurations or models.

[1] In this work we will not consider Formal Languages as an isolated object, i.e. we postpone their study to further development, when the meaning associated to them will, in most cases, be relative to their arithmetical interpretation.

The interpretation process is carried out in the same way as the direct *perception* of reality, nevertheless the generated *configurations* are acknowledged as possible relative to the models we previously acquired by the process of acculturation. The use of languages creates, sometimes, a lack of contact with reality.




In the process of interpretation, the *linguistic* units, that is, symbols, words and sentences are interpreted as objects (real or imaginary, concrete or abstract) as well as relations or activities among these objects.

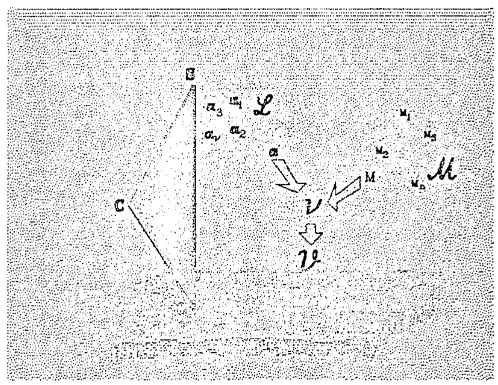
Evaluation systems together with languages are called *Semantical Systems*, and are made up of 4 components:

$\mathcal{L}$- A language

$\mathcal{M}$- A class of models

$\mathcal{V}$- A set of abstract values

$\varphi$- An evaluation function



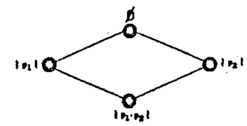

We must note that by using the word *function* for the evaluation process we are making a simplification. We are suposing that the interpretation of a sentence *generates* only one configuration and that the underlying evaluation process produces a definite abstract value. The function $\varphi$ might not be computable, i.e. there may not be an algorithmic way to produce a value for it.

## 2 Propositional Semantical Systems

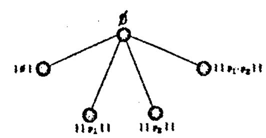
**Example 2.1** *The propositional semantical system* $\mathcal{S}^p_n$ *has the following components:*

. Language $\mathcal{L}^p_n$ defined by the grammar: $G = \{SA \longrightarrow p_1 \mid p_2 \mid \cdots \mid p_n\,,\; S \longrightarrow SA\,,\; S \longrightarrow (S \wedge S)\,,\; S \longrightarrow (S \vee S)\,,\; S \longrightarrow (S \Rightarrow S)\,,\; S \longrightarrow (S \Leftrightarrow S)\,,\; S \longrightarrow \neg S\}$

. Models $\mathcal{M}_n^p = \wp(\{p_1, p_2, \cdots, p_n\})$. In the case of the propositional system $\mathcal{S}_2^p$ $\mathcal{M}_n^p = \{\emptyset, \{p_1\}, \{p_2\}, \{p_1, p_2\}\}$, which, together with inclusion, form a lattice.

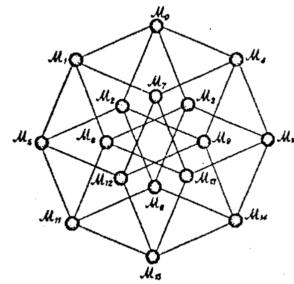


Models here are considered as sets and not as *individuals* or elements of a set.



We are interested in the subclasses of models $\mathcal{M}_2^p$. If we consider only the sets which represent the models above we do not have a lattice anymore but a partial order:

Considering all the subclasses we have $2^4 = 16$ subsets of $\mathcal{M}_2^p$, that also form a lattice regarding inclusion.



In the figure:

$\mathcal{M}_0 = \emptyset$
$\mathcal{M}_1 = \{\emptyset\}$
$\mathcal{M}_2 = \{\{p_1\}\}$
$\mathcal{M}_3 = \{\{p_2\}\}$
$\mathcal{M}_4 = \{\{p_1\}, \{p_2\}\}$

the remaining ones are obtained by the union of the above, for instance $\mathcal{M}_5 = \mathcal{M}_0 \cup \mathcal{M}_1 = \{\emptyset, \{p_1\}\}$

. Abstract Values $\mathcal{V} = \{0, 1\}$

. Evaluation Function $\varphi_n^p =$ defined by:

$\varphi_n^p(p_i, M) = 1$ if and only if $p_i \in M$
$\varphi_n^p(\neg\alpha, M) = 1 - \varphi_n^p(\alpha, M)$
$\varphi_n^p((\alpha \wedge \beta), M) = \varphi_n^p(\alpha, M).\varphi_n^p(\beta, M)$
$\varphi_n^p((\alpha \vee \beta), M) = max\{\varphi_n^p(\alpha, M), \varphi_n^p(\beta, M)\}$
$\varphi_n^p((\alpha \Rightarrow \beta), M) = max\{1 - \varphi_n^p(\alpha, M), \varphi_n^p(\beta, M)\}$
$\varphi_n^p((\alpha \Leftrightarrow \beta), M) = 1$ if and only if $\varphi_n^p(\alpha, M) = \varphi_n^p(\beta, M)$

Propositional semantical systems are usually presented in the *Propositional Calculus* without explicitly mentioning the *models* or *state of affairs*. The evaluation function is shown through the use of *truth tables* that are defined for the logical connectives. There are various kinds of knowledge that can be represented by the propositional semantical systems and actually all the digital electronics depends greatly upon or at least results from the discipline known as Boolean Algebra, which is but another syntatical representation of the propositional semantical systems. The propositional semantical systems have finite sets of models and therefore there are computable evaluation functions $\varphi_n^p$ for them.

## 3 First Order Semantical Systems

The first order language is generated from an alphabet, constituted by the following sets:

$\mathcal{V}$ - Variables $\{x_1, x_2, \cdots, x_n, \cdots\}$

$\mathcal{C}$ - Constants $\{a_1, a_2, \cdots, a_n, \cdots\}$

$\mathcal{F}$ - Functions $\{f_1, f_2, \cdots, f_n, \cdots\}$

$\mathcal{P}$ - Predicates $\{P_1, P_2, \cdots, P_n, \cdots\}$

From this alphabet *terms* and *formulas* are constructed by induction:

$T_1$ - Variables and constants are terms.

$T_2$ - If $t_1, t_2, \cdots, t_n$ are terms and $f$ a function then $f(t_1, t_2, \cdots, t_n)$ is a term.

$T_3$ - The only terms are those expressions which follows from $T_1$ and $T_2$ above.

Atomic formulas are defined by

$A_1$ - If $t_1, t_2, \cdots, t_n$ are terms and $P$ a predicate then $P(t_1, t_2, \cdots, t_n)$ is an atomic formula.

$A_2$ - The only atomic formulas are those expressions which follow from $A_1$ above.

Formulas are defined by

$F_1$ - Atomic formulas are formulas.

$F_2$ - If $\alpha$ and $\beta$ are formulas then

. $\neg\alpha$, $(\alpha \vee \beta)$, $(\alpha \wedge \beta)$, $(\alpha \rightarrow \beta)$ and $(\alpha \leftrightarrow \beta)$ are formulas.

$F_3$ If $x$ is a variable and $\alpha$ is a formula then $\forall x\, \alpha$ and $\exists x\, \alpha$ are formulas.

$F_4$ - The only formulas are those expressions which follow from $F_1$, $F_2$ and $F_3$ above.

The models are certain mathematical structures like $\mathcal{E} = < D, \mathcal{R}, \Phi >$, where:

$D$ - Is a set, called the domain of the structure.

$\mathcal{R}$ - Is a set of relations on $D$.

$\Phi$ - Is a set of functions defined on $D$

For a particular first order language, its interpretation with respect to an adequate structure, is a function $I$ which assigns:

- to all constants $a \in \mathcal{C}$ an element $a^I \in D$,
- to the functions $f \in \mathcal{F}$ a function $f^I : D^n \longrightarrow D$, and
- to the predicates $P \in \mathcal{P}$ a relation $P^I \subseteq D^n$

An interpreted structure $\mathcal{E}$ by $I$ is denoted by $\mathcal{E}^I$ or simply by $I$. The valuation function $\varphi$ is defined considering not only the sentences of the language, but all formulas, by the introduction of a valuation of variables $s : \mathcal{V} \longrightarrow D$, which is extended to all terms of the language by:

$$\hat{s}(a) = a^I, \text{ for every } a \in \mathcal{C}$$
$$\hat{s}(x) = s(x), \text{ for every } x \in \mathcal{V}$$
$$\hat{s}(f(t_1, \cdots, t_n)) = f^I(\hat{s}(t_1), \cdots, \hat{s}(t_n))$$

We will use $s$ instead of $\hat{s}$ in order to simplify notation. We also define modifications on $s$ given by:

$$s_x^d(y) = \begin{cases} d & \text{if } y = x \\ s(y) & \text{otherwise} \end{cases}$$

With this we can define the s-valuation function $\varphi_s$

- $\varphi_s(P(t_1, t_2, \cdots, t_n), I) = 1 \Leftrightarrow < s(t_1), \cdots, s(t_n) > \in P^I$
- $\varphi_s(\neg\alpha, I) = 1 - \varphi_s(\alpha, I) = 1$
- $\varphi_s((\alpha \wedge \beta), I) = \varphi_s(\alpha, I).\varphi_s(\beta, I)$
- $\varphi_s((\alpha \vee \beta), I) = max\{\varphi_s(\alpha, I), \varphi_s(\beta, I)\}$
- $\varphi_s((\alpha \Rightarrow \beta), I) = max\{1 - \varphi_s(\alpha, I), \varphi_s(\beta, I)\}$
- $\varphi_s((\alpha \Leftrightarrow \beta), I) = 1$ if and only if $\varphi_s(\alpha, I) = \varphi_s(\beta, I)$
- $\varphi_s(\forall x \alpha, I) = \prod_{d \in D} \varphi_{s_x^d}(\alpha, I)$
- $\varphi_s(\exists x \alpha, I) = \coprod_{d \in D} \varphi_{s_x^d}(\alpha, I)$

If $\alpha$ is a sentence, i.e. there is no occurrence of variables outside the scope of a quantifier $\forall$, $\exists$ we can drop $s$ as subscripts of $\varphi$.

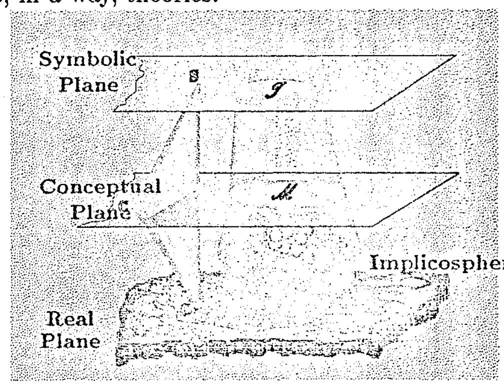
The domains $D$ are sets of arbitrary cardinality, and the evaluation functions are partially computable. Both systems, propositional and the first order language systems, are *two-valued* systems, i.e. $\mathcal{V} = \{0, 1\}$, and despite the fact that in the latter $\varphi$ is not computable, they are considered total functions, i.e. for all $< \alpha, M > \in \mathcal{L} \times \mathcal{M}$ the value $\varphi(\alpha, M)$ is defined. This, of course is not the most general situation, but in this work we will consider semantical systems in which $\varphi$ is total with respect to a certain subclass of $\mathcal{L} \times \mathcal{M}$ systems in which $\varphi$ is total and two-valued. This subclass $\varphi^{-1}(\mathcal{V}) \subseteq \mathcal{L} \times \mathcal{M}$ is a relation that holds between certain elements of the language $\mathcal{L}$ and certain models of $\mathcal{M}$. The set of sentences of $\mathcal{L}$ is denoted by $\mathcal{T}^{\mathcal{L}}_{\infty,\mathcal{M}} = \mathcal{D}(\varphi^{-1}(\mathcal{V})) \subseteq \mathcal{L}$ and the set of proper models of $\mathcal{M}$ is denoted by $\mathcal{W}^{\mathcal{M}}_{\infty,\mathcal{L}} = \mathcal{R}(\varphi^{-1}(\mathcal{V})) \subseteq \mathcal{M}$

# 4 Planes of Meaning

Douglas Hoffstadter, in [Hoff], introduced the concept of *Implicosphere* as related to the concept of creativity. Implicosphere can be imagined as a kind of a fuzzy collection of impressions one has about an object, concept or state, concrete or abstract. They are like clouds of impressions. This cloud becomes thicker and more complex as we gain familiarity with such entities. The *Implicosphere* of a concept includes its conections with other *Implicospheres*, and those conections are overlaps between *implicospheres*.

This is a very important idea because sentences do have meanings. Even though a sentence can be considered as an assemble of units (words), some of them carrying no semantical values - without independent meaning- , they might carry implications which depend more on the context in which they occur. We could say that sentences, even atoms or just symbols, can generate an infinity of inferences. *Implicospheres* are, in a way, theories.

By methodological reasons we will *project* such implicospheres in an imaginary plane **conceptual plane**, and the corresponding theories in another plane, **symbolic plane**. There is yet another plane, which we will call the **real plane**, where the objects, concepts or states actually *happen*. These three planes are connected, directly or indirectly, in a fashion similar to the meaning triangle.




These theories are formed through interactions inside a social group and they are conceptual entities which are developed in a very dynamical process. The interplay between the real plane and the conceptual plane is the main trade of the Natural and Social Sciences. Mathematicians, Logicians and Philosophers deal mostly with the interplay between the conceptual and symbolic planes. These planes are not as determined as shown in the presented pictures. As a matter of fact, knowledge is structured, and concepts varyes in their level of abstraction. There is a whole infinity of planes, and primitive concepts belong to lower levels of abstraction, following from them the derived concepts. The separation we did, in our presentation, in conceptual and symbolic planes is methodological, and the difference resides only in the degree of development of the semantical system. If we consider the primitive concepts of a theoretical discipline in a certain level, then derived concepts are in higher conceptual planes which are in some sense more symbolic, because they need of symbolic mechanisms such as definitions or derivation to be understood besides the understanding of the more basic or primitives.

We will only analyse the interplay between the conceptual plane (CP) and the symbolic plane (SP). We can say that a certain state of affairs is observed and acquires meaning through a *cultural filter* of concepts which are, in general, already *represented* in a semantical system. Then we assume that in a semantical system

$$< \mathcal{L}, \mathcal{M}, \mathcal{V}, \varphi >$$

the class $\mathcal{M}$ of models is situated on the conceptual plane **CP** and the language on the symbolic plane **SP**. So, if we consider a subclass $\mathcal{N}$ of $\mathcal{M}$, it corresponds to the understanding of a certain state of affairs in $\mathcal{M}$, that is represented in the language $\mathcal{L}$ by a subclass $\mathcal{T}$ of sentences or meaningful expressions of $\mathcal{L}$.

The correspondence between the subclasses $\mathcal{N}$ of $\mathcal{W}^{\mathcal{M}}_{\infty,\mathcal{L}}$ and the subclasses $\mathcal{T}$ of the language $\mathcal{T}^{\mathcal{L}}_{\infty,\mathcal{M}}$ have been examined for some semantical systems. These studies were dependent on the state of organization or syntatical structure of the language $\mathcal{L}$, and on restriction imposed on the class of models $\mathcal{M}$. In the case of the first order semantical systems it is usual to define functions $Mod : \mathcal{T}^{\mathcal{L}}_{\infty,\mathcal{M}} \longrightarrow \mathcal{W}^{\mathcal{M}}_{\infty,\mathcal{L}}$ and $Th : \mathcal{W}^{\mathcal{M}}_{\infty,\mathcal{L}} \longrightarrow \mathcal{T}^{\mathcal{L}}_{\infty,\mathcal{M}}$, which are defined considering the whole class of relational structures and the whole first order language. Besides, the second function $Th$ is sintatically defined from the function $Cn$. The relation between these functions, after Gödel's completeness theorem, is given by

$$Cn(S) = Th(Mod(S))$$

In this work we define functions **Mod**, **Th** without any dependence on syntatical considerations, moreover we relativize both to subclasses of $\mathcal{T}^{\mathcal{L}}_{\infty,\mathcal{M}}$ and $\mathcal{W}^{\mathcal{M}}_{\infty,\mathcal{L}}$.

In order to understand, abstractly, i.e. outside syntatical considerations, the goals and contributions of this work as a general framework to the discussion of *mathematical* and *computational systems* we refer the reader to Tarski's work [Tar1]. There Tarski presents four axioms related to two primitives notions: a set S of sentences and logical consequence Cn as an operator defined on S.

The first axiom is about the cardinality of the set S of sentences:

$A_1$. The set S is at most enumerable.

This axiom is intuitively reasonable, given that the set of sentences must fulfill certain requirements of communication. We could be more rigorous by saying:

$A_1'$. The set S is recursively enumerable.

which means that concerning the Deductive Sciences, the languages are formally, ideally, defined through formal grammars.
Even stronger, would be:

$A_1''$. The set S is recursive.

which implies the existence of recognition algorithms for the languages of Sciences.
The second axiom refers to the operational character of the function Cn:

$A_2$. If $A \subseteq S$ then $A \subseteq Cn(A) \subseteq S$

The third axiom states that once the consequences of a set of sentences are obtained no further knowledge can be gathered by re-applying the operator Cn:

$A_3$. If $A \subseteq S$ then $Cn(Cn(A)) = Cn(A)$

The last axiom states that all consequences obtained from a set of sentences A must also be obtainable from finite subsets of A:

$$A_4. \text{ If } A \subseteq S \text{ then } Cn(A) = \bigcup_{X \in Fin(A)} Cn(X)$$

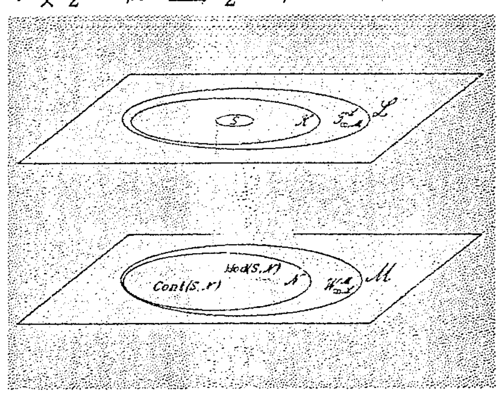
Despite the series of important results derived by Tarski, from the four axioms above, their meaning has been overlooked, at least with respect to their generality, in further research. Here we started with the notion of semantical systems [Carn1] and we shall prove $A_2$ and $A_3$ with the aid of very simple mathematical tools. By starting with simpler assumptions than the above axioms we shall attempt to clarify the results obtained by Tarski, as well as some others which make up the necessary theoretical basis for better evaluating some of the recent achievements in the field of Applied Logic.

# 5 Basic Functions

This section introduces four functions which represent the relationship between $\mathcal{L}$ and $\mathcal{M}$. These functions transpose some concepts which are traditionally treated in Model Theory to the wider context of a two-valued semantical system.

The first two functions represent *projections* from the symbolic plane to the conceptual plane, i.e. from language to models or interpretations:

$$(a) \cdots \mathbf{Mod} : 2^{\mathcal{T}^{\mathcal{L}}_{\infty,\mathcal{M}}} \times 2^{\mathcal{W}^{\mathcal{M}}_{\infty,\mathcal{L}}} \longrightarrow 2^{\mathcal{W}^{\mathcal{M}}_{\infty,\mathcal{L}}}$$
$$(b) \cdots \mathbf{Cont} : 2^{\mathcal{T}^{\mathcal{L}}_{\infty,\mathcal{M}}} \times 2^{\mathcal{W}^{\mathcal{M}}_{\infty,\mathcal{L}}} \longrightarrow 2^{\mathcal{W}^{\mathcal{M}}_{\infty,\mathcal{L}}}$$



- The function **Cont** applied to a set $S$ of sentences in $\mathcal{L}$ and to a subfamily $\mathcal{N}$ of models in $\mathcal{W}^{\mathcal{M}}_{\infty,\mathcal{L}}$ returns a subfamily of models in $\mathcal{N}$.
- The function **Mod** applied to a set $S$ of sentences in $\mathcal{L}$ and to a subfamily $\mathcal{N}$ of models in $\mathcal{M}$ returns a subfamily of models in $\mathcal{N}$.

The application of **Mod** results in the representation of the particular state of affairs described by $S$ regarding the world $\mathcal{N}$.

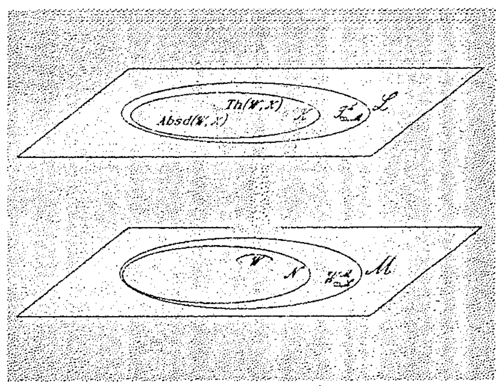
The application of **Cont** results in the representation of the state of affairs which happens outside the world $\mathcal{N}$, that is, which is contradicted by $S$. In the figure we represented the case in which $\varphi$ is a total function, i.e. $\mathbf{Mod}(S, \mathcal{N})$ and $\mathbf{Cont}(S,\mathcal{N})$ are complementary. The intuitive meaning of this situation is that all models can be referred by the language $\mathcal{T}^{\mathcal{L}}_{\infty,\mathcal{M}}$, so if $\varphi$ is a total function no model is left outside linguistic consideration. So the language $\mathcal{T}^{\mathcal{L}}_{\infty,\mathcal{M}}$ has enough *expressive power.*

The effects of the application of these two functions are interpretative: given a set of sentences of the language $\mathcal{L}$ of a semantical system $\mathcal{S}$, we generate *configurations* or models according to our understanding of the evaluation function $\varphi$. It must be clear that we have not classified $\varphi$ rom the point of view of its computability or even with respect to its domain, as being a total or parcial function. **Mod** and **Cont** can represent the abstract activity of a mathematician interpreting a collection of sentences, for instance equations, positively by **Mod** and negatively by **Cont**.

The following functions are also *projections*, but now from the conceptual plane to the symbolic plane. They represent formalizations.

$$(c) \cdots \mathbf{Th} : 2^{\mathcal{W}^{\mathcal{M}}_{\infty,\mathcal{L}}} \times 2^{\mathcal{T}^{\mathcal{L}}_{\infty,\mathcal{M}}} \longrightarrow 2^{\mathcal{T}^{\mathcal{L}}_{\infty,\mathcal{M}}}$$
$$(d) \cdots \mathbf{Absd} : 2^{\mathcal{W}^{\mathcal{M}}_{\infty,\mathcal{L}}} \times 2^{\mathcal{T}^{\mathcal{L}}_{\infty,\mathcal{M}}} \longrightarrow 2^{\mathcal{T}^{\mathcal{L}}_{\infty,\mathcal{M}}}$$

- The function **Th** applied to a set $\mathcal{N}$ of models in $\mathcal{W}^{\mathcal{M}}_{\infty,\mathcal{L}}$ and to a subset of $\mathcal{K}$, a dialect $\mathcal{K}$, returns a set of sentences in $\mathcal{K}$.
- The function **Absd** applied to a set $\mathcal{N}$ of models in $\mathcal{W}^{\mathcal{M}}_{\infty,\mathcal{L}}$ and to a subset of $\mathcal{T}^{\mathcal{L}}_{\infty,\mathcal{M}}$, a dialect $\mathcal{K}$ returns a set of sentences in $\mathcal{K}$.




The application of **Th** results in a description in $\mathcal{K}$ of the state of affairs $\mathcal{W}$ in $\mathcal{N}$. The application of **Absd** results in a description of sentences in $\mathcal{K}$ that do not occur in the state of affairs $\mathcal{W}$ in $\mathcal{N}$. In the following we consider only bivalued semantical systems $\mathcal{S} = < \mathcal{L}, \mathcal{M}, \{0, 1\}, \varphi >$

**Definição 5.1** *Let $\mathcal{S}$ be a bivalued semantical system, then*

i $\mathbf{Mod}(\alpha, \mathcal{N}) = \{M \in \mathcal{N} \mid \varphi(\alpha, M) = 1\}$
ii $\mathbf{Cont}(\alpha, \mathcal{N}) = \{M \in \mathcal{N} \mid \varphi(\alpha, M) = 0\}$
iii $\mathbf{Th}(M, \mathcal{K}) = \{\alpha \in \mathcal{K} \mid \varphi(\alpha, M) = 1\}$
iv $\mathbf{Absd}(M, \mathcal{K}) = \{\alpha \in \mathcal{K} \mid \varphi(\alpha, M) = 0\}$ □

Note: In $(i), \cdots, (iv)$ we used $\alpha$ instead of $\{\alpha\}$ and $M$ instead of $\{M\}$ as usual in the literature.

**Example 5.1** *For the propositional semantical system* $\mathcal{S}^p_2$:
$\mathcal{M} = \{\emptyset, \{p_1\}, \{p_2\}, \{p_1, p_2\}\}$, $\mathbf{Mod}(p_1, \mathcal{M}) = \{\{p_1\}\{p_1, p_2\}\}$
$\mathbf{Cont}(p_1, \mathcal{M}) = \{\emptyset, \{p_2\}\}$, $\{p_2, (p_2 \vee p_1)\} \subset \mathbf{Th}(\{p_2\}, \mathcal{L}^p_2)$
and $\{\neg p_2, (\neg p_2 \vee p_1)\} \subset \mathbf{Absd}(\{p_2\}, \mathcal{L}^p_2)$ □

**Definição 5.2** *Let* $\mathcal{S}$ *be a bivalued semantical system,* $\mathcal{N} \subseteq \mathcal{W}^{\mathcal{M}}_{\infty,\mathcal{L}}$ *and* $\mathcal{K} \subseteq \mathcal{T}^{\mathcal{L}}_{\infty,\mathcal{M}}$, *then* [2]

$$(I) \cdots \mathbf{Mod}(S, \mathcal{N}) = \bigcap_{\alpha \in S} \mathbf{Mod}(\alpha, \mathcal{N})$$
$$(II) \cdots \mathbf{Cont}(S, \mathcal{N}) = \bigcup_{\alpha \in S} \mathbf{Cont}(\alpha, \mathcal{N})$$
$$(III) \cdots \mathbf{Th}(\mathcal{N}, \mathcal{K}) = \bigcap_{M \in \mathcal{N}} \mathbf{Th}(M, \mathcal{K})$$
$$(IV) \cdots \mathbf{Absd}(\mathcal{N}, \mathcal{K}) = \bigcup_{M \in \mathcal{N}} \mathbf{Absd}(M, \mathcal{K}) \quad \square$$

We assume here that if $\{X_i\}$ is a family of subclasses of a class $X$ then:

$$(a_1) \cdots x \in \bigcap_{i \in A} X_i \text{ iff for all } i \in A,\ x \in X_i$$
$$(a_2) \cdots x \in \bigcup_{i \in A} X_i \text{ iff for some } i \in A,\ x \in X_i$$
$$(a_3) \cdots \bigcup_{i \in \emptyset} X_i = \emptyset$$
$$(a_4) \cdots \bigcap_{i \in \emptyset} X_i = X$$
$$(a_5) \cdots \bigcap_{i \in A \cup B} X_i = \bigcap_{i \in A} X_i \cap \bigcap_{i \in B} X_i$$

**Lema 5.1** *If* $S \subseteq \mathcal{K} \subseteq \mathcal{T}^{\mathcal{L}}_{\infty,\mathcal{M}}$ *and* $\mathcal{N} \subseteq \mathcal{W}^{\mathcal{M}}_{\infty,\mathcal{L}}$, *then:*

$(a_1)$ $\mathbf{Mod}(S,\mathcal{N}) \cap \mathbf{Cont}(S,\mathcal{N}) = \emptyset$
$(a_2)$ $\mathbf{Mod}(S,\mathcal{N}) \cup \mathbf{Cont}(S,\mathcal{N}) = \mathcal{N}$
$(a_3)$ $\mathbf{Mod}(S,\mathcal{N}) = \mathcal{N} - \mathbf{Cont}(S,\mathcal{N})$
$(b_1)$ $\mathbf{Th}(\mathcal{N},\mathcal{K}) \cap \mathbf{Absd}(\mathcal{N},\mathcal{K}) = \emptyset$
$(b_2)$ $\mathbf{Th}(\mathcal{N},\mathcal{K}) \cup \mathbf{Absd}(\mathcal{N},\mathcal{K}) = \mathcal{K}$
$(b_3)$ $\mathbf{Th}(\mathcal{N},\mathcal{K}) = \mathcal{K} - \mathbf{Absd}(\mathcal{N},\mathcal{K})$ □

$(a_1)$ and $(b_1)$ are independent from $\varphi$ being a total function. There are important connections between this lemma and the notions, to be developed, of consistency and completeness. In the case of the semantical system of first order languages, $\varphi$ is assumed to be a total function, despite the fact that the class of models is constituted of structures over very abstract domains.

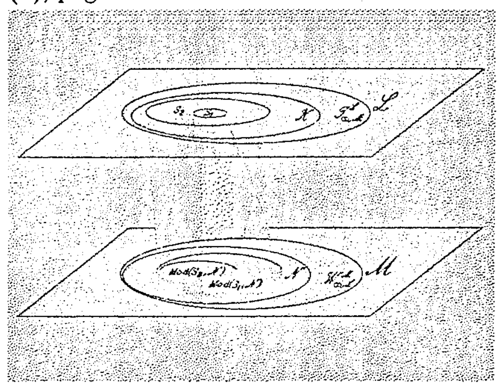
The following lemmas show the anti-monotonicity of **Mod** and **Th** and the monotonicity of **Cont** and **Absd** relative to the class $S$ of sentences.

**Lema 5.2** *Let* $S_1, S_2 \subseteq \mathcal{K} \subseteq \mathcal{T}^{\mathcal{L}}_{\infty,\mathcal{M}}$ and $\mathcal{N} \subseteq \mathcal{W}^{\mathcal{M}}_{\infty,\mathcal{L}}$. *If* $S_1 \subseteq S_2$ *then* $\mathbf{Mod}(S_2, \mathcal{N}) \subseteq \mathbf{Mod}(S_1, \mathcal{N})$.

[2]The intuitive justification for the use of $\bigcap$ in definitions $(I)$ and $(III)$ and for the use of $\bigcup$ in definitions $(II)$ and $(IV)$ is that the *meaning* associated with a class of sentences is the common *meaning*, the intersection of classes which represent the *meaning* of each individual sentence, this is in agreement with the traditional, classical or otherwise, usage of *comprehension* × *extension*, the same intuitive or at least conventional justification is given for classes of models.

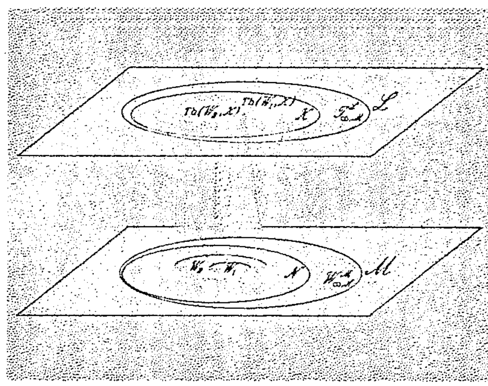
Proof: Immediate from definition $(I)$, page 13. □

The anti-monotonicity of **Mod** is intuitively justified by the fact that the extension of a description, i.e. the collection of objects corresponding to a description decreases with the increasing of the same description by the introduction of more details



**Lema 5.3** *Let* $\mathcal{N}_1, \mathcal{N}_2 \subseteq \mathcal{N} \subseteq \mathcal{W}^{\mathcal{M}}_{\infty,\mathcal{L}}$ *and* $\mathcal{K} \subseteq \mathcal{T}^{\mathcal{L}}_{\infty,\mathcal{M}}$. *If* $\mathcal{N}_1 \subseteq \mathcal{N}_2$ *then* $\mathbf{Th}(\mathcal{N}_2,\mathcal{K}) \subseteq \mathbf{Th}(\mathcal{N}_1,\mathcal{K})$. □

If we increase the amount of conceptual objects, i.e. models, we can say less (smaller number of sentences) about their common properties.



The monotonicity of **Cont** and **Absd** are shown by:

**Lema 5.4** *Let* $S_1, S_2 \subseteq \mathcal{T}^{\mathcal{L}}_{\infty,\mathcal{M}}$ *and* $\mathcal{N}_1,\mathcal{N}_2 \subseteq \mathcal{W}^{\mathcal{M}}_{\infty,\mathcal{L}}$. *If* $_1 \subseteq S_2$ *then* $\mathbf{Cont}(S_1,\mathcal{N}) \subseteq \mathbf{Cont}(S_2,\mathcal{N})$, *and if* $\mathcal{N}_1 \subseteq \mathcal{N}_2$ *then* $\mathbf{Absd}(N_1, \mathcal{K}) \subseteq \mathbf{Absd}(N_2, \mathcal{K})$
□

Monotonicity of the functions with respect to the second parameter comes from:

**Lema 5.5** *Let* $S \subseteq \mathcal{K} \subseteq \mathcal{T}^{\mathcal{L}}_{\infty,\mathcal{M}}$ *and* $\mathcal{W} \subseteq \mathcal{N} \subseteq \mathcal{W}^{\mathcal{M}}_{\infty,\mathcal{L}}$, *then:*

. $\mathbf{Th}(\mathcal{W},\mathcal{K}) = \mathbf{Th}(\mathcal{W},\mathcal{T}^{\mathcal{L}}_{\infty,\mathcal{M}}) \cap \mathcal{K}$

. $\mathbf{Absd}(\mathcal{W}, \mathcal{K}) = \mathbf{Absd}(\mathrm{W}, \mathcal{T}^{\mathcal{L}}_{\infty,\mathcal{M}}) \cap \mathcal{K}$

. $\mathbf{Mod}(S,\mathcal{N}) = \mathbf{Mod}(S, \mathcal{W}^{\mathcal{M}}_{\infty,\mathcal{L}}) \cap \mathcal{N}$

. $\mathbf{Cont}(S,\mathcal{N}) = \mathbf{Mod}(S, \mathcal{W}^{\mathcal{M}}_{\infty,\mathcal{L}}) \cap \mathcal{N}$ □

## 6 Preliminary results

In all that follows we will consider only semantical systems in which the evaluation function is not a constant function, the reason for this is given by:

**Lema 6.1** *Let* $\mathcal{S} = < \mathcal{L}, \mathcal{M}, \{0, 1\}, \varphi >$ *be a bivalued semantical system, and* $\varphi$ *a constant function, then:*

(a) . If for all $< \alpha, M > \in \mathcal{T}^{\mathcal{L}}_{\infty,\mathcal{M}} \times \mathcal{W}^{\mathcal{M}}_{\infty,\mathcal{L}}$ $\varphi(\alpha, M) = 0$ then:

- (a.1) For all $\mathcal{N} \subseteq \mathcal{W}^{\mathcal{M}}_{\infty,\mathcal{L}}$ $\mathbf{Mod}(\alpha,\mathcal{N}) = \emptyset$
- (a.2) For all $S \subseteq \mathcal{T}^{\mathcal{L}}_{\infty,\mathcal{M}}$ $\mathbf{Mod}(S,\mathcal{N}) = \emptyset$
- (a.3) For all $\mathcal{K} \subseteq \mathcal{T}^{\mathcal{L}}_{\infty,\mathcal{M}}$ $\mathbf{Th}(M,\mathcal{K}) = \emptyset$
- (a.4) For all $\mathcal{N} \subseteq \mathcal{W}^{\mathcal{M}}_{\infty,\mathcal{L}}$ and all $\mathcal{K} \subseteq \mathcal{T}^{\mathcal{L}}_{\infty,\mathcal{M}}$ $\mathbf{Th}(\mathcal{N},\mathcal{K}) = \emptyset$

(b) . If for all $< \alpha, M > \in \mathcal{T}^{\mathcal{L}}_{\infty,\mathcal{M}} \times \mathcal{W}^{\mathcal{M}}_{\infty,\mathcal{L}}$ $\varphi(\alpha, M) = 1$ then:

- (b.1) For all $\alpha \in \mathcal{T}^{\mathcal{L}}_{\infty,\mathcal{M}}$ $\mathbf{Mod}(\alpha,\mathcal{N}) = \mathcal{N}$
- (b.2) For all $S \subseteq \mathcal{T}^{\mathcal{L}}_{\infty,\mathcal{M}}$ $\mathbf{Mod}(S,\mathcal{N}) = \mathcal{N}$
- (b.3) For all $\mathcal{K} \subseteq \mathcal{T}^{\mathcal{L}}_{\infty,\mathcal{M}}$ $\mathbf{Th}(M,\mathcal{K}) = \mathcal{K}$
- (b.4) For all $\mathcal{N} \subseteq \mathcal{W}^{\mathcal{M}}_{\infty,\mathcal{L}} \subseteq \mathcal{T}^{\mathcal{L}}_{\infty,\mathcal{M}}$ and all $\mathcal{K}$ $\mathbf{Th}(\mathcal{N},\mathcal{K}) = \mathcal{K}$

Proof: Imediate. □

So semantical systems for which $\varphi$ is a constant function are not to be considered as proper semantical systems, they are *pseudo- semantical systems.*

**Lema 6.2** *Let* $\mathcal{N}_1, \mathcal{N}_2, \mathcal{W} \subseteq \mathcal{N} \subseteq \mathcal{W}^{\mathcal{M}}_{\infty,\mathcal{L}}$ *and* $S_1, S_2, S \subseteq \mathcal{T}^{\mathcal{L}}_{\infty,\mathcal{M}}$, *then:*

(a) $\mathbf{Mod}(\mathcal{S}_1 \cup \mathcal{S}_2, \mathcal{N}) = \mathbf{Mod}(\mathcal{S}_1, \mathcal{N}) \cap \mathbf{Mod}(\mathcal{S}_2, \mathcal{N})$
(b) $\mathbf{Mod}(S, \mathcal{N}_1 \cup \mathcal{N}_2) = \mathbf{Mod}(S, \mathcal{N}_1) \cup \mathbf{Mod}(S, \mathcal{N}_2)$
(c) $\mathbf{Th}(\mathcal{N}_1 \cup \mathcal{N}_2, \mathcal{K}) = \mathbf{Th}(\mathcal{N}_1, \mathcal{K}) \cap \mathbf{Th}(\mathcal{N}_2, \mathcal{K})$
(d) $\mathbf{Th}(\mathcal{W}, \mathcal{K}_1 \cup \mathcal{K}_2) = \mathbf{Th}(\mathcal{W}, \mathcal{K}_1) \cup \mathbf{Th}(\mathcal{W}, \mathcal{K}_2)$

Proof: (a),(b),(c) and (d) are imediate consequences of $(a_1)$, page 13, and lemma 7.5. □

**Lema 6.3** *If* $\mathcal{N} \subseteq \mathcal{W}^{\mathcal{M}}_{\infty,\mathcal{L}}$ $S, T \subseteq \mathcal{T}^{\mathcal{L}}_{\infty,\mathcal{M}}$, *and* $\mathcal{N}_1, \mathcal{N}_2 \subseteq \mathcal{W}^{\mathcal{M}}_{\infty,\mathcal{L}}$ *then:*

(*i*) $\mathbf{Mod}(S \cup T, \mathcal{N}) = \emptyset \Leftrightarrow \mathbf{Mod}(S, \mathcal{N}) \subseteq \mathbf{Cont}(T, \mathcal{N})$
(*ii*) $\mathbf{Th}(\mathcal{N}_1 \cup \mathcal{N}_2, \mathcal{K}) = \emptyset \Leftrightarrow \mathbf{Th}(\mathcal{N}_1, \mathcal{K}) \subseteq \mathbf{Absd}(\mathcal{N}_2, \mathcal{K})$

Proof: from lemma 8.2 (a) and hypothesis (*i*) $\mathbf{Mod}(S \cup T, \mathcal{N}) = \emptyset \Leftrightarrow \mathbf{Mod}(S, \mathcal{N}) \cap \mathbf{Mod}(T, \mathcal{N}) = \emptyset \Leftrightarrow \mathbf{Mod}(S, \mathcal{N}) \subseteq \mathbf{Cont}(T, \mathcal{N})$. The proof of (ii) is the dual of (i). □

Note: The last step in the proof of (i) depends on the complementarity of **Mod** and **Cont** and so on the admission that $\varphi$ is a total function. Also, the above result is apparently strange in face of hypothesis (ii) which seems to be unreasonable. It indicates a state of affairs where not even true sentences are possible. Nevertheless we could admit that the language $\mathcal{L}$ of the semantical system is inadequate for expressing valid sentences for all states of affairs.

**Lema 6.4** *Let* $\mathcal{W} \subseteq \mathcal{N} \subseteq \mathcal{W}^{\mathcal{M}}_{\infty,\mathcal{L}}$ *and* $\mathcal{K} \subseteq \mathcal{T}^{\mathcal{L}}_{\infty,\mathcal{M}}$, *then* $\alpha \in \mathbf{Th}(\mathcal{W},\mathcal{K}) \Leftrightarrow \mathcal{W} \subseteq \mathbf{Mod}(\alpha,\mathcal{N})$

Proof: $\alpha \in \mathbf{Th}(\mathcal{W},\mathcal{K}) \Leftrightarrow \alpha \in \bigcap_{M \in \mathcal{W}} \mathbf{Th}(M,\mathcal{K}) \Leftrightarrow$ for all $M \in \mathcal{W}$, $\alpha \in \mathbf{Th}(M,\mathcal{K}) \Leftrightarrow$ for all $M \in \mathcal{W}$ $\varphi(M, \alpha) = 1 \Leftrightarrow$ for all $M \in \mathcal{W}$, $M \in \mathbf{Mod}(\alpha,\mathcal{N}) \Leftrightarrow \mathcal{W} \subseteq \mathbf{Mod}(\alpha,\mathcal{N})$ □
In the same way we can prove:

**Lema 6.5** *Let* $S \subseteq \mathcal{K} \subseteq \mathcal{T}^{\mathcal{L}}_{\infty,\mathcal{M}}$ *and* $M \in \mathcal{N} \subseteq \mathcal{W}^{\mathcal{M}}_{\infty,\mathcal{L}}$, *then* $M \in \mathbf{Mod}(S,\mathcal{N}) \Leftrightarrow S \subseteq \mathbf{Th}(M,\mathcal{K})$. □

**Lema 6.6** *Let* $\mathcal{W} \subseteq \mathcal{N} \subseteq \mathcal{W}^{\mathcal{M}}_{\infty,\mathcal{L}}$ *and* $\alpha \in \mathcal{K} \subseteq \mathcal{T}^{\mathcal{L}}_{\infty,\mathcal{M}}$, *then* $\alpha \in \mathbf{Absd}(\mathcal{W},\mathcal{K}) \Leftrightarrow \mathcal{W} \cap \mathbf{Cont}(\alpha,\mathcal{N}) \neq \emptyset$

Proof: $\alpha \in \mathbf{Absd}(\mathcal{W},\mathcal{K}) \Leftrightarrow \alpha \in \bigcup_{M \in \mathcal{W}} \mathbf{Absd}(M,\mathcal{K})$ $\Leftrightarrow$ exists $M \in \mathcal{W}$ such that $\alpha \in \mathbf{Absd}(M,\mathcal{K}) \Leftrightarrow$ exists $M \in \mathcal{W}$ such that $\varphi(M, \alpha) = 0 \Leftrightarrow$ exists $M \in \mathcal{W}$, and $M \in \mathbf{Cont}(\alpha,\mathcal{N}) \Leftrightarrow \mathcal{W} \cap \mathbf{Cont}(\alpha,\mathcal{N}) \neq \emptyset$ □
Similarly we can prove:

**Lema 6.7** *Let* $S \subseteq \mathcal{K} \subseteq \mathcal{T}^{\mathcal{L}}_{\infty,\mathcal{M}}$ *and* $\mathcal{N} \subseteq \mathcal{W}^{\mathcal{M}}_{\infty,\mathcal{L}}$ *then* $M \in \mathbf{Cont}(S,\mathcal{N})$ *if and only if* $S \cap \mathbf{Absd}(M,\mathcal{K}) \neq \emptyset$ □

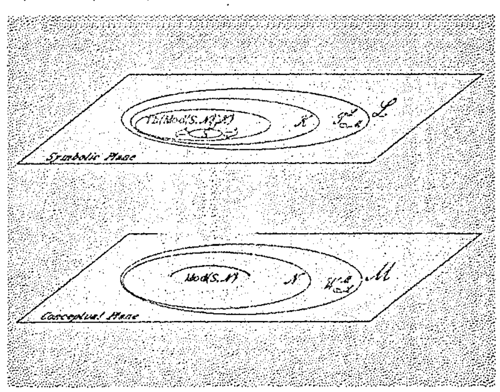
# 7 Semantical Consequence

The concept of consequence is usually associated to the concepts of cause and effect. To account for the occurence of an event $A$ there exists a sequence of events $B_1, B_2, \cdots, B_n, \cdots$, ending with $A$ such that for each $B_i$ there exists a sequence $B_1, B_2, \cdots, B_{i-1}$ that accounts for it. In this way, the events $B_i$ are like links of a chain, or generally, like nodes of a tree or of a directed graph. At an arbitrary point of an explanation this graph could be seen as a hierarchical structure, that is, a subgraph without loops. This intuitive notion of an explanation could be a model of what is known as the Hypothetical-Deductive method.

The causality between consecutive nodes is generally not as simple as a conditional expression such as **If** < *premiss* > **then** < *conclusion* >, but relies upon a wider context to which the concepts (model level) and the symbols (language level) representing the regarded situation belong. Thus the notion of consequence must depend on representational features, such as the language available for the description of the events, and on abstractional features such as the models that convey only certain aspects of the overall state of affairs. Considering our notion of two-valued semantical system $\mathcal{S} = < \mathcal{L}, \mathcal{M}, \{0,1\}, \varphi >$, to define consequence we must be concerned with the dialect $\mathcal{K}$ of $\mathcal{L}$, actually used to describe a certain state of affairs[3] $\mathcal{N}$ as part of the overall state of affairs $\mathcal{M}$ of $\mathcal{S}$.

In the previous section we have seen that $\textbf{Mod}(S,\mathcal{N})$ is a subclass of $\mathcal{N}$ which satisfies the class $S$ of sentences of a dialect $\mathcal{K}$ of $\mathcal{L}$, which means that $\text{Mod}(S,\mathcal{N})$ is the class of all situations where the events represented by $S$ occur. Intuitively, a sentence $\beta$ of $\mathcal{K}$ is a consequence of $S$ if whenever all sentences of $S$ occur then $\beta$ occurs.

**Definição 7.1** $\textbf{CnS}(S,\mathcal{N},\mathcal{K}) = \textbf{Th}(\textbf{Mod}(S,\mathcal{N}),\mathcal{K})$ □



The projection of $S$ on the conceptual plane generates $\text{Mod}(S,\mathcal{N})$, which is the class of all interpretations of the classe $S$ of sentences with respect to $\mathcal{N}$.
The representation or formalization of these interpretations obtained by the projection of $\text{Mod}(S,\mathcal{N})$, back on the symbolic plane, contains more sentences than in the original class $S$. This is the meaning of the consequences of $S$ with respect to $\mathcal{N}$

The following result is more *operational* and intuitive:

**Corolario 7.1** $\alpha \in \textbf{CnS}(S,\mathcal{N},\mathcal{K}) \Leftrightarrow \textbf{Mod}(S,\mathcal{N}) \subseteq \textbf{Mod}(\alpha,\mathcal{N})$

Proof: From the definition 9.1 $\alpha \in \textbf{CnS}(S,\mathcal{N},\mathcal{K}) \Leftrightarrow \alpha \in \text{Th}(\text{Mod}(S,\mathcal{N}),\mathcal{K})$, so from lemma 8.4 $\alpha \in \textbf{CnS}(S,\mathcal{N},\mathcal{K}) \Leftrightarrow \textbf{Mod}(S,\mathcal{N}) \subseteq \textbf{Mod}(\alpha,\mathcal{N})$ □

**Exemplo 7.1** *In* $\mathcal{S}^{p}_{3}$ $p_2 \in \textbf{CnS}(\{p_1,(p_1 \vee p_2)\},\{\{p_1,p_2\},\{p_2,p_3\}\},\mathcal{L}^{p}_{n})$, *however* $p_2 \notin \textbf{CnS}(\{p_1,(p_1 \vee p_2)\},\mathcal{M}^{p}_{3},\mathcal{L}^{p}_{n})$ □

We shall later discuss the variation of **CnS** regarding the language $\mathcal{K}$ and $\mathcal{N}$.

[3]In Mathematical Logic the context does not affect the notion of consequence. Instead, the language must have considerable expressive power (first order languages and its extensions) and a sufficiently comprehensive class of models (relational structure).

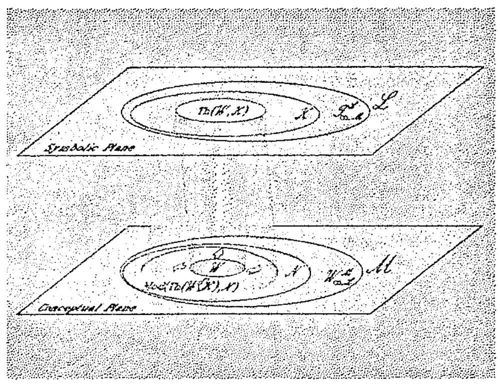
**Lema 7.1** $S \subseteq$ **CnS**$(S,\mathcal{N},\mathcal{K})$

Proof: By the corollary 7.1 we have only to show that for all $\alpha$, if $\alpha \in S$ then $\text{Mod}(S,\mathcal{N}) \subseteq \text{Mod}(\alpha,\mathcal{N})$, and this is a direct consequence of definition 5.2. □
The dual concept of **CnS** is obtained by:

**Definição 7.2** **CnM**$(\mathcal{W},\mathcal{K},\mathcal{N})$ = **Mod**(**Th**$(\mathcal{W},\mathcal{K}),\mathcal{N}$) □

**Corolario 7.2** $M \in$ **CnM**$(\mathcal{W},\mathcal{K},\mathcal{N}) \Leftrightarrow$ **Th**$(\mathcal{W},\mathcal{K}) \subseteq$ **Th**$(M,\mathcal{K})$

Proof: Similar to corollary 7.1. □



The projection of $\mathcal{W}$ on the symbolic plane generates $\text{Th}(\mathcal{W},\mathcal{K})$, which is the formalization of the classe $\mathcal{W}$ of models with respect to lingK.
The representation or interpretation of these sentences obtained by the projection of $\text{Th}(\mathcal{W},\mathcal{K})$, back on the conceptual plane, contains more models than in the original class $\mathcal{W}$. This is the meaning of the consequences of $\mathcal{W}$ with respect to $\mathcal{K}$

**Lema 7.2** $\mathcal{W} \subseteq$ **CnM**$(\mathcal{W},\mathcal{K},\mathcal{N})$

Proof: Similar to lemma 7.1. □

# 8 Behavior of CnS and CnM.

In this section we shall study the behavior of the functions **CnS** and **CnM** as we vary their arguments.

**Lema 8.1** $S \subseteq T \Rightarrow$ **CnS**$(S,\mathcal{N},\mathcal{K}) \subseteq$ **CnS**$(T,\mathcal{N},\mathcal{K})$

Proof: Follows directly from lemmas 7.2 and 7.3. □

**Lema 8.2** *If* $\mathcal{N}_1 \subseteq \mathcal{N}_2$ *then* **CnS**$(S,\mathcal{N}_2,\mathcal{K}) \subseteq$ **CnS**$(S,\mathcal{N}_1,\mathcal{K})$

Proof: It follows immediately from lemma 7.1 □

An intuitive interpretation for lemma 8.2 is that the smaller our specific knowledge of a general state of affairs (*world*), the smaller the class of consequences we can derive. The class $S$ is a *formalization* obtained by *generalization* or inductive inference from a number of particular pieces of knowledge of a world of which we have only an imprecise definition. If we want consequences which are undoubtably acceptable whichever the world $\mathcal{N}$ concerning $S$. To make sure we do so we must employ **CnS** regarding the wider world $\mathcal{W}^{\mathcal{M}}_{\infty,\mathcal{L}}$, as shown in the following lemma, whose proof is immediate.

**Lema 8.3** $\mathbf{CnS}(\mathcal{S},\mathcal{W}^{\mathcal{M}}_{\infty,\mathcal{L}},\mathcal{K}) = \bigcap_{\mathcal{W} \subseteq \mathcal{W}^{\mathcal{M}}_{\infty,\mathcal{L}}} \mathbf{CnS}(\mathcal{S},\mathcal{W},\mathcal{K})$ □

The dual results are:

**Lema 8.4** *If* $\mathcal{K}_1 \subseteq \mathcal{K}_2$ *then* $\mathbf{CnS}(\mathcal{S},\mathcal{N},\mathcal{K}_1) \subseteq \mathbf{CnS}(\mathcal{S},\mathcal{K},\mathcal{K}_2)$

Proof: Immediate. □

**Lema 8.5** *If* $\mathcal{W}_1 \subseteq \mathcal{W}_2$ *then* $\mathbf{CnM}(\mathcal{W}_1,\mathcal{K},\mathcal{N}) \subseteq \mathbf{CnM}(\mathcal{W}_2,\mathcal{K},\mathcal{N})$

Proof: Dual to lemma 8.1. □

**Lema 8.6** *If* $\mathcal{K}_1 \subseteq \mathcal{K}_2$ *then* $\mathbf{CnM}(\mathcal{W},\mathcal{K}_2,\mathcal{N}) \subseteq \mathbf{CnM}(\mathcal{W},\mathcal{K}_1,\mathcal{N})$

Proof: Similar to lemma 8.2 □

**Lema 8.7** *If* $\mathcal{N}_1 \subseteq \mathcal{N}_2$ *then* $\mathbf{CnM}(\mathcal{W},\mathcal{K},\mathcal{N}_1) \subseteq \mathbf{CnM}(\mathcal{W},\mathcal{K},\mathcal{N}_2)$

Proof: Immediate. □

**Lema 8.8** $\mathbf{CnS}(\mathcal{S} \cup \mathcal{T},\mathcal{N},\mathcal{K}) = \mathbf{CnS}(\mathbf{CnS}(\mathcal{S},\mathcal{N},\mathcal{K}) \cup \mathbf{CnS}(\mathcal{T},\mathcal{N},\mathcal{K}),\mathcal{N},\mathcal{K})$

Proof: From lemma 7.1 $\mathcal{S} \cup \mathcal{T} \subseteq \mathcal{S} \cup \mathbf{CnS}(\mathcal{T},\mathcal{N},\mathcal{K})$ and $\mathcal{S} \cup \mathcal{T} \subseteq \mathbf{CnS}(\mathcal{S},\mathcal{N},\mathcal{K}) \cup \mathbf{CnS}(\mathcal{T},\mathcal{N},\mathcal{K})$, by the monotonicity of **CnS** (lemma 8.1):
(1) $\cdots$ $\mathbf{CnS}(\mathcal{S} \cup \mathcal{T},\mathcal{N},\mathcal{K}) \subseteq \mathbf{CnS}(\mathbf{CnS}(\mathcal{S},\mathcal{N},\mathcal{K}) \cup \mathbf{CnS}(\mathcal{T},\mathcal{N},\mathcal{K}),\mathcal{N},\mathcal{K})$.
Again, from lemma 8.1 $\mathbf{CnS}(\mathcal{S},\mathcal{N},\mathcal{K}) \subseteq \mathbf{CnS}(\mathcal{S} \cup \mathcal{T},\mathcal{N},\mathcal{K})$ and $\mathbf{CnS}(\mathcal{T},\mathcal{N},\mathcal{K}) \subseteq \mathbf{CnS}(\mathcal{S} \cup \mathcal{T},\mathcal{N},\mathcal{K})$, and so:
(2) $\cdots$ $\mathbf{CnS}(\mathcal{S},\mathcal{N},\mathcal{K}) \cup \mathbf{CnS}(\mathcal{T},\mathcal{N},\mathcal{K}) \subseteq \mathbf{CnS}(\mathcal{S} \cup \mathcal{T},\mathcal{N},\mathcal{K})$
From (1) and (2) $\mathbf{CnS}(\mathbf{CnS}(\mathcal{S},\mathcal{N},\mathcal{K}) \cup \mathbf{CnS}(\mathcal{T},\mathcal{N},\mathcal{K}),\mathcal{N},\mathcal{K}) = \mathbf{CnS}(\mathcal{S} \cup \mathcal{T},\mathcal{N},\mathcal{K})$. □

**Lema 8.9** $\mathbf{CnM}(\mathcal{W}_1 \cup \mathcal{W}_2,\mathcal{K},\mathcal{N}) = \mathbf{CnM}(\mathbf{CnM}(\mathcal{W}_1,\mathcal{K},\mathcal{N}) \cup \mathbf{CnM}(\mathcal{W}_2,\mathcal{K},\mathcal{N}),\mathcal{K},\mathcal{N})$.

Proof: Dual of the one before. □

# 9 Closure Properties

The results to be presented in this section are corollaries of the previous definitions and results. Nevertheless, they are essential for the reader to acquire a geometrical view of the relationships between languages and models

**Lema 9.1** $\mathbf{Mod}(\mathcal{S},\mathcal{N}) = \mathbf{Mod}(\mathbf{Th}(\mathbf{Mod}(\mathcal{S},\mathcal{N}),\mathcal{K}),\mathcal{N})$

Proof: From lemma 7.1 $\mathcal{S} \subseteq \mathbf{Th}(\mathbf{Mod}(\mathrm{S},\mathcal{N}),\mathcal{K})$, so from lemma 7.2:
(1) $\mathbf{Mod}(\mathbf{Th}(\mathbf{Mod}(\mathrm{S},\mathcal{N}),\mathcal{K}),\mathcal{N}) \subseteq \mathbf{Mod}(\mathcal{S},\mathcal{N})$,
so by lemma 8.2: (2) $\mathbf{Mod}(\mathcal{S},\mathcal{N}) \subseteq \mathbf{Mod}(\mathbf{Th}(\mathbf{Mod}(\mathrm{S},\mathcal{N}),\mathcal{K}),\mathcal{N})$, therefore from (1) and (2) $\mathbf{Mod}(\mathcal{S},\mathcal{N}) = \mathbf{Mod}(\mathbf{Th}(\mathbf{Mod}(\mathrm{S},\mathcal{N}),\mathcal{K}),\mathcal{N})$. □

Proof: follows immediately from $\mathcal{T}^{\mathcal{L}}_{0,\mathcal{M}} = \bigcap_{\mathcal{W} \subseteq \mathcal{W}^{\mathcal{M}}_{\infty,\mathcal{L}}} \mathrm{Th}(\mathcal{W},\mathcal{T}^{\mathcal{L}}_{\infty,\mathcal{M}})$ □

**Corolario 9.4** *$\mathcal{T}^{\mathcal{L}}_{\infty,\mathcal{M}}$ is the largest theory and $\mathcal{W}^{\mathcal{M}}_{\infty,\mathcal{L}}$ is the largest world.*

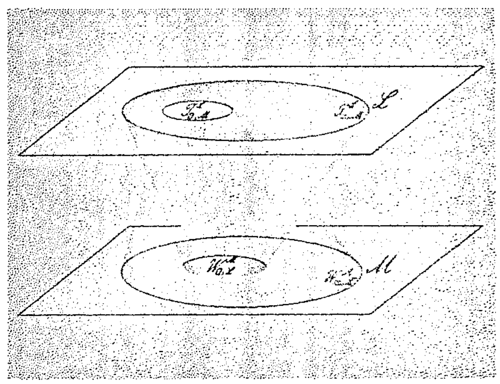
Proof: $\mathcal{T}^{\mathcal{L}}_{\infty,\mathcal{M}} = \mathrm{Th}(\mathcal{W}^{\mathcal{M}}_{0,\mathcal{L}},\mathcal{T}^{\mathcal{L}}_{\infty,\mathcal{M}})$ and $\mathcal{W}^{\mathcal{M}}_{\infty,\mathcal{L}} = \mathrm{Mod}(\mathcal{T}^{\mathcal{L}}_{0,\mathcal{M}},\mathcal{W}^{\mathcal{M}}_{\infty,\mathcal{L}})$. □

The picture shows the relations between the classes:
- $\mathcal{T}^{\mathcal{L}}_{0,\mathcal{M}}$ and $\mathcal{W}^{\mathcal{M}}_{\infty,\mathcal{L}}$, given by

$$\mathcal{T}^{\mathcal{L}}_{0,\mathcal{M}} = \mathrm{Th}(\mathcal{W}^{\mathcal{M}}_{\infty,\mathcal{L}},\mathcal{T}^{\mathcal{L}}_{\infty,\mathcal{M}})$$

and between the classes:
- $\mathcal{W}^{\mathcal{M}}_{0,\mathcal{L}}$ and $\mathcal{T}^{\mathcal{L}}_{\infty,\mathcal{M}}$, given by

$$\mathcal{W}^{\mathcal{M}}_{0,\mathcal{L}} = \mathrm{Mod}(\mathcal{T}^{\mathcal{L}}_{\infty,\mathcal{M}},\mathcal{W}^{\mathcal{M}}_{\infty,\mathcal{L}})$$



Despite the facts that, neither every set of sentences $S$ is a theory, nor every class of models $\mathcal{W}$ is a world, there exists as many theories as there are subclasses of $\mathcal{W}^{\mathcal{M}}_{\infty,\mathcal{L}}$ and as many worlds as subsets of $\mathcal{T}^{\mathcal{L}}_{\infty,\mathcal{M}}$ . Thus, if in a semantical system $\mathcal{S}$ the cardinality of $\mathcal{T}^{\mathcal{L}}_{\infty,\mathcal{M}}$ is $\nu$ there will be $2^{\nu}$ worlds, and if the cardinality of $\mathcal{W}^{\mathcal{M}}_{\infty,\mathcal{L}}$ is $\eta$ there will be $2^{\eta}$ theories.

## 10 True and False sentences

In the last section we defined two very importante classes, $\mathcal{T}^{\mathcal{L}}_{0,\mathcal{M}}$ and $\mathcal{W}^{\mathcal{M}}_{0,\mathcal{L}}$.

- $\mathcal{T}^{\mathcal{L}}_{0,\mathcal{M}}$ is the smallest theory and its sentences $\alpha$ are **true** sentences in the sense that for all models $M \in \mathcal{W}^{\mathcal{M}}_{\infty}$, $\varphi(\alpha, M) = 1$

- $\mathcal{W}^{\mathcal{M}}_{0,\mathcal{L}}$ is the smallest world and its models $M$ are **universal** models, in the sense that for all $\alpha \in \mathcal{T}^{\mathcal{L}}_{\infty,\mathcal{M}}$, $\varphi(\alpha, M) = 1$. Two other classes are also important:

- $\mathcal{F}^{\mathcal{L}}_{0} = \bigcap_{M \in \mathcal{W}^{\mathcal{M}}_{\infty}} \mathrm{Absd}(M, \mathcal{T}^{\mathcal{L}}_{\infty,\mathcal{M}})$, called the class of **false sentences.**

- $\mathcal{N}^{\mathcal{M}}_{0} = \bigcap_{\alpha \in \mathcal{T}^{\mathcal{L}}_{\infty,\mathcal{M}}} \mathrm{Cont}(\alpha,\mathcal{W}^{\mathcal{M}}_{\infty})$ called the class of **empty models.** □

In this section we will examine the relationship between these classes. For this purpose we present a few results.

**Lema 10.1** *For all $\gamma \in \mathcal{T}^{\mathcal{L}}_{\infty,\mathcal{M}}$, $\gamma \in \mathcal{F}^{\mathcal{L}}_{0}$ iff* $\mathrm{Mod}(\gamma,\mathcal{W}^{\mathcal{M}}_{\infty}) = \emptyset$.

**Lema 9.2** $\mathrm{Th}(\mathcal{W},\mathcal{K}) = \mathrm{Th}(\mathrm{Mod}(\mathrm{Th}(\mathcal{W},\mathcal{K}),\mathcal{N}),\mathcal{K})$

Proof: Dual to lemma 8.1. □

The expressions $\mathrm{Th}(\mathcal{W},\mathcal{K}) = \mathrm{Th}(\mathrm{Mod}(\mathrm{Th}(\mathcal{W},\mathcal{K}),\mathcal{N}),\mathcal{K})$ and $\mathrm{Mod}(S,\mathcal{N}) = \mathrm{Mod}(\mathrm{Th}(\mathrm{Mod}(S,\mathcal{N}),\mathcal{K}),\mathcal{N})$ could be re-written generating the expressions:

(A) $\mathrm{Mod}(S,\mathcal{N}) = \mathrm{Mod}(\mathrm{CnS}(S,\mathcal{N},\mathcal{K}),\mathcal{N})$
(B) $\mathrm{Mod}(S,\mathcal{N}) = \mathrm{CnM}(\mathrm{Mod}(S,\mathcal{N}),\mathcal{K},\mathcal{N})$
(C) $\mathrm{Th}(\mathcal{W},\mathcal{K}) = \mathrm{Th}(\mathrm{CnM}(\mathcal{W},\mathcal{K},\mathcal{N}),\mathcal{K})$
(D) $\mathrm{Th}(\mathcal{W},\mathcal{K}) = \mathrm{CnS}(\mathrm{Th}(\mathcal{W},\mathcal{K}),\mathcal{N},\mathcal{K})$

(A) says that $\mathrm{CnS}(S,\mathcal{N},\mathcal{K})$ is the largest class of sentences $\mathcal{K}$ that contain $S$ and preserves its models, this suggests the following:

**Definição 9.1** *$\mathcal{T}$ is a theory in $\mathcal{K}$ and $\mathcal{N}$ iff* $\mathcal{T} = \mathrm{CnS}(\mathcal{T},\mathcal{N},\mathcal{K})$ □

**Lema 9.3** $\mathcal{T}^{\mathcal{L}}_{\infty,\mathcal{M}}$ *is a theory in* $\mathcal{W}^{\mathcal{M}}_{\infty,\mathcal{L}}$ *and* $\mathcal{T}^{\mathcal{L}}_{\infty,\mathcal{M}}$.

Proof: From lemma 7.1 $\mathcal{T}^{\mathcal{L}}_{\infty,\mathcal{M}} = \mathrm{CnS}(\mathcal{T}^{\mathcal{L}}_{\infty,\mathcal{M}},\mathcal{W}^{\mathcal{M}}_{\infty,\mathcal{L}},\mathcal{T}^{\mathcal{L}}_{\infty,\mathcal{M}})$ □

**Lema 9.4** *$\mathcal{T}$ is a theory in $\mathcal{N}$ and $\mathcal{K}$ if and only if there exists $\mathcal{W}_{\mathcal{T}}$ such that* $\mathcal{T} = \mathrm{Th}(\mathcal{W}_{\mathcal{T}},\mathcal{K})$.

Proof: It suffices to take $\mathcal{W}_{\mathcal{T}} = \mathrm{Mod}(\mathcal{T},\mathcal{N})$. □
(C) suggests the following definition:

**Definição 9.2** *$\mathcal{W}$ is a* **world** *in $\mathcal{K}$ and $\mathcal{N}$ iff* $\mathcal{W} = \mathrm{CnM}(\mathcal{W},\mathcal{K},\mathcal{N})$ □

$\mathrm{CnM}(\mathcal{W},\mathcal{K},\mathcal{N})$ is the largest class of models that contain $\mathcal{W}$ and preservs its theory. This, again suggests the following:

**Lema 9.5** $\mathcal{W}^{\mathcal{M}}_{\infty,\mathcal{L}}$ *is a in* $\mathcal{T}^{\mathcal{L}}_{\infty,\mathcal{M}}$ *and* $\mathcal{W}^{\mathcal{M}}_{\infty,\mathcal{L}}$.

Proof: trivial. □

**Corolario 9.1** *$\mathcal{W}$ is a world in $\mathcal{K}$ and $\mathcal{N}$ if and only if there exists $\mathcal{T}_{\mathcal{W}}$, $\mathcal{T}_{\mathcal{W}} \subseteq \mathcal{K}$ such that* $\mathcal{W} = \mathrm{Mod}(\mathcal{T}_{\mathcal{W}},\mathcal{N})$

Proof: It suffices to take $\mathcal{T}_{\mathcal{W}} = \mathrm{Th}(\mathcal{W},\mathcal{K})$. □

**Corolario 9.2** $\mathcal{W}^{\mathcal{M}}_{0,\mathcal{L}} = \mathrm{Mod}(\mathcal{T}^{\mathcal{L}}_{\infty,\mathcal{M}},\mathcal{W}^{\mathcal{M}}_{\infty,\mathcal{L}})$ *is the smallest world*

Proof: follows immediately from $\mathcal{W}^{\mathcal{M}}_{0,\mathcal{L}} = \bigcap_{\mathcal{T} \subseteq \mathcal{T}^{\mathcal{L}}_{\infty,\mathcal{M}}} \mathrm{Mod}(\mathcal{T},\mathcal{W}^{\mathcal{M}}_{\infty,\mathcal{L}})$ □

**Corolario 9.3** $\mathcal{T}^{\mathcal{L}}_{0,\mathcal{M}} = \mathrm{Th}(\mathcal{W}^{\mathcal{M}}_{\infty,\mathcal{L}},\mathcal{T}^{\mathcal{L}}_{\infty,\mathcal{M}})$ *is the smallest theory*

Proof: $\gamma \in \mathcal{F}_0^{\mathcal{L}}$ iff $\gamma \in \bigcap_{M \in \mathcal{W}_\infty^{\mathcal{M}}}$ **Absd** $( M , \mathcal{T}_{\infty,\mathcal{M}}^{\mathcal{L}} )$ iff for all $M \in \mathcal{W}_\infty^{\mathcal{M}}$ $M \in$ **Absd** $( M , \mathcal{T}_{\infty,\mathcal{M}}^{\mathcal{L}} )$ iff for all $M \in \mathcal{W}_\infty^{\mathcal{M}}$ $\varphi( \gamma , M ) = 0$ iff for all $M \in \mathcal{W}_\infty^{\mathcal{M}}$ $M \in$ **Cont**$(\gamma,\mathcal{W}_\infty^{\mathcal{M}})$ iff **Cont**$(\gamma,\mathcal{W}_\infty^{\mathcal{M}}) = \mathcal{W}_\infty^{\mathcal{M}}$ iff **Mod**$(\gamma, \mathcal{W}_\infty^{\mathcal{M}}) = \emptyset$ □

**Lema 10.2** $\mathcal{F}_0^{\mathcal{L}} \neq \emptyset \quad iff \quad \mathcal{W}_{0,\mathcal{L}}^{\mathcal{M}} = \emptyset$

Proof ( $\longrightarrow$ ) : $\mathcal{F}_0^{\mathcal{L}} \neq \emptyset$ then there is $\gamma \in \mathcal{F}_0^{\mathcal{L}}$ , by the last lemma **Mod**$(\gamma, \mathcal{W}_\infty^{\mathcal{M}}) = \emptyset$ , as $\mathcal{W}_{0,\mathcal{L}}^{\mathcal{M}} \subseteq$ **Mod**$(\gamma, \mathcal{W}_\infty^{\mathcal{M}})$, thus $\mathcal{W}_{0,\mathcal{L}}^{\mathcal{M}} = \emptyset$ .
( $\longleftarrow$ ) $\mathcal{W}_{0,\mathcal{L}}^{\mathcal{M}} = \emptyset$ , as $\mathcal{W}_{0,\mathcal{L}}^{\mathcal{M}} = \bigcap_{\alpha \in \mathcal{T}_{\infty,\mathcal{M}}^{\mathcal{L}}}$ **Mod**$(\alpha, \mathcal{W}_\infty^{\mathcal{M}})$ there is $\gamma \in \mathcal{T}_{\infty,\mathcal{M}}^{\mathcal{L}}$ such that **Mod**$(\gamma, \mathcal{W}_\infty^{\mathcal{M}}) = \emptyset$ , so by from lemma 10.1 $\gamma \in \mathcal{F}_0^{\mathcal{L}}$ and so $\mathcal{F}_0^{\mathcal{L}} \neq \emptyset$ □
The last result shows that the existence of contradictions or falses sentences in a semantical system depends on the non existence of universal models. The next two lemmas shows the dual situation.

**Lema 10.3** *For all* $M \in \mathcal{W}_\infty^{\mathcal{M}}$, $M \in \mathcal{N}_0^{\mathcal{M}}$ *iff* **Th**$(M,\mathcal{T}_{\infty,\mathcal{M}}^{\mathcal{L}}) = \emptyset$ .

Proof: Dual to lemma 12.1. □

**Lema 10.4** $\mathcal{N}_0^{\mathcal{M}} \neq \emptyset \quad iff \quad \mathcal{T}_{0,\mathcal{M}}^{\mathcal{L}} = \emptyset$

Proof: Dual to lemma 12.2. □
There are empty models if and only if there are no true sentences. Relations between $\mathcal{T}_{0,\mathcal{M}}^{\mathcal{L}}$ and $\mathcal{W}_{0,\mathcal{L}}^{\mathcal{M}}$ are more difficult to envisage at this stage of our development. In the follwoing sections we shall use the notation:

- $\mathcal{W}_{0,\mathcal{L}}^{\mathcal{N}} = \mathcal{W}_{0,\mathcal{L}}^{\mathcal{M}} \cap \mathcal{N}$, $\mathcal{W}_{\infty}{}_{\mathcal{K}}^{\mathcal{N}} = \mathcal{W}_\infty^{\mathcal{M}} \cap \mathcal{N}$
- $\mathcal{T}_{0,\mathcal{N}}^{\mathcal{K}} = \mathcal{T}_{0,\mathcal{M}}^{\mathcal{L}} \cap \mathcal{K}$, $\mathcal{T}_{\infty}{}_{\mathcal{M}}^{\mathcal{K}} = \mathcal{T}_{\infty,\mathcal{M}}^{\mathcal{L}} \cap \mathcal{K}$

## 11 Negation

In this section we will present several forms of negation. The relevance of this study is that through abstract definitions we can distinguish each form of negation from the others. By doing so, we might be able to discuss certain mathematical concepts regardless of syntatical considerations.

**Definição 11.1** **NegC**$( \alpha , \mathcal{N}, \mathcal{K} ) = \{ \beta \in \mathcal{K} \mid$ **Mod**$(\alpha,\mathcal{N}) =$ **Cont**$(\beta,\mathcal{N}) \}$

$$\textbf{NegC}( S , \mathcal{N}, \mathcal{K} ) = \bigcup_{\alpha \in S} \textbf{NegC}( \alpha , \mathcal{N}, \mathcal{K} )$$ □

As for the propositional semantical systems the above class (a set in this case) of sentences coincides with the class of sentences logically equivalent to the negation of $\alpha$. For instance, in the propositional semantical system $\mathcal{S}^{\mathrm{p}}_{n}$ we have $\mathrm{Mod}(\alpha, \mathcal{M}^{\mathrm{p}}_{n}) = \mathrm{Cont}(\neg\alpha, \mathcal{M}^{\mathrm{p}}_{n})$, and so $\mathrm{NegC}(\alpha, \mathcal{M}^{\mathrm{p}}_{n}, \mathcal{L}^{\mathrm{p}}_{n}) = \{\beta \in \mathcal{L}^{\mathrm{p}}_{n} \mid \neg\alpha \equiv \beta\}$. Note that our definition of negation (NegC) is relative to a certain state of affairs $\mathcal{N}$, and to a particular dialect $\mathcal{K}$. Therefore, by modifying either $\mathcal{N}$ or $\mathcal{K}$ we change the value of NegC, that is, different classes of sentences are obtained. Our definition satisfies the following criteria [*Ebb*] for classical negation:

For all $\tau$ and all $\varphi \in \mathcal{L}[\tau]$ there is a sentence $\psi \in \mathcal{L}[\tau]$ such that
$Mod^{\tau}_{\mathcal{L}}(\psi) = Str[\tau] - Mod^{\tau}_{\mathcal{L}}(\varphi)$

In [*Ebb*], the emphasis is syntatical, so the relativization is on notational considerations, $\tau$ is the vocabulary and $\mathcal{L}$ is a possible extension for the first order language.

**Example 11.1** *In* $\mathcal{S}^{\mathrm{p}}_{2}$ *we have* $\mathrm{Mod}(\mathrm{p}_1, \{\{\mathrm{p}_1\}, \{\mathrm{p}_2\}\}) = \{\{\mathrm{p}_1\}\} = \mathrm{Cont}(\mathrm{p}_2, \{\{\mathrm{p}_1\}, \{\mathrm{p}_2\}\})$, an so $\mathrm{p}_2 \in \mathrm{NegC}(\mathrm{p}_1, \{\{\mathrm{p}_1\}, \{\mathrm{p}_2\}\}, \mathcal{L}^{\mathrm{p}}_{2})$. *Note that* $\neg\mathrm{p}_1 \in \mathrm{NegC}(\mathrm{p}_1, \{\{\mathrm{p}_1\}, \{\mathrm{p}_2\}\}, \mathcal{L}^{\mathrm{p}}_{2})$, then in a sense $\neg\mathrm{p}_1$ is equivalent to $\mathrm{p}_2$, i.e. they have the same models, i.e., $\mathrm{Mod}(\neg\mathrm{p}_1, \{\{\mathrm{p}_1\}, \{\mathrm{p}_2\}\}) = \{\mathrm{p}_2\} = \mathrm{Mod}(\mathrm{p}_2, \{\{\mathrm{p}_1\}, \{\mathrm{p}_2\}\})$)

At first the relativization of negation seems strange, but we can consider the sentence $\mathrm{p}_1$ · 'John is married to Ann' as being the negation of the sentence $\mathrm{p}_2$ · 'John is married to Mary' in a world in which Ann and Mary are different women and monogomy is a certain fact, only one of the two sentences is true. An important property of the classical negation is that if $\beta$ is the negation of $\alpha$ then the class containing both $\beta$ and $\alpha$ has no models:

**Lema 11.1** *If* $\beta \in \mathrm{NegC}(\alpha, \mathcal{N}, \mathcal{K})$ then $\mathrm{Mod}(\{\alpha, \beta\}, \mathcal{N}) = \emptyset$

Proof: We know that: (1) · $\mathrm{Mod}(\{\alpha, \beta\}, \mathcal{N}) = \mathrm{Mod}(\alpha, \mathcal{N}) \cap \mathrm{Mod}(\beta, \mathcal{N})$, also $\beta \in \mathrm{NegC}(\alpha, \mathcal{N}, \mathcal{K}) \Leftrightarrow$ (2) · $\mathrm{Mod}(\alpha, \mathcal{N}) = \mathrm{Cont}(\beta, \mathcal{N})$, then from (1) and (2) : $\mathrm{Mod}(\{\alpha, \beta\}, \mathcal{N}) = \mathrm{Cont}(\beta, \mathcal{N}) \cap \mathrm{Mod}(\beta, \mathcal{N}) = \emptyset$ □

**Example 11.2** In $\mathcal{S}^{\mathrm{p}}_{2}$ we have: $\mathrm{Mod}(\{\mathrm{p}_1, \mathrm{p}_2\}, \{\{\mathrm{p}_1\}, \{\mathrm{p}_2\}\}) = \emptyset$ □

If we allow all *possible* state of affairs to be considered, for instance in the case of marriage we allow poligamy, the sentences $p_1$ and $p_2$ above will not be contradictory.

**Lema 11.2** $\mathrm{NegC}(\alpha, \mathcal{W}^{\mathcal{M}}_{\infty}, \mathcal{K}) \subseteq \mathrm{NegC}(\alpha, \mathcal{N}, \mathcal{K})$

Proof: $\beta \in \mathrm{NegC}(\alpha, \mathcal{W}^{\mathcal{M}}_{\infty}, \mathcal{K}) \Leftrightarrow$ (1) · $\mathrm{Mod}(\alpha, \mathcal{W}^{\mathcal{M}}_{\infty}) = \mathrm{Cont}(\beta, \mathcal{W}^{\mathcal{M}}_{\infty})$. As $\mathrm{Mod}(\alpha, \mathcal{N}) = \mathrm{Mod}(\alpha, \mathcal{W}^{\mathcal{M}}_{\infty}) \cap \mathcal{N}$ and $\mathrm{Cont}(\beta, \mathcal{N}) = \mathrm{Cont}(\beta, \mathcal{W}^{\mathcal{M}}_{\infty}) \cap \mathcal{N}$, then from (1), $\mathrm{Mod}(\alpha, \mathcal{N}) = \mathrm{Cont}(\beta, \mathcal{N})$, thus $\beta \in \mathrm{NegC}(\alpha, \mathcal{N}, \mathcal{K})$ □

Note that if we consider families of models $\mathcal{N}_1$ and $\mathcal{N}_2$, then: If $\mathcal{N}_1 \subseteq \mathcal{N}_2$ then $\mathbf{Mod}(\alpha, \mathcal{N}_1) \subseteq \mathbf{Mod}(\alpha, \mathcal{N}_2)$ and If $\mathcal{N}_1 \subseteq \mathcal{N}_2$ then $\mathbf{Cont}(\alpha, \mathcal{N}_2) \subseteq \mathbf{Mod}(\alpha, \mathcal{N}_1)$ then we cannot derive uniform variation of **NegC** either increasing or decreasing. On the other hand the variation regarding dialects is more predictable as the following lemma shows.

**Lema 11.3** *If* $\mathcal{K}_1, \mathcal{K}_2 \subseteq \mathcal{T}^{\mathcal{L}}_{\infty,\mathcal{M}}$ *then* $\mathbf{NegC}(\alpha, \mathcal{N}, \mathcal{K}_1) \subseteq \mathbf{NegC}(\alpha, \mathcal{N}, \mathcal{K}_2)$

Proof: It follows immediately from the definition of **NegC**. □

We can thus conclude that NegC increases monotonically with the dialect $\mathcal{K}$ of $\mathcal{T}^{\mathcal{L}}_{\infty,\mathcal{M}}$. The next result shows that in the semantical systems where $\mathcal{W}^{\mathcal{M}}_{0,\mathcal{L}} \neq \emptyset$ no sentence of the language $\mathcal{T}^{\mathcal{L}}_{\infty,\mathcal{M}}$ admits the classical negation.

**Lema 11.4** *If* $\alpha \in \mathcal{T}^{\mathcal{L}}_{\infty,\mathcal{M}}$ *and* $\mathcal{W}^{\mathcal{M}}_{0,\mathcal{L}} \neq \emptyset$ *then* $\mathrm{NegC}(\alpha, \mathcal{W}^{\mathcal{M}}_{\infty}, \mathcal{T}^{\mathcal{L}}_{\infty,\mathcal{M}}) = \emptyset$

Proof: We know that $\mathcal{W}^{\mathcal{M}}_{0,\mathcal{L}} = \bigcap_{S \subseteq \mathcal{T}^{\mathcal{L}}_{\infty,\mathcal{M}}} \mathrm{Mod}(S, \mathcal{W}^{\mathcal{M}}_{\infty})$ then $(1) \cdots$ for all $S \subseteq \mathcal{T}^{\mathcal{L}}_{\infty,\mathcal{M}}$ $\mathrm{Mod}(S, \mathcal{W}^{\mathcal{M}}_{\infty}) \neq \emptyset$, if we assume that $\mathbf{NegC}(\alpha, \mathcal{W}^{\mathcal{M}}_{\infty}, \mathcal{T}^{\mathcal{L}}_{\infty,\mathcal{M}}) \neq \emptyset$, then there is a $\beta \in \mathcal{T}^{\mathcal{L}}_{\infty,\mathcal{M}}$ such that $\beta \in \mathbf{NegC}(\alpha, \mathcal{W}^{\mathcal{M}}_{\infty}, \mathcal{T}^{\mathcal{L}}_{\infty,\mathcal{M}})$, so by lemma 11.1, $\mathrm{Mod}(\{\alpha, \beta\}, \mathcal{W}^{\mathcal{M}}_{\infty}) = \emptyset$, which contradicts (1). □

**Corolario 11.1** *If* $\alpha \in \mathcal{T}^{\mathcal{L}}_{\infty,\mathcal{M}}$ *and* $\mathcal{W}^{\mathcal{M}}_{0,\mathcal{L}} \neq \emptyset$ *then* $\mathrm{NegC}(\alpha, \mathcal{W}^{\mathcal{M}}_{\infty}, \mathcal{K}) = \emptyset$

Proof: It follows from the previous lemmas. □

Another property of the classical negation is that if a sentence $\beta$ is a negation of $\alpha$ then $\alpha$ is a negation of $\beta$. To express it we have the following lemma:

**Lema 11.5** $\beta \in \mathbf{NegC}(\alpha, \mathcal{N}, \mathcal{K}) \Leftrightarrow \alpha \in \mathbf{NegC}(\beta, \mathcal{N}, \mathcal{K})$

Proof: $\beta \in \mathbf{NegC}(\alpha, \mathcal{N}, \mathcal{K}) \Leftrightarrow \mathbf{Mod}(\beta, \mathcal{N}) = \mathbf{Cont}(\alpha, \mathcal{N}) \Leftrightarrow$
$\alpha \in \mathbf{NegC}(\beta, \mathcal{N}, \mathcal{K})$ □

At this point of our discussions we must make a conceptual division, regarding two extreme types of semantical systems:

**Symmetrical** Those in which for all $\alpha \in \mathcal{T}^{\mathcal{L}}_{\infty,\mathcal{M}}$, for all $\mathcal{N} \subseteq \mathcal{W}^{\mathcal{M}}_{\infty}$ and for all $\mathcal{K} \subseteq \mathcal{T}^{\mathcal{L}}_{\infty,\mathcal{M}}$ $\mathbf{NegC}(\alpha, \mathcal{N}, \mathcal{K}) \neq \emptyset$

**Positive** Those in which for all $\alpha \in \mathcal{T}^{\mathcal{L}}_{\infty,\mathcal{M}}$, for all $\mathcal{N} \subseteq \mathcal{W}^{\mathcal{M}}_{\infty}$ and for all $\mathcal{K} \subseteq \mathcal{T}^{\mathcal{L}}_{\infty,\mathcal{M}}$ $\mathbf{NegC}(\alpha, \mathcal{N}, \mathcal{K}) = \emptyset$

There are reasons related to the concepts of completeness and consistency for considering these extreme types of semantical systems: such systems have typical metamathematical behavior. From lemma 11.1 we can see that semantical systems in which $\mathcal{W}^{\mathcal{M}}_{0,\mathcal{L}} \neq \emptyset$ are positive semantical systems.

The relation between classical negation, false sentences and true sentences in symmetrical semantical systems is given by:

**Lema 11.6** *Let $\mathcal{S}$ be a symmetrical semantical system. Then*
$$\mathcal{F}_0^{\mathcal{L}} = \mathbf{NegC}(\mathcal{T}_{0,\mathcal{M}}^{\mathcal{L}}, \mathcal{W}_\infty^{\mathcal{M}}, \mathcal{T}_{\infty,\mathcal{M}}^{\mathcal{L}})$$

Proof: From definition 11.1 $\gamma \in \mathbf{NegC}(\mathcal{T}_{0,\mathcal{M}}^{\mathcal{L}}, \mathcal{W}_\infty^{\mathcal{M}}, \mathcal{T}_{\infty,\mathcal{M}}^{\mathcal{L}})$ if and only if $\gamma \in \bigcup_{\alpha \in \mathcal{T}_{0,\mathcal{M}}^{\mathcal{L}}} \mathbf{NegC}(\mathcal{T}_{0,\mathcal{M}}^{\mathcal{L}}, \mathcal{W}_\infty^{\mathcal{M}}, \mathcal{T}_{\infty,\mathcal{M}}^{\mathcal{L}})$ if and only if there is $\alpha \in \mathcal{T}_{0,\mathcal{M}}^{\mathcal{L}}$ such that $\mathrm{Mod}(\alpha, \mathcal{W}_\infty^{\mathcal{M}}) = \mathbf{Cont}(\gamma, \mathcal{W}_\infty^{\mathcal{M}})$, as for all $\alpha \in \mathcal{T}_{0,\mathcal{M}}^{\mathcal{L}}$ $\mathrm{Mod}(\alpha, \mathcal{W}_\infty^{\mathcal{M}}) = \mathcal{W}_\infty^{\mathcal{M}}$, $\gamma \in \mathbf{NegC}(\mathcal{T}_{0,\mathcal{M}}^{\mathcal{L}}, \mathcal{W}_\infty^{\mathcal{M}}, \mathcal{T}_{\infty,\mathcal{M}}^{\mathcal{L}})$ if and only if $\mathbf{Cont}(\gamma, \mathcal{W}_\infty^{\mathcal{M}}) = \mathcal{W}_\infty^{\mathcal{M}}$, therefore $\gamma \in \mathbf{NegC}(\mathcal{T}_{0,\mathcal{M}}^{\mathcal{L}}, \mathcal{W}_\infty^{\mathcal{M}}, \mathcal{T}_{\infty,\mathcal{M}}^{\mathcal{L}})$ if and only if $\mathrm{Mod}(\gamma, \mathcal{W}_\infty^{\mathcal{M}}) = \emptyset$, then by lemma 12.1 $\gamma \in \mathbf{NegC}(\mathcal{T}_{0,\mathcal{M}}^{\mathcal{L}}, \mathcal{W}_\infty^{\mathcal{M}}, \mathcal{T}_{\infty,\mathcal{M}}^{\mathcal{L}})$ if and only if $\gamma \in \mathcal{F}_0^{\mathcal{L}}$, therefore $\mathcal{F}_0^{\mathcal{L}} = \mathbf{NegC}(\mathcal{T}_{0,\mathcal{M}}^{\mathcal{L}}, \mathcal{W}_\infty^{\mathcal{M}}, \mathcal{T}_{\infty,\mathcal{M}}^{\mathcal{L}})$. □

Similarly, we can prove:

**Lema 11.7** *Let $\mathcal{S}$ be a symmetrical semantical system. Then*
$$\mathcal{T}_{0,\mathcal{M}}^{\mathcal{L}} = \mathbf{NegC}(\mathcal{F}_0^{\mathcal{L}}, \mathcal{W}_\infty^{\mathcal{M}}, \mathcal{T}_{\infty,\mathcal{M}}^{\mathcal{L}})$$ □

## 12 Other kinds of negation

In classical negation, as seen in the previous section, $\beta$ is a negation of $\alpha$ if and only if $\mathrm{Mod}(\alpha, \mathcal{N}) = \mathrm{Cont}(\beta, \mathcal{N})$, this is a very restrictive condition. The intuitive meaning of it is that the class of situations where $\alpha$ holds is exactly the class of situations where $\beta$ does not hold. So, the class of situations where $\alpha$ does not hold is the class of situations where $\beta$ does hold. If we allow the class of situations where $\alpha$ does not hold, i.e. $\mathbf{Cont}(\alpha, \mathcal{N})$ to be a subclass of the class of situations where $\beta$ does hold, we are to accept that there are, eventually, common models for both $\alpha$ and $\beta$.

**Definição 12.1** $\mathbf{NegD}(\alpha, \mathcal{N}, \mathcal{K}) = \{ \beta \in \mathcal{K} \mid \mathbf{Cont}(\alpha, \mathcal{N}) \subsetneq \mathrm{Mod}(\beta, \mathcal{N}) \}$
$\mathbf{NegD}(S, \mathcal{N}, \mathcal{K}) = \bigcup_{\alpha \in S} \mathbf{NegD}(\alpha, \mathcal{N}, \mathcal{K})$ □

Is **NegD**, really a kind of negation? The following lemma is a partial positive answer:

**Lema 12.1** *If $\beta \in \mathbf{CnS}(\alpha, \mathcal{N}, \mathcal{K})$ then* $\mathbf{NegD}(\alpha, \mathcal{N}, \mathcal{K}) \subseteq \mathbf{NegD}(\beta, \mathcal{N}, \mathcal{K})$

Proof: $\beta \in \mathbf{CnS}(\alpha, \mathcal{N}, \mathcal{K})$ if and only if (1) $\cdots$ $\mathbf{Cont}(\beta, \mathcal{N}) \subseteq \mathbf{Cont}(\alpha, \mathcal{N})$, and so by def ... $\gamma \in \mathbf{NegD}(\alpha, \mathcal{N}, \mathcal{K})$ if and only if $\mathbf{Cont}(\alpha, \mathcal{N}) \subseteq \mathrm{Mod}(\gamma, \mathcal{N})$, then from (1) $\mathbf{Cont}(\beta, \mathcal{N}) \subseteq \mathrm{Mod}(\gamma, \mathcal{N})$, so by def ... $\gamma \in \mathbf{NegD}(\beta, \mathcal{N}, \mathcal{K})$ □
another property of negation is given by:

**Lema 12.2** If $\beta \in \mathbf{NegD}(\alpha, \mathcal{N}, \mathcal{K})$ then $\alpha \in \mathbf{NegD}(\beta, \mathcal{N}, \mathcal{K})$

Proof: $\beta \in \mathbf{NegD}(\alpha, \mathcal{N}, \mathcal{K})$ if and only if $\mathbf{Cont}(\alpha,\mathcal{N}) \subseteq \mathbf{Mod}(\beta,\mathcal{N})$ if and only if $\mathbf{Cont}(\beta,\mathcal{N}) \subseteq \mathbf{Mod}(\alpha,\mathcal{N})$ if and only if $\alpha \in \mathbf{NegD}(\beta, \mathcal{N}, \mathcal{K})$. □

The following lemma shows that **NegD** is the theory of the class of models $M$ that *falsifies* $\alpha$, i.e. $\varphi(\alpha, M) = 0$.

**Lema 12.3** $\mathbf{NegD}(\alpha, \mathcal{N}, \mathcal{K}) = \mathbf{Th}(\mathbf{Cont}(\alpha,\mathcal{N}),\mathcal{K})$

Proof: $\beta \in \mathbf{NegD}(\alpha, \mathcal{N}, \mathcal{K})$ if and only if $\mathbf{Cont}(\alpha,\mathcal{N}) \subseteq \mathbf{Mod}(\beta,\mathcal{N})$ if and only if $\beta \in \mathbf{Th}(\mathbf{Cont}(\alpha,\mathcal{N}),\mathcal{K})$ (according to lemma 8.4). □

The next lemma shows that the only sentences commom to $\mathbf{CnS}(\alpha,\mathcal{N},\mathcal{K})$ and $\mathbf{NegD}(\alpha, \mathcal{N}, \mathcal{K})$ are the ones in $\mathcal{T}^{\mathcal{L}}_{0,\mathcal{N}}$.

**Lema 12.4** $\mathbf{NegD}(\alpha, \mathcal{N}, \mathcal{K}) \cap \mathbf{CnS}(\alpha,\mathcal{N},\mathcal{K}) = \mathcal{T}^{\mathcal{K}}_{0,\mathcal{N}}$

Proof: $\mathbf{NegD}(\alpha, \mathcal{N}, \mathcal{K}) \cap \mathbf{CnS}(\alpha,\mathcal{N},\mathcal{K}) = \mathbf{Th}(\mathbf{Cont}(\alpha,\mathcal{N}),\mathcal{K}) \cap \mathbf{Th}(\mathbf{Mod}(\alpha,\mathcal{N}),\mathcal{K}) = \mathbf{Th}(\mathbf{Cont}(\alpha,\mathcal{N}) \cup \mathbf{Mod}(\alpha,\mathcal{N}),\mathcal{K}) = \mathbf{Th}(\mathcal{N},\mathcal{K}) = \mathcal{T}^{\mathcal{K}}_{0,\mathcal{N}}$ □

**Lema 12.5** *If* $\gamma \in \mathbf{NegC}(\alpha,\mathcal{N},\mathcal{K})$ *then* $\mathbf{NegD}(\alpha,\mathcal{N},\mathcal{K}) = \mathbf{Th}(\mathbf{Mod}(\gamma,\mathcal{N}),\mathcal{K})$

Proof: From lemma 11.3 $\mathbf{NegD}(\alpha, \mathcal{N}, \mathcal{K}) = \mathbf{Th}(\mathbf{Cont}(\alpha,\mathcal{N}),\mathcal{K})$, from $\gamma \in \mathbf{NegC}(\alpha, \mathcal{N}, \mathcal{K})$ we have $\mathbf{Cont}(\alpha,\mathcal{N}) = \mathbf{Mod}(\gamma,\mathcal{N})$ so by substitution we have the proof. So $\mathbf{NegD}(\alpha, \mathcal{N}, \mathcal{K})$ is the class of consequences of the (classical) negations of $\alpha$. □

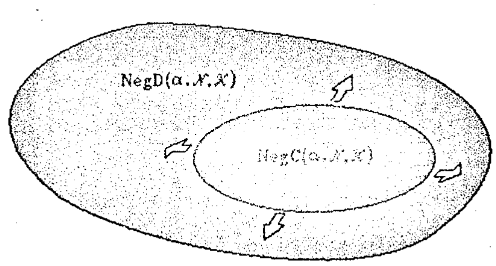
From lemma 12.4 $\mathcal{T}^{\mathcal{K}}_{0,\mathcal{N}} \subseteq \mathbf{NegD}(\alpha, \mathcal{N}, \mathcal{K})$, this is quite strange, for true sentences are denials of a description $\alpha$. So we introduce a modification in the definition of **NegD**:

$$\mathbf{NegD}(\alpha, \mathcal{N}, \mathcal{K}) = \mathbf{Th}(\mathbf{Cont}(\alpha,\mathcal{N}),\mathcal{K}) - \mathcal{T}^{\mathcal{K}}_{0,\mathcal{N}}$$

The intuitive meaning of **NegD** can be seen if we consider the sentences

$p \cdots$ John is a man *and* $q \cdots$ John is imortal

then $q$ is a negation (**NegD**) of $p$, as $q$ is a consequence of $\neg p$.



**Lema 12.6** *If* $\mathcal{W}^{\mathcal{N}}_{0,\mathcal{K}} \neq \emptyset$ *then for all* $\alpha \in \mathcal{K}$ $\mathbf{NegD}(\alpha, \mathcal{N}, \mathcal{K}) = \emptyset$

Proof: From corollary 11.1 $\mathbf{NegC}(\alpha, \mathcal{N}, \mathcal{K}) = \emptyset$ then $\mathbf{NegD}(\alpha, \mathcal{N}, \mathcal{K}) - \mathcal{T}^{\mathcal{K}}_{0,\mathcal{N}} = \mathbf{CnS}(\mathbf{NegC}(\alpha, \mathcal{N}, \mathcal{K}), \mathcal{N}, \mathcal{K}) - \mathcal{T}^{\mathcal{K}}_{0,\mathcal{N}} = \mathbf{CnS}(\emptyset, \mathcal{N}, \mathcal{K}) - \mathcal{T}^{\mathcal{K}}_{0,\mathcal{N}} = \mathcal{T}^{\mathcal{K}}_{0,\mathcal{N}} - \mathcal{T}^{\mathcal{K}}_{0,\mathcal{N}} = \emptyset$ □

The next negation is easier to accept as such, as we shall see.

**Definição 12.2** $\mathrm{NegE}(\alpha, \mathcal{N}, \mathcal{K}) = \{ \beta \in \mathcal{K} \mid \mathbf{Mod}(\alpha, \mathcal{N}) \subseteq \mathbf{Cont}(\beta, \mathcal{N}) \}$

$$\mathrm{NegE}(S, \mathcal{N}, \mathcal{K}) = \bigcup_{\alpha \in S} \mathrm{NegE}(\alpha, \mathcal{N}, \mathcal{K})$$

□

The following is one very clear property of negation:

**Lema 12.7** *If* $\beta \in \mathrm{NegE}(\alpha, \mathcal{N}, \mathcal{K})$ *then* $\mathrm{Mod}(\{\alpha, \beta\}, \mathcal{N}) = \emptyset$

Proof: We know that (1) $\cdot$ $\mathbf{Mod}(\{\alpha, \beta\}, \mathcal{N}) = \mathbf{Mod}(\alpha, \mathcal{N}) \cap \mathbf{Mod}(\beta, \mathcal{N})$, as $\beta \in \mathrm{NegE}(\alpha, \mathcal{N}, \mathcal{K})$ if and only if (2) $\cdot$ $\mathrm{Mod}(\alpha, \mathcal{N}) \subseteq \mathbf{Cont}(\beta, \mathcal{N})$ then from (1) and (2) $\mathrm{Mod}(\{\alpha, \beta\}, \mathcal{N}) \subseteq \mathbf{Cont}(\beta, \mathcal{N}) \cap \mathbf{Mod}(\beta, \mathcal{N}) = \emptyset$, thus $\mathrm{Mod}(\{\alpha, \beta\}, \mathcal{N}) = \emptyset$ □

The next lemma shows that in any positive semantical system NegE is empty:

**Lema 12.8** *If* $\mathcal{W}^{\mathcal{M}}_{0,\mathcal{L}} \neq \emptyset$ *then* $\mathrm{NegE}(\alpha, \mathcal{N}, \mathcal{K}) = \emptyset$

Proof: Similar to the proof of corollary 11.1. □

**Lema 12.9** *If* $\beta \in \mathrm{NegE}(\alpha, \mathcal{N}, \mathcal{K})$ *then* $\alpha \in \mathrm{NegE}(\beta, \mathcal{N}, \mathcal{K})$

Proof: $\beta \in \mathrm{NegE}(\alpha, \mathcal{N}, \mathcal{K}) \Leftrightarrow \mathbf{Mod}(\alpha, \mathcal{N}) \subseteq \mathbf{Cont}(\beta, \mathcal{N}) \Leftrightarrow \mathbf{Mod}(\beta, \mathcal{N}) \subseteq \mathbf{Cont}(\alpha, \mathcal{N}) \Leftrightarrow \alpha \in \mathrm{NegE}(\beta, \mathcal{N}, \mathcal{K})$. □

**Lema 12.10** $\mathrm{NegC}(\alpha, \mathcal{N}, \mathcal{K}) \subseteq \mathrm{NegE}(\alpha, \mathcal{N}, \mathcal{K})$

Proof: $\beta \in \mathbf{NegC}(\alpha, \mathcal{N}, \mathcal{K}) \Leftrightarrow \mathrm{Mod}(\alpha, \mathcal{N}) = \mathbf{Cont}(\beta, \mathcal{N})$ then $\mathrm{Mod}(\alpha, \mathcal{N}) \subseteq \mathbf{Cont}(\beta, \mathcal{N}) \Leftrightarrow \beta \in \mathrm{NegE}(\alpha, \mathcal{N}, \mathcal{K})$ □

**Lema 12.11** $\mathrm{NegC}(\alpha, \mathcal{N}, \mathcal{K}) \subseteq \mathrm{NegD}(\alpha, \mathcal{N}, \mathcal{K})$

Proof: Similar to the proof of lemma 11.11 □

**Lema 12.12** $\mathbf{NegD}(\alpha, \mathcal{N}, \mathcal{K}) \cap \mathbf{NegE}(\alpha, \mathcal{N}, \mathcal{K}) = \mathbf{NegC}(\alpha, \mathcal{N}, \mathcal{K})$

Proof: Immediate. □

**Definição 12.3** $\mathrm{NegED}(\alpha, \mathcal{N}, \mathcal{K}) = \mathrm{NegE}(\alpha, \mathcal{N}, \mathcal{K}) \cup \mathrm{NegD}(\alpha, \mathcal{N}, \mathcal{K})$ □

**Lema 12.13** (1) $\cdots$ $\mathbf{NegE}(\alpha, \mathcal{N}, \mathcal{K}) \subseteq \mathbf{NegED}(\alpha, \mathcal{N}, \mathcal{K})$
(2) $\cdots$ $\mathbf{NegD}(\alpha, \mathcal{N}, \mathcal{K}) \subseteq \mathbf{NegED}(\alpha, \mathcal{N}, \mathcal{K})$

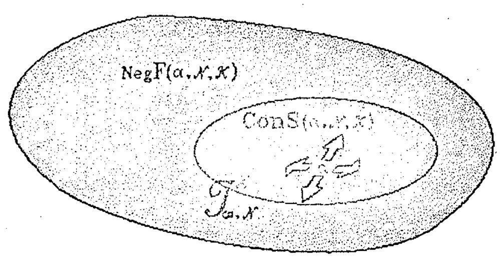
Proof: Immediate. □

The last kind of negation we will consider is very important. It is the semantical counterpart of Post's negation by non demonstrability.

**Definição 12.4** $\mathbf{NegF}(\alpha, \mathcal{N}, \mathcal{K}) = \mathbf{Absd}(\mathbf{Mod}(\alpha,\mathcal{N}), \mathcal{K})$
$\mathbf{NegF}(\mathcal{S}, \mathcal{N}, \mathcal{K}) = \mathbf{Absd}(\mathbf{Mod}(\mathcal{S},\mathcal{N}), \mathcal{K})$ □

**Lema 12.14** $\mathbf{NegF}(\alpha, \mathcal{N}, \mathcal{K}) = \mathcal{T}^{\mathcal{K}}_{\infty,\mathcal{N}} - \mathbf{CnS}(\alpha,\mathcal{N},\mathcal{K})$

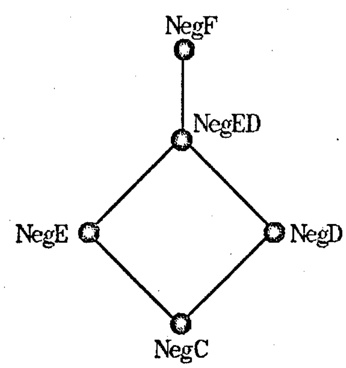
Proof: Obvious. □




**Lema 12.15** If $\mathbf{Mod}(\alpha,\mathcal{N}) \neq \emptyset$ then $\mathbf{NegE}(\alpha, \mathcal{N}, \mathcal{K}) \subseteq \mathbf{NegF}(\alpha, \mathcal{N}, \mathcal{K})$

Proof: $\beta \in \mathbf{NegE}(\alpha, \mathcal{N}, \mathcal{K}) \Leftrightarrow \mathbf{Mod}(\alpha,\mathcal{N}) \subseteq \mathbf{Cont}(\beta,\mathcal{N}) \Leftrightarrow \mathbf{Mod}(\alpha,\mathcal{N}) \cap \mathbf{Cont}(\beta,\mathcal{N}) = \mathbf{Mod}(\alpha,\mathcal{N})$, then if $\mathbf{Mod}(\alpha,\mathcal{N}) \neq \emptyset$ then $\mathbf{Mod}(\alpha,\mathcal{N}) \cap \mathbf{Cont}(\beta,\mathcal{N}) \neq \emptyset$, and thus from lemma 8.6 $\beta \in \mathbf{Absd}(\mathbf{Mod}(\alpha,\mathcal{N}), \mathcal{K}) = \mathbf{NegF}(\alpha, \mathcal{N}, \mathcal{K}$. □

The figure shows that the defined types of negation form a lattice regarding inclusion



# 13 Complete Theories

Intuitively a theory $\mathcal{T}$ is complete when it fully describes a certain state of affairs ( $\text{Mod}(\mathcal{T},\mathcal{N})$). This means that its sentences are sufficient to determine both the situations which satisfies it - its models - and the complementary situations - its empirical content. Therefore if a sentence $\alpha$ does not belong to $\mathcal{T}$ then if $\mathcal{T}$ is a complete theory the models which satisfy $\alpha$, $\text{Mod}(\alpha,\mathcal{N})$, do not satisfy $\mathcal{T}$, and so these models are among those which do not satisfy $\mathcal{T}$, i.e. $\text{Mod}(\alpha,\mathcal{N}) \subseteq \text{Cont}(\mathcal{T},\mathcal{N})$, or equivalentely $\text{Mod}(\mathcal{T},\mathcal{N}) \subseteq \text{Cont}(\alpha,\mathcal{N})$. Formally we have the following (tentative) definition:

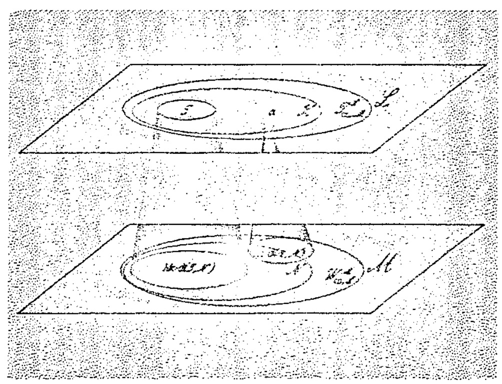
**Definição 13.1** *Let $\mathcal{T}$ be a theory in $\mathcal{N}$ and $\mathcal{K}$. We say that $\mathcal{T}$ is complete$_1$ in $\mathcal{K}$ and $\mathcal{N}$, if and only if for all $\alpha \in \mathcal{K}$ if $\alpha \notin \mathcal{T}$ then* $\text{Mod}(\mathcal{T},\mathcal{N}) \subseteq \text{Cont}(\alpha,\mathcal{N})$.
□

In the case of the first order semantical system the definition above coincides with the usual one:
*$\mathcal{T}$ is complete iff for every $\alpha$, $\alpha \in \mathcal{T}$ or $\neg\alpha \in \mathcal{T}$*

**Lema 13.1** *$\mathcal{T}$ is a complete$_1$ theory in $\mathcal{K}$ and $\mathcal{N}$ if and only if for every $\alpha \notin \mathcal{T}$* $\text{Mod}(\mathcal{T},\mathcal{N}) \cap \text{Mod}(\alpha,\mathcal{N}) = \emptyset$

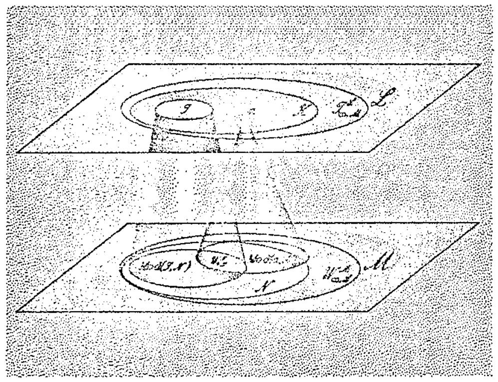
Proof: It is an immediate consequence of definition 11.1. □

The situation presented above correspond to the case when $\mathcal{W}^{\mathcal{M}}_{0,\mathcal{L}} = \emptyset$, i.e. there are no universal models. If there are universal models ($\mathcal{W}^{\mathcal{M}}_{0,\mathcal{L}} \neq \emptyset$) then this models are in $\mathcal{T}$ and $\text{Mod}(\alpha, \mathcal{W}^{\mathcal{M}}_{\infty,\mathcal{L}})$.



We will also consider the following definition:

**Definição 13.2** *Let $\mathcal{T}$ be a theory in $\mathcal{N}$ and $\mathcal{K}$. We say that $\mathcal{T}$ is complete$_2$ in $\mathcal{K}$ and $\mathcal{N}$, if and only if for all $\alpha \in \mathcal{K}$ if $\alpha \notin \mathcal{T}$ then* $\text{Mod}(\mathcal{T},\mathcal{N}) \cap \text{Mod}(\alpha,\mathcal{N}) = \mathcal{W}^{\mathcal{N}}_{0,\mathcal{K}}$. □



If $\mathcal{W}^{\mathcal{M}}_{0,\mathcal{L}} = \emptyset$ the two definitions coincide.

**Lema 13.2** *$\mathcal{T}$ is complete$_1$ in $\mathcal{N}$ and $\mathcal{K}$ if and only if $\mathcal{T}$ is complete$_2$ and $\mathcal{W}^{\mathcal{N}}_{0,\mathcal{K}} = \emptyset$.*

Proof: Immediate. □

We will generally adopt definition 15.2 as it is more comprehensive, but we can alternatively use definition 15.1 when the semantical system is symmetrical.

**Corolario 13.1** *$\mathcal{T}^{\mathcal{K}}_{\infty,\mathcal{N}}$ is a complete theory in $\mathcal{K}$ and $\mathcal{N}$.*

Proof: It follows immediately from both definitions. □
The next corollary shows that our definition subsumes Tarski's definition of completeness.

**Corolario 13.2** *$\mathcal{T}$ is a complete theory in $\mathcal{K}$ and $\mathcal{N}$ if and only if for every $\alpha \in \mathcal{T}^{\mathcal{K}}_{\infty,\mathcal{N}}$ if $\alpha \notin \mathcal{T}$ then $\mathrm{CnS}(\mathcal{T} \cup \{\alpha\},\mathcal{N},\mathcal{K}) = \mathcal{T}^{\mathcal{K}}_{\infty,\mathcal{N}}$.*

Proof: For all $\alpha \in \mathcal{K}$, $\mathrm{CnS}(\mathcal{T} \cup \{\alpha\},\mathcal{N},\mathcal{K}) = \mathrm{Th}(\mathrm{Mod}(\mathcal{T} \cup \{\alpha\},\mathcal{N}),\mathcal{K}) = \mathrm{Th}(\mathrm{Mod}(\mathcal{T},\mathcal{N}) \cap \mathrm{Mod}(\alpha,\mathcal{N}),\mathcal{K})$, if $\alpha \notin \mathcal{T}$ then by definition 11.2 $\mathrm{Mod}(\mathcal{T},\mathcal{N}) \cap \mathrm{Mod}(\alpha,\mathcal{N}) = \mathcal{W}^{\mathcal{N}}_{0,\mathcal{K}}$, so $\mathrm{CnS}(\mathcal{T} \cup \{\alpha\},\mathcal{N},\mathcal{K}) = \mathrm{Th}(\mathcal{W}^{\mathcal{N}}_{0,\mathcal{K}},\mathcal{K}) = \mathcal{T}^{\mathcal{K}}_{\infty,\mathcal{N}}$. □

**Lema 13.3** *Let $\mathcal{N} \subseteq \mathcal{W}^{\mathcal{M}}_{\infty,\mathcal{L}}$ and $\mathcal{T}_1$ and $\mathcal{T}_2$ theories in $\mathcal{K} \subseteq \mathcal{T}^{\mathcal{L}}_{\infty}$. Then if:*

$\mathcal{T}_1$ is complete in $\mathcal{K}$ and $\mathcal{N}$, and
$\mathcal{T}_1 \subseteq \mathcal{T}_2$, and
$\mathrm{Mod}(\mathcal{T}_2,\mathcal{N}) \neq \emptyset$

then $\mathcal{T}_1 = \mathcal{T}_2$ or $\mathcal{T}_2 = \mathcal{T}^{\mathcal{K}}_{\infty,\mathcal{N}}$.
Proof: Let us suppose that $\mathcal{T}_1 \neq \mathcal{T}_2$ then there exists $\alpha \in \mathcal{T}_2$ such that $\alpha \notin \mathcal{T}_1$, then as $\mathcal{T}_1$ is complete in $\mathcal{K}$ and $\mathcal{N}$: (4) $\cdot$ $\mathrm{Mod}(\mathcal{T}_1,\mathcal{N}) \cap \mathrm{Mod}(\alpha,\mathcal{N}) = \mathcal{W}^{\mathcal{N}}_{0,\mathcal{K}}$. From hypothesis 2, (5) $\cdot$ $\mathrm{Mod}(\mathcal{T}_2,\mathcal{N}) \subseteq \mathrm{Mod}(\mathcal{T}_1,\mathcal{N})$, and as $\mathcal{T}_2$ is a theory and $\alpha \in \mathcal{T}_2$ (6) $\cdot$ $\mathrm{Mod}(\mathcal{T}_2,\mathcal{N}) \subseteq \mathrm{Mod}(\alpha,\mathcal{N})$, so, from (5) and (6), $\mathrm{Mod}(\mathcal{T}_2,\mathcal{N}) \subseteq \mathrm{Mod}(\mathcal{T}_1,\mathcal{N}) \cap \mathrm{Mod}(\alpha,\mathcal{N}) = \mathcal{W}^{\mathcal{N}}_{0,\mathcal{K}}$, thus $\mathrm{Mod}(\mathcal{T}_2,\mathcal{N}) = \mathcal{W}^{\mathcal{N}}_{0,\mathcal{K}}$, therefore $\mathcal{T}_2 = \mathcal{T}^{\mathcal{K}}_{\infty,\mathcal{N}}$. □
The previous lemma shows that if a theory is complete then no other proper theory contains it.

**Definição 13.3** *$\mathcal{W}$ is an elementary class in $\mathcal{N}$ relative to $\mathcal{K}$ ($EC^{\mathcal{N}}_{\mathcal{K}}$) iff for all $M \in \mathcal{W}$, $Th(M,\mathcal{K}) = Th(\mathcal{W},\mathcal{K})$.* □

**Lema 13.4** *If $\mathcal{W}^{\mathcal{N}}_{0,\mathcal{K}} \neq \emptyset$ then $\mathcal{W}^{\mathcal{N}}_{0,\mathcal{K}}$ is $EC^{\mathcal{N}}_{\mathcal{K}}$.*

Proof: For any $M \in \mathcal{W}^{\mathcal{N}}_{0,\mathcal{K}}$ $\mathcal{T}^{\mathcal{K}}_{\infty,\mathcal{N}} \subseteq \text{Th}(M,\mathcal{K}) = \mathcal{T}^{\mathcal{K}}_{\infty,\mathcal{N}}$. □
The models of an elementary class regarding a dialect $\mathcal{K}$ are undistiguishable by this dialect. We say that they are equivalent, or more precisely:

**Definição 13.4** *Let $M_1$, $M_2 \in \mathcal{W}^{\mathcal{M}}_{\infty,\mathcal{L}}$, we say that $M_1$ is equivalent to $M_2$ in $\mathcal{K}$, ($M_1 \equiv_{\mathcal{K}} M_2$), we say that $M_1$ is equivalent to $M_2$ in $\mathcal{K}$, ($M_1 \equiv_{\mathcal{K}} M_2$) if and only if $\text{Th}(M_1,\mathcal{K}) = \text{Th}(M_2,\mathcal{K})$.* □

**Lema 13.5** *Let $\mathcal{S}$ be a semantical system, $\mathcal{N} \subseteq \mathcal{W}^{\mathcal{M}}_{\infty,\mathcal{L}}$ and $\mathcal{K} \subseteq \mathcal{T}^{\mathcal{L}}_{\infty,\mathcal{M}}$. $\equiv_{\mathcal{K}}$ is an equivalence relation in $\mathcal{N}$.*

Proof: Trivial. □
The following is a very useful little result:

**Lema 13.6** *Let $\mathcal{S}$ be a semantical system, $\mathcal{N} \subseteq \mathcal{W}^{\mathcal{M}}_{\infty,\mathcal{L}}$ and $\mathcal{K} \subseteq \mathcal{T}^{\mathcal{L}}_{\infty,\mathcal{M}}$. Then for every $\mathcal{W} \subseteq \mathcal{W}^{\mathcal{N}}_{\infty,\mathcal{K}}$ $\text{Th}(\mathcal{W},\mathcal{K}) = \text{Th}(\mathcal{W} - \mathcal{W}^{\mathcal{N}}_{0,\mathcal{K}},\mathcal{K})$*

Proof: If $\mathcal{S}$ is symmetrical, as $\mathcal{W}^{\mathcal{N}}_{0,\mathcal{K}}$ is empty there is nothing to prove. In any case $\text{Th}(\mathcal{W},\mathcal{K}) = \bigcap_{M \in \mathcal{W}} \text{Th}(M,\mathcal{K}) = \left(\bigcap_{M \in \mathcal{W} - \mathcal{W}^{\mathcal{N}}_{0,\mathcal{K}}} \text{Th}(M,\mathcal{K})\right) \cap \bigcap_{M \in \mathcal{W}^{\mathcal{N}}_{0,\mathcal{K}}} \text{Th}(M,\mathcal{K}) = \text{Th}(\mathcal{W} - \mathcal{W}^{\mathcal{N}}_{0,\mathcal{K}},\mathcal{K})$ □
The next result tells us that a complete theory $\mathcal{T}$ has a nice property with respect to its models $\text{Mod}(\mathcal{T},\mathcal{N})$, which is, we can identify for such theories, besides $\mathcal{W}^{\mathcal{N}}_{0,\mathcal{K}}$, other elementary classes relative to $\mathcal{K}$.

**Teorema 13.1** *Let $\mathcal{N} \subseteq \mathcal{W}^{\mathcal{M}}_{\infty,\mathcal{L}}$ and $\mathcal{T} \in \mathcal{K} \subseteq \mathcal{T}^{\mathcal{L}}_{\infty}$. $\mathcal{T}$ is complete in $\mathcal{K}$ and $\mathcal{N}$ if and only if $\text{Mod}(\mathcal{T},\mathcal{N}) - \mathcal{W}^{\mathcal{N}}_{0,\mathcal{K}}$ is $EC^{\mathcal{N}}_{\mathcal{K}}$.*

Proof: ($\Leftarrow$) : Let us assume that $\text{Mod}(\mathcal{T},\mathcal{N}) - \mathcal{W}^{\mathcal{N}}_{0,\mathcal{K}}$ is $EC^{\mathcal{N}}_{\mathcal{K}}$. So, by definition, for every $M \in \text{Mod}(\mathcal{T},\mathcal{N}) - \mathcal{W}^{\mathcal{N}}_{0,\mathcal{K}}$: (1)$\cdot$ $\text{Th}(M,\mathcal{K}) = \text{Th}(\text{Mod}(\mathcal{T},\mathcal{N}) - \mathcal{W}^{\mathcal{N}}_{0,\mathcal{K}},\mathcal{K})$ and from lemma 13.6 (2)$\cdot$ $\text{Th}(M,\mathcal{K}) = \text{Th}(\text{Mod}(\mathcal{T},\mathcal{N}),\mathcal{K}) = \mathcal{T}$. Suppose that $\alpha \notin \mathcal{T}$ then from (2) for every $M \in \text{Mod}(\mathcal{T},\mathcal{N}) - \mathcal{W}^{\mathcal{N}}_{0,\mathcal{K}}$ we have that $\alpha \notin \text{Th}(M,\mathcal{K})$, i.e. $M \in \text{Cont}(\alpha,\mathcal{N})$ and so $\text{Mod}(\mathcal{T},\mathcal{N}) - \mathcal{W}^{\mathcal{N}}_{0,\mathcal{K}} \subseteq \text{Cont}(\alpha,\mathcal{N}) \leftrightarrow \text{Mod}(\mathcal{T},\mathcal{N}) \cap \text{Mod}(\alpha,\mathcal{K}) = \mathcal{W}^{\mathcal{N}}_{0,\mathcal{K}}$, from this we conclude that $\mathcal{T}$ is complete in $\mathcal{K}$ and $\mathcal{N}$.

($\Rightarrow$): Now let us assume that $\mathcal{T}$ is complete in $\mathcal{K}$ and $\mathcal{N}$, then we have to prove that $\text{Mod}(\mathcal{T},\mathcal{N}) - \mathcal{W}^{\mathcal{N}}_{0,\mathcal{K}}$ is $EC^{\mathcal{N}}_{\mathcal{K}}$. Therefore we need to show that for every $M \in$

$\text{Mod}(\mathcal{T},\mathcal{N}) - \mathcal{W}^{\mathcal{N}}_{0,\mathcal{K}}$ $\text{Th}(M,\mathcal{K}) = \text{Th}(\text{Mod}(\mathcal{T},\mathcal{N}) - \mathcal{W}^{\mathcal{N}}_{0,\mathcal{K}},\mathcal{K})$, but then from lemma 13.6 we have to show that for every $M \in \text{Mod}(\mathcal{T},\mathcal{N}) - \mathcal{W}^{\mathcal{N}}_{0,\mathcal{K}}$ $\text{Th}(M,\mathcal{K}) = \mathcal{T}$. As $\mathcal{T} \subseteq \text{Th}(M,\mathcal{K})$, it remains to be shown that $\text{Th}(M,\mathcal{K}) \subseteq \mathcal{T}$. Suppose that for some $N \in \text{Mod}(\mathcal{T},\mathcal{N}) - \mathcal{W}^{\mathcal{N}}_{0,\mathcal{K}}$ $\text{Th}(M,\mathcal{K}) \not\subseteq \mathcal{T}$, then there is $\alpha \in \mathcal{K}$ such that $(4) \cdot \alpha \in \text{Th}(M,\mathcal{K})$ and $(5) \cdot \alpha \notin \mathcal{T}$. From (5) and from the fact that $\mathcal{T}$ is a complete theory in $\mathcal{K}$ and $\mathcal{N}$ we have that $\text{Mod}(\mathcal{T},\mathcal{N}) \cap \text{Mod}(\alpha,\mathcal{N}) = \mathcal{W}^{\mathcal{N}}_{0,\mathcal{K}}$. As $M \in \text{Mod}(\mathcal{T},\mathcal{N}) - \mathcal{W}^{\mathcal{N}}_{0,\mathcal{K}}$, we have that $M \notin \text{Mod}(\alpha,\mathcal{N})$, i.e. $\alpha \notin \text{Th}(M,\mathcal{K})$, which contradicts (1) above. □

Theorem 15.1 is very important, because it shows that for complete theories $\mathcal{T}$ in a language $\mathcal{K}$, all the properties expressed in $\mathcal{K}$, of a particular model of $\mathcal{T}$ are properties of all models in $\text{Mod}(\mathcal{T},\mathcal{N})$. This is the meaning of $\text{Mod}(\mathcal{T},\mathcal{K}) - \mathcal{W}^{\mathcal{N}}_{0,\mathcal{K}}$ being $EC^{\mathcal{N}}_{\mathcal{K}}$.

The results presented so far are valid for any semantical system $\mathcal{S}$. In the next section we will study the case of symmetrical semantical systems.

## 13.1 Symmetrical Semantical Systems

We saw that for complete theories $\mathcal{T}$ $\text{Mod}(\mathcal{T},\mathcal{N}) - \mathcal{W}^{\mathcal{N}}_{0,\mathcal{K}}$ is $EC^{\mathcal{N}}_{\mathcal{K}}$, since for symmetrical semantical systems $\mathcal{W}^{\mathcal{N}}_{0,\mathcal{K}}$ is empty, we have:

**Corolario 13.3** *Let $\mathcal{S}$ be a symmetrical semantical system. If $\mathcal{T}$ is complete in $\mathcal{K}$ and $\mathcal{N}$ then* $\text{Mod}(\mathcal{T},\mathcal{N})$ *is an elementary class relative to $\mathcal{K}$ in $\mathcal{N}$.*

Proof:Immediate after theorem 13.1. □

The next result shows that the usual (first order) definition of completeness is subsumed by our definition 13.2.

**Lema 13.7** *Let $\mathcal{S}$ be a symmetrical semantical system. $\mathcal{T}$ is complete in $\mathcal{N}$ and $\mathcal{K}$ if and only if for all $\alpha \in \mathcal{K}$, if $\alpha \notin \mathcal{T}$ then* $\text{NegC}(\alpha, \mathcal{K}, \mathcal{N}) \subseteq \mathcal{T}$

Proof: $(\Rightarrow)$ Let us suppose that $\mathcal{T}$ is complete in $\mathcal{N}$ and $\mathcal{K}$.
Then $(1) \cdots$ if $\alpha \notin \mathcal{T}$ then $\text{Mod}(\mathcal{T},\mathcal{N}) \subseteq \text{Cont}(\alpha,\mathcal{N})$. As $\mathcal{S}$ is symmetrical $\text{NegC}(\alpha, \mathcal{K}, \mathcal{N}) \neq \emptyset$, so let $\beta \in \text{NegC}(\alpha, \mathcal{K}, \mathcal{N})$ then $\text{Mod}(\alpha,\mathcal{N}) = \text{Cont}(\beta,\mathcal{N})$, or equivalentely $(2) \cdots \text{Mod}(\beta,\mathcal{N}) = \text{Cont}(\alpha,\mathcal{N})$. From (1) and (2) $\text{Mod}(\mathcal{T},\mathcal{N}) \subseteq \text{Mod}(\beta,\mathcal{N})$, as $\mathcal{T}$ is a theory, $\beta \in \mathcal{T}$ and so $\text{NegC}(\alpha, \mathcal{K}, \mathcal{N}) \subseteq \mathcal{T}$.
$(\Leftarrow)$ Let us suppose now that if $\alpha \notin \mathcal{T}$ then $\text{NegC}(\alpha, \mathcal{K}, \mathcal{N}) \subseteq \mathcal{T}$.

If $M \in \text{Mod}(\mathcal{T},\mathcal{N})$ then for all $\beta \in \mathcal{T}$, $M \in \text{Mod}(\beta,\mathcal{N})$. This holds in particular for $\beta \in \text{NegC}(\alpha, \mathcal{K}, \mathcal{N})$. Thus $M \in \text{Mod}(\beta,\mathcal{N})$, therefore as $\text{Mod}(\beta,\mathcal{N}) = \text{Cont}(\alpha,\mathcal{N})$, $M \in \text{Cont}(\alpha,\mathcal{N})$, then $\text{Mod}(\mathcal{T},\mathcal{N}) \subseteq \text{Cont}(\alpha,\mathcal{N})$, and therefore $\mathcal{T}$ is complete. □

The next two theorems are generalizations of two important results of first order Model Theory. We will see that they are also very important in our framework.

**Teorema 13.2** *Let $\mathcal{S}$ be a symmetrical semantical system, $M \in \mathcal{N} \subseteq \mathcal{W}^{\mathcal{M}}_{\infty,\mathcal{L}}$, then* $\mathrm{Th}(M,\mathcal{K})$ *is a complete theory in $\mathcal{N}$ and $\mathcal{K}$.*

Proof: Let $\alpha \notin \mathrm{Th}(M,\mathcal{K})$, then $\varphi(\alpha, M) = 0$, so (1) $\cdot$ $M \in \mathrm{Cont}(\alpha,\mathcal{N})$. As $\mathcal{S}$ is a symmetrical semantical system, by lemma 13.7 it is sufficient to prove that $\mathrm{NegC}(\alpha, \mathcal{K}, \mathcal{N}) \subseteq \mathrm{Th}(M,\mathcal{K})$. Let $\beta \in \mathrm{NegC}(\alpha, \mathcal{K}, \mathcal{N})$, then (2) $\cdot$ $\mathrm{Mod}(\beta,\mathcal{N}) = \mathrm{Cont}(\alpha,\mathcal{N})$, from (1) and (2) we have $M \in \mathrm{Mod}(\beta,\mathcal{N})$, and thus $\beta \in \mathrm{Th}(M,\mathcal{K})$.□

**Teorema 13.3** *Let $\mathcal{S}$ be a symmetrical semantical system. If $\mathcal{W} \subseteq \mathcal{N}$ is $EC^{\mathcal{N}}_{\mathcal{K}}$ then* $\mathrm{Th}(\mathcal{W},\mathcal{K})$ *is a complete theory in $\mathcal{N}$ and $\mathcal{K}$.*

Proof: As $\mathcal{W}$ is elementary in $\mathcal{K}$, for all $M \in \mathcal{W}$ $\mathrm{Th}(\mathcal{W},\mathcal{K}) = \mathrm{Th}(M,\mathcal{K})$, thus by lemma 13.2 $\mathrm{Th}(\mathcal{W},\mathcal{K})$ is complete in $\mathcal{N}$ and $\mathcal{K}$. □

The relation $\equiv_{\mathcal{K}}$ gave us some understanting about the models of a semantical system. There is another relation which will give us deeper results about models and *topological* properties of $\mathcal{W}^{\mathcal{M}}_{\infty,\mathcal{L}}$, the subjacency relation.

**Definição 13.5** *Let $\mathcal{K} \subseteq \mathcal{T}^{\mathcal{L}}_{\infty,\mathcal{M}}$ $M_1$, $M_2 \in \mathcal{W}^{\mathcal{M}}_{\infty,\mathcal{L}}$, we say that $M_1$ is subjacent to $M_2$ ($M_1 \sqsubseteq_{\mathcal{K}} M_2$) in $\mathcal{K}$ if and only if* $\mathrm{Th}(M_1,\mathcal{K}) \subseteq \mathrm{Th}(M_2,\mathcal{K})$. □

## 13.2 The relation $\sqsubseteq$

The subjacency relation, $\sqsubseteq$, has several advantages over the notion of equivalence of models. The first of which is given by:

**Lema 13.8** *Let $\mathcal{S}$ be a semantical system, $M_1$, $M_2 \in \mathcal{N} \subseteq \mathcal{W}^{\mathcal{M}}_{\infty,\mathcal{L}}$.*
$M_1 \equiv_{\mathcal{K}} M_2 \leftrightarrow M_1 \sqsubseteq_{\mathcal{K}} M2$ *and* $M_2 \sqsubseteq_{\mathcal{K}} M1$

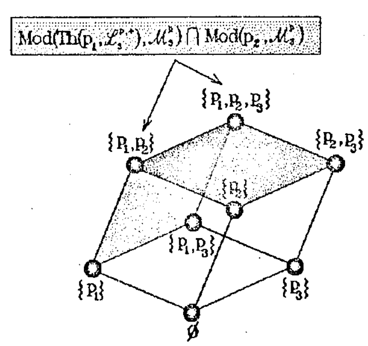
Proof: Immediate. □

**Example 13.1** *If in the propositional semantical systems we consider only the dialects $\mathcal{L}^{+,\mathrm{p}}_n$ that do not contain negation the notion of subjacency coincides with the $\subseteq$, therefore in the system $\mathcal{S}^{p+}_3$,*

$$\{p_1\} \sqsubseteq \{p_1, p_2\}$$

As a matter of fact we have that: $< \mathcal{M}^{\mathrm{p}}_3, \sqsubseteq >$ is a lattice, as shown below:

Lattice $< \mathcal{M}^{\mathrm{p}}_3, \sqsubseteq >$



We can see in the lattice $< \mathcal{M}_3^p, \sqsubseteq >$ that for each model $M \in \mathcal{M}_3^p$ the class of models of the theory of $M$ are all the models above $M$, including $M$ itself.
So,

(1) $\cdots$ $\mathrm{Mod}(\mathrm{Th}(\{p_1\}, \mathcal{L}_3^{p,+}), \mathcal{M}_3^p) = \{\{p_1\}, \{p_1, p_2\}, \{p_1, p_3\}, \{p_1, p_2, p_3\}\}$

(2) $\cdots$ $\mathrm{Mod}(\mathrm{Th}(\{p_2\}, \mathcal{M}_3^p), \mathcal{L}_3^{p,+}) = \{\{p_2\}, \{p_1, p_2\}, \{p_2, p_3\}, \{p_1, p_2, p_3\}\}$

Moreover $\mathrm{Mod}(p_2, \mathcal{M}_3^p) = \mathrm{Mod}(\mathrm{Th}(\{p_2\}, \mathcal{L}_3^{p,+}), \mathcal{M}_3^p)$

As $p_2 \notin \mathrm{Th}(\{p_1\}, \mathcal{L}_3^{p,+})$, we can see that $\mathrm{Th}(\{p_1\}, \mathcal{L}_3^{p,+})$ is not complete, for
$\mathrm{Mod}(\mathrm{Th}(\{p_1\}, \mathcal{L}_3^{p,+}), \mathcal{M}_3^p) \cap \mathrm{Mod}(p_2, \mathcal{M}_3^p) = \{\{p_1, p_2\}, \{p_1, p_2, p_3\}\}$
So, the question is: which models in $\mathcal{M}_3^p$ have complete theories relative to the language $\mathcal{L}_3^{p,+}$? A partial answer to this question for the semantical system $\mathcal{S}_3^{p,+}$ is that the only complete theories are those of the models

$$\{p_1, p_2\}, \{p_1, p_3\}, \{p_2, p_3\}$$

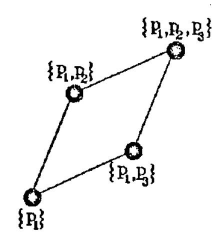
because above these models there is only the model $\{p_1, p_2, p_3\}$ which is the only one in $\mathcal{W}_{0,\mathcal{L}}^{\mathcal{M}}$ for the considered semantical system.

The models in $\mathcal{M}_3^p$ are finite. So, one would expect their theories to be complete because as such they can be *completly* described by enumerating all of its elements. It is surprising that only few of them are complete. The semantical system $\mathcal{S}_3^{p,+}$ was obtained from $\mathcal{S}_3^p$ by restriction of the language and consequently of the evaluation function $\varphi_3^p$. The system $\mathcal{S}_3^{p,+}$ is very poor as a *declarative* system as it lacks *expressive power*.

Let us suppose we want to describe the model $\{p_1\}$ in $\mathcal{S}_3^{p,+}$. A strong, natural and intuitive candidate would be the sentence $p_1$ of $\mathcal{L}_3^{p,+}$, but then as we just saw this sentence has other models besides $\{p_1\}$ which we intended to describe (completly).

What seemed intuitive and natural is a mistake generated by *reasoning* in the language of proposicional semantical systems and strongly restricting its language.

Looking at the piece of the lattice containing the models of $p_1$ we can see that $\{p_1\}$ is the minimum (relative to $\subseteq$) of them.




The sentence $p_1 \wedge p_2 \wedge p_3$, in $\mathcal{S}_3^{p,+}$, describes completely the top model of the lattice because this is the only model for that sentence.

One way to strengthen the *expressive power* of the semantical system $\mathcal{S}_3^{p,+}$ is modifying its evaluation function so that it may *capture* this notion about minimum model. There are some difficulties in this project because this notion is quite elusive

as it depends on a complete description of the universe of discourse. For instance, in the present case, we *know* completely all the models of $\mathcal{S}_3^{p,+}$ and the order relation $\sqsubseteq$, as presented in the last page. Intentional considerations, such as of minimum model is out the scope of this work.
It is very useful to postulate a *representative* of the world $\mathcal{W}^{\mathcal{N}}_{0,\mathcal{K}}$, even when it is empty.

**Postulate 1** - We represent by $\top^{\mathcal{N}}_{\mathcal{K}}$, or simply by $\top$, any model in $\mathcal{W}^{\mathcal{N}}_{0,\mathcal{K}}$[4]. □

**Lema 13.9** *For any semantical system* $\mathcal{S}$, $\mathrm{Th}(\top^{\mathcal{N}}_{\mathcal{K}},\mathcal{K}) = \mathcal{T}^{\mathcal{K}}_{\infty,\mathcal{N}}$.

Proof: Immediate. □
The need to postulate a bottom model is partially justified by the existence of empty models in propositional semantical systems. The world $\mathcal{W}^{\mathcal{N}}_{\infty,\mathcal{K}}$ is usually not a class of equivalence, so we can not take one of its models as representative of it.
**Postulate 2** - $\bot^{\mathcal{N}}_{\mathcal{K}}$, or simply $\bot$ is a model which *corresponds* to $\mathcal{W}^{\mathcal{N}}_{\infty,\mathcal{K}}$, in the sense that $\mathrm{Th}(\bot,\mathcal{K}) = \mathcal{T}^{\mathcal{K}}_{0,\mathcal{N}}\mathrm{Th}(\mathcal{W}^{\mathcal{N}}_{\infty,\mathcal{K}},\mathcal{K})$. □
It is important to observe that if we consider the whole language of $\mathcal{S}_2^{p}$ then

$$\mathrm{Th}(\{p_1\},\mathcal{L}_2^p) \not\sqsubseteq \mathrm{Th}(\{p_1, p_2\},\mathcal{L}_2^p)$$

Thus such models are not comparable regarding the notion of subjacency.
The following result shows that $< \mathcal{W}^{\mathcal{N}}_{\infty,\mathcal{K}}, \sqsubseteq >$ is a *flat* lattice:

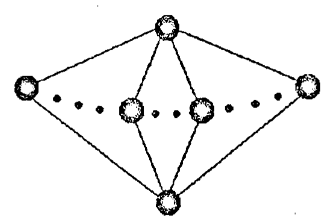
**Lema 13.10** *Let* $\mathcal{S}$ *be a symmetrical semantical system. Then:*

(1) *for all* $M \in \mathcal{W}^{\mathcal{N}}_{\infty,\mathcal{K}}$ $\bot \sqsubseteq M \sqsubseteq \top$

(2) *for all* $M, N \in \mathcal{W}^{\mathcal{N}}_{\infty,\mathcal{K}} - \{\bot, \top\}$ $M \not\sqsubseteq N$

Proof: (1) For all $M \in \mathcal{W}^{\mathcal{N}}_{\infty,\mathcal{K}}$ we have $\mathcal{T}^{\mathcal{K}}_{0,\mathcal{N}} \subseteq \mathrm{Th}(M,\mathcal{K}) \subseteq \mathcal{T}^{\mathcal{K}}_{\infty,\mathcal{N}}$. (2) follows from the fact that in a symmetrical semantical system $\mathrm{Th}(M,\mathcal{K})$ is complete in $\mathcal{K}$ and $\mathcal{N}$, and lemma 15.3. □

Lattice $< \mathcal{M}, \sqsubseteq >$ for symmetrical systems



[4] In the case that $\mathcal{W}^{\mathcal{N}}_{0,\mathcal{K}}$ is empty (symmetrical systems), $\top$ is the model for contradictions and is used analagously to **F** in Classical Logic.

In the above lattice as well as in the one we presented for the positive system $\mathcal{S}_3^{p+}$ we can see that the models that possess complete theories with respect to $\sqsubseteq$ are the biggest models immediately below $\top$. The reason for this is lemma 15.3 that is valid for any semantical systems. The example $\mathcal{S}_3^{p+}$ serves only as a motivation for the analysis of the topology of the positive semantical systems relative to $\sqsubseteq$.

**Note:** Notice that the evaluation function of $\mathcal{S}_3^{p}$ is the same of $\mathcal{S}_3^{p+}$ but the topology with respect to $\sqsubseteq$ changed. What remained the same in the two topologies is that above all the models sits $\top$, corresponding to the whole language as theory. Also, the level immediately below the top contains complete theories, both in the symmetrical case and in the analysed case of $\mathcal{S}_3^{p+}$. As for the general case, it is a matter to be analysed in the study of positive semantical systems.

# 14 Conclusions

Applications of the results of this work are already under way and we have been working in two direction:

**Positive Semantical Systems** The positive semantical systems are our bridge to the study of the formal systems. Smullyan's Theory of Formal Systems have been studied within our general framework and we have improved our understanding of Gödel's Incompleteness Theorem. A report on this subject is being prepared.

**Intensional Semantical Systems** From the simple fact that all definitions are relative to subclasses of $\mathcal{W}^{\mathcal{M}}_{\infty,\mathcal{L}}$ and subclasses of $\mathcal{T}^{\mathcal{L}}_{\infty,\mathcal{M}}$, a natural thing is to investigate what happens when one considers worlds and theories instead of general classes of models or sentences.

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