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Abstract

The aim of this work is to provide a denotational semantics for an arbitrary level λ-Calculus, i.e., a typed λ-Calculus which has rules as types. This system is shown to be Curry-Howard isomorphic to a Natural Deduction system, where not only assumptions, but rules, can be discharged.

Key-words : typed λ-Calculus, Denotational Semantics, Natural Deduction.

Resumo

Este trabalho apresenta uma semântica denotacional para um λ-Calculus de nível arbitrário onde pode-se ter regras como tipos. Pode-se mostrar que este λ-Calculus é Curry-Howard isomórfico a um sistema de Dedução Natural onde regras podem ser descarregadas.

Palavras-Chave : λ-Calculus tipificado, Semântica Denotacional, Dedução Natural.

A Denotational Semantics for an Arbitrary Level Typed λ-Calculus

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Abstract

The aim of this work is to provide a denotational semantics for an arbitrary level λ-Calculus, i.e., a typed λ-Calculus which has rules as types. This system is shown to be Curry-Howard isomorphic to a Natural Deduction system, where not only assumptions, but rules, can be discharged.

1 Introduction

A natural extension of Natural Deduction (N.D.) was introduced by Schroeder-Heister ([2]) where not only formulae but also rules could be used as hypothesis. This fact immediately allowed for the possibility of rules of arbitrary levels that could discharge not only assumption-formulae but also

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assumption-rules. This extension of N.D. was used in the definition of abstract introduction and elimination schemes for which the set of intuitionistic sentential operators $\{\bot, \rightarrow, \wedge, \vee\}$ was shown to be complete, i.e., for any sentential operator Φ if the introduction and elimination rules for Φ are instances of the abstract schemes then, there is an intuitionistic formula F constructed out of the intuitionistic sentential operators and the sentential variables occurring in $\Phi(A_1, \ldots, A_n)$ such that $\vdash_{I+\Phi} \Phi(A_1, \ldots, A_n) \leftrightarrow F$. It was also possible to define an *abstract reduction* and under certain conditions to obtain abstract normalization and strong normalization results.

The aim of the present paper is to define a typed λ-Calculus which has types of arbitrary levels (formula-as-types and rule-as-types) and to provide it with an abstract denotational semantics. We shall also show that this typed λ-Calculus is *Curry-Howard* isomorphic to Schroeder-Heister's abstract N.D.

In section 2 we describe the syntax of the typed λ-Calculus of arbitrary levels. In section 3 we introduce the abstract denotational model for the system. In section 4 we show the semantic adequacy of the reduction rules. Finally, in section 5, we show the *Curry-Howard* isomorphism.

2 The System λ_R

The calculus is defined in the usual way by the presentation of Type construction, Term construction and Reduction rules.

2.1 Types

1. A_1, B_1, C_1, ..., A_n, B_n, C_n, ... are types of level 0.
2. If X_1, ..., X_n are types of level 0 and Φ is a type constructor then $\Phi(X_1, \ldots, X_n)$ is a type of level 0 too. .
3. If $T_1, \ldots, T_n$ are types ,B is a type of level 0,and n is the greatest level among them, then $T_1^\dagger, \ldots, T_n^\dagger/B$ is a type of level $n+1$. Here $T^\dagger$ denotes the string obtained from T throught the replacement of replacing all the occurences of $/$ for $\rightarrow$ and of $\rightarrow$ for $/$ simultaneously.

Examples :

- A/B is a type of level 1.
- $A \to B, D/C$ is a type of level 2.
- $(A/B, D) \to C/E$ is a type of level 3.

Obs: Note that the parenteses are needed in order to correctly present the hierarchy of the type formation. They were not include in the definition of the types for the sake of clarity.

We also define two auxiliary functions ρ_1 and ρ_2 from $Types$ into 2_{Types}.

In addition to the definition of terms, we have a notational function which retrives some of the elements of a term. Thus :

- $\rho_1(B) = \emptyset$ if B is of level 0.
- $\rho_2(B) = \{B\}$ if B is of level 0.
- $\rho_2(T_1^\dagger, \ldots, T_n^\dagger/B) = \{B\}$
- $\rho_1(T_1^\dagger, \ldots, T_n^\dagger/B) = \rho_2(T_1) \cup \ldots \cup \rho_2(T_n)$

2.2 Contexts

The concept of context plays the same rôle, in λ-Calculus , as the set of hypothesis (not yet discharged) on which the conclusion of a deduction depends, in Natural Deduction. Thus, if $T_1, \ldots, T_n$ are types and $x_1, \ldots, x_n$ are variables then a context is a sequence (sometimes viewed as a set) of the form :

$$x_1 : T_1; x_2 : T_2; \ldots; x_n : T_n$$

We will use the capital greek letters Δ, Θ and Γ to denote contexts.

2.3 Patterns

The concept of pattern serves to denote, syntactically, a rule which may be used in the construction of terms which represent a deduction having this rule occurring (as an assumed rule). It is worth noting that even when a rule is used as a hypothesis we should take it as being applied to its premises resulting in a deduction of its conclusion. However, as an individual object, a rule depends only on itself. Thus, a pattern is of the form $x[x : T] : T$ where, x is a variable and

T is a type of any level. Note that a pattern of level 0 is nothing but a formula assumption. So, we regard a pattern, in general, as a rule assumption. However, for the purpose of simplicity in the following definitions, we shall consider separately assumption rule from assumption formulae considering as patterns only the former.

2.4 Terms

Let us assume that we have the following rule in N.D. style:

$$\frac{\begin{array}{ccc} H_1^1,\ldots,H_1^{j_1} & & H_n^1,\ldots,H_n^{j_n} \\ | & \cdots & | \\ G_1 & & G_n \end{array}}{A}$$

We can relate each hypothesis $H_i^{j_p}$ either to a term $x : H_i^{j_p}[x : H_i^{j_p}]$ or to a pattern of the corresponding type. Thus, the term related to the deduction of the corresponding G_p has free occurences of x. The term corresponding to the deduction of A must then have x as a bound variable, since the type corresponding to x is discharged by the application of the rule. Well, we may think of a more computational kind of rule, where the mechanism of discharging is applied only to the deduction of the appropriated premisse. Thus, our λ-Calculus should take into account that the binding of a variable in a term which is formed by an application of the "rule" is only related to the term representing the deduction of the corresponding premisse. This selective way of binding variables is then represented by a list (or lists) of variables where each member of the list is related, in the correspondent ordering, to the term where this variables is are going to be bound.

1. $x : X[x : X]$ is a term for any type X of level 0 and any variable x. Thus, note that a pattern, of level 0, is a term.
2. If $t_1 : T_1[\Theta_1], \ldots, t_n : T_n[\Theta_n]$ are terms and x is a pattern of type $T_1^*, \ldots, T_n^*/X$, then :

   $$[\vec{x}_1 \mid \ldots \mid \vec{x}_n] < x, t_1, \ldots, t_n >: X[x : T_1^*, \ldots, T_n^*/X, \Theta_1 - [\vec{x}_1], \ldots, \Theta_n - [\vec{x}_n]]$$

   is a term where, $\vec{x}_i$ is a list of variables whose types are in $\rho_1(T_i)$.

We can now represent the introduction and elimination rules of N.D. for an arbitrary constant Φ in our λ-Calculus as following.

Let Φ be a constant which has n introduction schemas each of them similar to the rule shown at the beginning of this subsection. Its n schemas are, each of them , represented by the following terms of Λ_Φ-abstraction.

If $\vec{x^i_j} : \vec{T^i_j}$ and $t^i_j(\vec{x^i_j}) : G^i_j[\Gamma^i_j]$ are terms $(0 \leq j \leq \rho_i)$ then

$$\Lambda_\Phi[\vec{x^i_1}, \ldots, \vec{x^i_{\rho_i}}](t^i_1, \ldots, t^i_{\rho_i}) : \Phi(F_1, \ldots, F_n)[\Gamma^i_1, \ldots, \Gamma^i_{\rho_i}]$$

is a term.

Related to these n introduction schemas we have the following elimination schema.

Let

1. $\vec{y_i} \equiv [y^i_1 : T^i_{1,1}, \ldots, T^i_{1,m_1}/G^i_1, \cdots, y^i_{\rho_i} : T^i_{\rho_i,1}, \ldots, T_{\rho_i,m^i_{\rho_i}}/G^i_{\rho_i}]$
2. $v_i : A[\Delta_i]$
3. $v : \Phi(T_1, \ldots, T_n)[\Delta]$

Then

$$App_\Phi[\vec{y_1}, \ldots, \vec{y_n}] < v, v_1, \ldots, v_n >: A[\Delta, \Delta_i - \vec{y_i}]$$

is a term.

Fact 1 *It is worth notice that by the way we construct our terms for representing introduction and elimination rules, we have a relationship between them such that:*

> *For each variable of type rule R in a term resulted from an application of a Φ-Introduction , we have that any application of the Φ-Elimination has, as a subterm, the associated term representing a deduction of $\rho_2(R)$ (its conclusion) from $\rho_1(R)$, among others. We may note that the other way around also holds.*

This fact plays a central role in the reduction rules shown below.

2.5 Reduction Rules

A term is a redex if it has one of the following forms:

$$App_\Phi[\vec{y_1},\ldots,\vec{y_n}] < \Lambda_\Phi[\vec{x^i_1},\ldots,\vec{x^i_{\rho_i}}](t^i_1,\ldots,t^i_{\rho_i}), v_1,\ldots,v_n >: A[\Gamma]$$

$$App_\Phi[\Delta] < App_\Psi[\Theta] < t_1,\ldots,t_k >, v_1,\ldots,v_n >: A[\Gamma]$$

The first one is called operational redex, while the other one is called structural redex and plays the rôle of permutative reductions in N.D. systems (cf. [1]).

Before we define reduction rules for these redex, we will have a brief look at what happens in N.D. In this case we must consider which introduction rule the Λ_Φ operator represent. For the sake of clarity we will write down the reduction rule with Λ_Φ representing the first introduction rule for Φ (with the relation to the ordering of the minor premises in the elimination rule). Thus, if Δ_i is the context:

$$x_1 : (\Theta^i_1)^\dagger/G^i_1,\ldots,x_{\mu_1}(\Theta^i_{\mu_1})^\dagger/G^i_{\mu_1}$$

which is obviously the result of the $\dagger$ operator when applied to the corresponding configuration for the first Φ introduction rule. Remember the relationship between the elimination and introduction rules. According to this relationship we must have that in the first Φ introduction rule, the set of types discharged (variables bounded to the new term) is Θ^1_1 for the first premisse, Θ^1_2 for the second and so on. Let's consider then the following redex (associated to this first introduction rule).

$$\mathbf{App}_\Phi[\emptyset \mid \Delta_1 \mid \ldots \mid \Delta_n](\Lambda_\Phi[\Theta^1_2 \mid \ldots \Theta^1_k](t'_1,\ldots,t'_k), d_1,\ldots,d_n) : A[\Gamma]$$

Consider the process of filling all x_j (of type $(\Theta^1_j)^\dagger/G^1_j$) with the term t'_j (of type G^1_j and context Θ^1_j), then the reduction of the above redex is the result of appliyng this process of filling (which is a generalization of substitution) to d_1. We shall now detail this filling process of variables (representing rules) for terms (representing the associated deductions).

To replace (fill) a variable x of type $\Theta^\dagger/G$ by a term u of type G and context Θ, Γ in a term $t(< x, d_1,\ldots,d_n >)$ (here we explicitly show the occurence of x) is equivalent, in N.D., to replace a derived rule by the deduction that justifies it. Let Π be the following derivation:

$$\frac{\begin{matrix}\Pi_1 & & \Pi_n \\ H_1 & \cdots & H_n\end{matrix}}{\begin{matrix}\alpha \\ \Sigma \\ \beta\end{matrix}}$$

Given a deduction Π^*:

$$\begin{matrix}H_1 \dots H_n \\ \Pi^* \\ \alpha\end{matrix}$$

We can replace the application of the derived rule:

$$\frac{H_1 \dots H_n}{\alpha}$$

by Π^*, obtaining in this way the following derivation:

$$\begin{matrix}\Pi_1 & & \Pi_n \\ H_1 & \cdots & H_n \\ & \Pi^* & \\ & \alpha & \\ & \Sigma & \\ & \beta & \end{matrix}$$

In the mechanism of λ-Calculus the process of substitution discussed above can be expressed in the following, way. Given that:

1. $$t(< x, d_1, \dots, d_n >) : B[\Upsilon]$$
2. $x : \Theta^\dagger / G$ with $\Theta = x_1 : F_1/H_1, \dots, x_n : F_n/H_n$,
3. $$u(x_1, \dots, x_n) : G[x_1 : F_1/H_1, \dots, x_n : F_n/H_n, \Gamma]$$

We can obtain a new term

$$t(u(d_1, \dots, d_n)) : B[\Upsilon, \Gamma]$$

the replacing of x by the term u....

It is clear by the construction of the calculus that a variable of the type of a rule, if it occurs in a term, then it must occur in the context of an application of a rule. within the scope of an Application. The case which has the variable as not being of a type of a rule is treated as an usual substitution.

The reader can think of what happens if some of the x_j is of type of a rule in the example above when we are treating the case of introductions and eliminations. In this case, if x_j, for some j, is of type of rule, it must be of type $\Delta^{\dagger}/H_j$ where Δ is the context of d_j, in the elimination rule, which in turn is of type H_j. Thus, in this case the process of substitution continues until all the variables that occurs in similar contexts are replaced by the corresponding terms. We will denote by t^+ the result of this process. Thus, the contractum of an operational redex is :

$$v_i(< y^i_j, u^i_{j,1}, \ldots, u^i_{j,m_j} > \leftarrow (t^i_j)^+) : A[\Gamma]$$

also denoted by:

$$v_i(y^i_j \leftarrow (t^i_j)^+) : A[\Gamma],$$

for the sake of clarity.

The contractum of an structural redex

$$App_{\Phi}[\Delta] < App_{\Psi}[\Theta] < t_1, \ldots, t_k >, v_1, \ldots, v_n >: A[\Gamma]$$

is as following:

$$App_{\Psi}[\Theta](t_1, App_{\Phi}[\Delta](t_2, u_2, \ldots, u_n), \ldots, App_{\Phi}[\Delta](t_k, u_2, \ldots, u_n))$$

We use the symbol $\rhd$ to denote that t reduces t' in one step and $\rhd^*$ denotes the transitive-reflexive closure of $\rhd$.

We show below a short example of a reduction for the case where x_j is of type of rule in N.D. in our calculus. Consider the following deduction which has $\Phi(F)$ as a maximal formula.

$$\dfrac{\dfrac{\begin{array}{c}\Pi_1\\ [\dfrac{A}{B}]\\ \Pi_2\\ C\end{array}}{\Phi(F)} \qquad \begin{array}{c}[A]\\ \Pi_3\\ [\dfrac{B}{C}]\\ \Pi_4\\ D\end{array}}{D}$$

1. $M = \{M_T : T \text{ is a type}\}$.

2. $*$ *is a mapping that assigns to each term*

$$t : T[x_1 : T_1, \ldots, x_n : T_n]$$

a mapping t^*

$$M_{T_1} \times \ldots \times M_{T_n} \to M_T$$

3. $x : T[x : T]^*(c) = c \in M_T$.

4.

$$App_\Phi[\vec{y_1}, \ldots, \vec{y_n}] < \Lambda_\Phi[\vec{x_1^i}, \ldots, \vec{x_{\rho_i}^i}](t_1^i, \ldots, t_{\rho_i}^i), v_1, \ldots, v_n >: A[\Gamma]^*(c_1, \ldots, c_m) =$$

$$v_i^*(c_1, \ldots, c_m, < y_j^i, u_{j,1}^i, \ldots, u_{j,m_j}^i > \leftarrow ((t_j^i)^*)^+(c_1, \ldots, c_m))$$

Its clear that not all abstract model structures preserve the denotation under the substitution. The following notion of canonical model defines what this preservation of denotations means.

Definition 3.2 *A* Canonical Model *of* λ_R *is an abstract model structure that satisfies the following substitution condition:*

Let :

$$t(x : T_1^\dagger, \ldots, T_n^\dagger / T) : T'[\Gamma]$$

,and

$$s : T[y_1 : T_1, \ldots, y_n : T_n]$$

be terms such that t and s satisfy the variable condition. *Then :*

$$t(x \leftarrow s^+)^*(c_1, \ldots, c_n) = t^*(c_1, \ldots, c_n, (s^*)^+(c_1, \ldots, c_m))$$

The variable condition is nothing but a condition that assures that the filling process $.^+$ can be carried out. Note that this condition is always satified when we perform the filling process as a substep of the reduction step described in the previous subsection. It is a conseguence of the fact stated at the second section.

The following Lemma justify the notion of canonical model. Its proof is a direct consequence of the definitions and the satisfaction of the variable condition

It reduces to

$$\begin{array}{c} \Pi_1 \\ A \\ \Pi_3 \\ B \\ \Pi_2 \\ C \\ \Pi_4 \\ D \end{array}$$

In λ_R the reduction is shown as following.

$$\Lambda_\Phi[x]u([] < x, t >) : \Phi(F)[]$$

represents Π_2, where t represents Π_1

$$t'(y) : B[y : A]$$

represents Π_3

$$v([y] < z, t'(y) >) : D[z : A \to B/C]$$

represents Π_4. Thus,

$$App_\Phi[z] < \Lambda_\Phi[x]u([] < x, t >), v([y] < z, t'(y) >) >: D[] \quad \rhd^*$$

$$v(z \leftarrow u(x \leftarrow t'(y \leftarrow t)))$$

3 The Denotational Semantics

The denotational semantics for the system λ_R follows the usual structure of denotational semantics for typed λ-Calculi. We shall define the function [[.]] which takes each type and each term into their denotations. Our model is a syntactical one, i.e., the denotation of a type is the set of normal terms which have itself as type. Thus, in this way, we a have a straightforward preservation of denotation under our concept of "filling" variables with terms. Before we describe the Term model we define the abstract notion of a model for a typed λ-Calculus.

Definition 3.1 *An abstract model structure is a pair $< M, * >$ such that:*

Lemma 3.1 *Let $< M, * >$ be a canonical model for λ_R. Then, for any t, t' in λ_R , if $t \rhd^* t'$ then $t^* = t'^*$.*

We now introduce our Term Model. We define a function $[[.]]$ that assigns to each term $t : T[\Gamma]$ the set of terms to which it reduces.

$$[[t : T[\Gamma]]] = \{t' : t \rhd^* t'\}$$

The canonical term model $M = < M, * >$ is defined as:

- $M_T = \{[[t]]/t : T\}$ for T of level 0.
- $M_T = \{t/t : T'[x_1 : T_1, \ldots, x_n : T_n]\}$ for $T = T_1^\dagger, \ldots, T_n^\dagger/T'$.
- Let $c_i \in M_{T_i}$. Then,

$$(t : T[x_1 : T_1, \ldots, x_n : T_n])^*(c_1, \ldots, c_n) = [[t[x_i \leftarrow c_i^\dagger] : T]]$$

4 The categorical semantics for λ_R

Types will be interpreted as objects in a category while terms will be interpreted as morphisms.

4.1 Auxiliary Results

In this subsection, for the sake of notation and technical background, we touch the categorical theoretic notions used in this work as well prove some auxiliary results, namely Lemma I and Lemma II. What follows should hold in a cartesian closed category with finite co-products.

For any pair of morfisms $g : A \longrightarrow C$ and $f : B \longrightarrow C$ we use the notation $[g, f]$ to denote the (unique) morfism from the object $A + B$ into C with the co-product property, that is :

$$[g, f] \circ i_A = g$$

and

$$[g, f] \circ i_B = f$$

For any objects A, B and C, and the exponential B_A we have the existence of a natural isomorfism from $Hom(C \times A, B)$ into $Hom(C, B_A)$ which we will be denoted by $\widehat{g}$, that is, for any $g : C \times A \longrightarrow B$ we have $\widehat{g} : C \longrightarrow B^A$, such that

$$ev \circ < \widehat{g}, I_A >= g$$

where ev is the evaluator morfism for the exponential B_A.

In the sequel we state and prove two results which will be very useful when proving the adequacy of the semantics defined in the next subsection.

Lemma I. Let **C** be a cartesian closed category with finite co-products, $f : E \longrightarrow A$, $g_A : A \times C \longrightarrow F$ and $g_B : B \times D \longrightarrow F$ be **C**-morfims. Let also *dist* be the morfism :

$$< [i_{A\times\widehat{(C\times D)}}, i_{B\times\widehat{(C\times D)}}], I_{C\times D} >$$

from $(A + B) \times (C \times D)$ into $(A \times (C \times D)) + (B \times (C \times D))^{(C\times D)}$ and ev be the evaluator morfism of the exponential $(A \times (C \times D)) + (B \times (C \times D))^{(C\times D)}$. Then :

$$\begin{array}{l}
E \times (C \times D) \\
\quad \Big\downarrow \; < i_A \circ f, I_{C\times D} > \\
(A + B) \times (C \times D) \\
\quad \Big\downarrow \; ev \circ dist \\
(A \times (C \times D)) + (B \times (C \times D)) \\
\quad \Big\downarrow \; [g_A \circ < I_A, \pi_C >, g_B \circ < I_B, \pi_D >] \\
F
\end{array}$$

is equal to the morfism

$$E \times (C \times D) \xrightarrow{<f,\pi_C>} A \times C \xrightarrow{g_A} F$$

Proof. we first note, that, by the definition of co-product :

$$\begin{array}{ccc} A \times (C \times D) & \xrightarrow{<i_A, I_{C\times D}>} & (A+B) \times (C \times D) \\ & & \Big\downarrow < [\widehat{i_{A\times(C\times D)}}, \widehat{i_{B\times(C\times D)}}], I_{C\times D} > \\ & & ((A \times (C \times D)) + (B \times (C \times D)))^{(C\times D)} \times (C \times D) \end{array}$$

is the same as :

$$A\times(C\times D) \xrightarrow{<\widehat{i_{A\times(C\times D)}}, I_{C\times D}>} ((A\times(C\times D))+(B\times(C\times D)))^{(C\times D)}\times(C\times D)$$

and hence, by the definition of exponential we have:

$$\begin{array}{ccc} A \times (C \times D) & \xrightarrow{<\widehat{i_{A\times(C\times D)}}, I_{C\times D}>} & ((A \times (C \times D)) + (B \times (C \times D)))^{(C\times D)} \times (C \times D) \\ & & \Big\downarrow ev \\ & & (A \times (C \times D)) + (B \times (C \times D)) \end{array}$$

equals to $i_{(A\times(B\times C))}$.

The reader should also note that

$$< i_A \circ f, I_{C\times D} >=< i_A \circ f, I_{C\times D} \circ I_{C\times D} >$$

Thus :

$$
\begin{array}{l}
E \times (C \times D) \\
\Big\downarrow < i_A \circ f, I_{C\times D} > \\
(A + B) \times (C \times D) \\
\Big\downarrow ev \circ dist \\
(A \times (C \times D)) + (B \times (C \times D)) \\
\Big\downarrow [g_A \circ < I_A, \pi_C >, g_B \circ < I_B, \pi_D >] \\
F
\end{array}
$$

is the same as

$$
E \times (C \times D) \overset{i_{A\times(C\times D)} \circ < f, I_{C\times D} >}{\longrightarrow} (A \times (C \times D)) + (B \times (C \times D)) \Big\downarrow [g_A \circ < I_A, \pi_C >, g_B \circ < I_B, \pi_D >] \; F
$$

which, in turn, is the same as $< f, I_{C\times D} > \circ (< I_A, \pi_C > \circ g_A)$. However, $< f, I_{C\times D} > \circ < I_A, \pi_C > = < f, \pi_C >$. Finally we can conclude that the whole morfism is the same as $< f, \pi_C > \circ g_A$.

Q.E.D.

We should observe that the above Lemma also holds with regard to B instead of A.

Lemma II. Let **C** be a cartesian closed category with finite coproducts. Consider the following diagram:

$$
\begin{array}{ccccc}
 & & E \times E_1 \times E_2 \times F_1 \times F_2 & & \\
 & & \Big\downarrow {\scriptstyle < h, I >} & & \\
 & & (D + C) \times (E_1 \times E_2 \times F_1 \times F_2) & & \\
 & & \Big\downarrow {\scriptstyle DISTRB} & & \\
D \times (E_1 \times E_2 \times F_1 \times F_2) \overset{i}{\hookrightarrow} & & \begin{array}{c} D \times (E_1 \times E_2 \times F_1 \times F_2) \\ + \\ C \times (E_1 \times E_2 \times F_1 \times F_2) \end{array} & & \overset{i}{\hookleftarrow} C \times (E_1 \times E_2 \times F_1 \times F_2 \\
 & \searrow {\scriptstyle f_1} & \Big\downarrow {\scriptstyle [f_1, f_2]} & {\scriptstyle f_2} \swarrow & \\
 & & (A + B) \times (F_1 \times F_2) & & \\
 & & \Big\downarrow {\scriptstyle DISTRB'} & & \\
A \times (F_1 \times F_2) & \overset{i}{\hookrightarrow} & A \times (F_1 \times F_2) + B \times (F_1 \times F_2) & \overset{i}{\hookleftarrow} & B \times (F_1 \times F_2) \\
 & \searrow {\scriptstyle g_1} & \Big\downarrow {\scriptstyle [g_1, g_2]} & {\scriptstyle g_2} \swarrow & \\
 & & H & &
\end{array}
$$

where $h : E \longrightarrow (D + C)$. Then we have that:

$$[g_1, g_2] \circ DISTRB' \circ [f_1, f_2] \circ DISTRB \circ < h, I > =$$

$$[[g_1, g_2] \circ DISTRB' \circ f_1, [g_1, g_2] \circ DISTRB' \circ f_2] \circ DISTRB \circ < h, I >$$

Proof. We only need to prove that:

$$[g_1, g_2] \circ DISTRB' \circ [f_1, f_2] = [[g_1, g_2] \circ DISTRB' \circ f_1, [g_1, g_2] \circ DISTRB' \circ f_2]$$

which is obtained by observing the following equations :

$$([g_1, g_2] \circ DISTRB' \circ [f_1, f_2]) \circ i_{C \times (E_1 \times E_2 \times F_1 \times F_2)} = [g_1, g_2] \circ DISTRB' \circ f_1$$

and

$$([g_1, g_2] \circ DISTRB' \circ [f_1, f_2]) \circ i_{D \times (E_1 \times E_2 \times F_1 \times F_2)} = [g_1, g_2] \circ DISTRB' \circ f_2$$

However, by the definition of co-product we have a unique morfism from $(D \times (E_1 \times E_2 \times F_1 \times F_2)) + (D \times (E_1 \times E_2 \times F_1 \times F_2))$ into H which is by definition :

$$[[g_1, g_2] \circ DISTRB' \circ f_1, [g_1, g_2] \circ DISTRB' \circ f_2]$$

Q.E.D.

4.2 The function [[.]]

Given a cartesian closed category **C** with finite co-products [[.]] takes as arguments types, patterns and terms, and yields values in **C** (either objects or morfisms).

Types

- If T is a basic type of level 0, then $[[T]]$ is an object of C.
- If $T_1^\dagger, \ldots, T_n^\dagger / Y$ is a type of level $n > 0$ then

$$[[T_1^\dagger, \ldots, T_n^\dagger / Y]] = [[Y]]^{[[T_1]] \times, \ldots, \times [[T_n]]} \in Obj(\mathbf{C})$$

- If $\Phi(F_1, \ldots, F_n)$ is a type of level 0 and has $R_1, \ldots, R_n$ as the types of all of its corresponding introduction rules, then

$$[[\Phi(F_1, \ldots, F_n)]] = [[R_1]] + \ldots + [[R_n]]$$

If $\Gamma = x_1 : T_1, \ldots, x_n : T_n$ is a context then we use $[[\Gamma]]$ to denote $\prod_{i=1,n}[[T_i]]$, that is, the product of the denotations of each type in the context.

Patterns

- $[[x : R[x : R]]] = I_{[[R]]}$

Terms

A term $t : T[\Gamma]$ will be interpreted as a morphism from $[[\Gamma]]$ into $[[T]]$.

1.

$$[[x : X[x : X]]] = I_{[[X]]}$$

2. Given that :
   (a) $x : T_1^\dagger, \ldots, T_n^\dagger/Y[x : T_1^\dagger, \ldots, T_n^\dagger/Y]$ is a pattern and,
   (b) $t_i : Y_i[\Gamma_i \cup \Theta_i]$ are terms, $0 < i < n+1$, and,
   (c) $T_i = \Theta_i/Y_i$ and,
   (d)

$$[[t_i : Y_i[\Gamma_i \cup \Theta_i]]] = f_i : [[\Gamma_i]] \times [[\Theta_i]] \longrightarrow [[Y_i]]$$

and hence by the property of exponentials that gives a natural isomorphim between $Hom_{\mathbf{C}}(a \times b, c)$ and $Hom_{\mathbf{C}}(a, c^b)$ we have a unique morphism $\widehat{f_i}$ from $[[\Gamma_i]]$ into $[[Y_i]]^{[[\Theta_i]]}$. Thus, $< \widehat{f_1}, \ldots, \widehat{f_n} >$ is a morphism from $[[\Gamma]]$ into

$$[[Y_1]]^{[[\Theta_1]]} \times \ldots \times [[Y_n]]^{[[\Theta_n]]}$$

Finally we define

$$[[[\Theta_1 \mid \ldots \Theta_n] < x, t_1, \ldots, t_n >: Y[x : T_1^\dagger, \ldots, T_n^\dagger/Y, \Gamma_1, \ldots, \Gamma_n]]]$$

as $ev_{[[R]]} \circ < I_{[[R]]}, < \widehat{f_1}, \ldots, \widehat{f_n} >>$, where R is $T_1^\dagger, \ldots, T_n^\dagger/Y$ and $ev_{[[R]]}$ is the corresponding evaluator morphism of its denotation, which is of course an exponential. Note that the denotation given above is indeed a morphism from $[[x : R, \Gamma_1, \ldots, \Gamma_n]]$ into $[[Y]]$ as the following diagram shows :

$[[Y]]^{[[T_1]] \times, \ldots, \times [[T_n]]} \times [[\Gamma_1]] \times \ldots \times [[\Gamma_n]]$

$\downarrow < I_{[[R]]}, < \widehat{f_1}, \ldots, \widehat{f_n} >>$ $\qquad\searrow [[[\Theta_1 \mid \ldots \Theta_n] < x, \ldots]]$

$[[Y]]^{[[T_1]] \times, \ldots, \times [[T_n]]} \times [[T_1]] \times, \ldots, \times [[T_n]] \xrightarrow{ev_{[[R]]}} [[Y]]$

3. Given that :

   (a) $t_i : Y_i[\Gamma_i \cup \Theta_i]$ are terms, $0 < i < n+1$, and,

   (b)

   $$[[t_i : Y_i[\Gamma_i \cup \Theta_i]]] = f_i : [[\Gamma_i]] \times [[\Theta_i]] \longrightarrow [[Y_i]]$$

   and hence as we discussed in the above item we have $< \widehat{f_1}, \ldots, \widehat{f_n} >$ as a morfism from from $[[\Gamma]]$ into

   $$[[Y_1]]^{[[\Theta_1]]} \times \ldots \times [[Y_n]]^{[[\Theta_n]]}$$

   we define

   $$[[\Lambda_\Phi[\Theta_1 \mid \ldots \mid \Theta_n] < t_1, \ldots, t_n >: Y[\Gamma_1, \ldots, \Gamma_n]]]$$

   as $i_{[[R_i]]} \circ < \widehat{f_1}, \ldots, \widehat{f_n} >$, where R_i is :

   $$\Theta_1 \rightarrow Y_1, \ldots, \Theta_n \rightarrow Y_n / \Phi(F_1, \ldots, F_n)$$

   the i-th introduction rule for Φ. So, $i_{[[R_i]]}$ is the injection morfism given by the co-product denotation of the type $\Phi(F_1, \ldots, F_n)$.

4. Finally we have the morfism associated to the elimination rules. Let

   (a) $\vec{y_i} \equiv [y_1^i : T_{1,1}^i, \ldots, T_{1,m_1}^i / G_1^i, \cdots, y_{\rho_i}^i : T_{\rho_i,1}^i, \ldots, T_{\rho_i, m_{\rho_i}^i} / G_{\rho_i}^i]$

   (b) $v_i : A[\Delta_i]$

   (c) $v : \Phi(F_1, \ldots, F_n)[\Delta]$

   (d) $DISTRB = ev \circ dist$ be the morfism that plays the role of distributivity law of the product with regard to the co-product. Remember that *dist* is here a generalization of the one used in the previous section.

   Given that :

   $$[[\Delta]] \xrightarrow{[[v]]} [[\Phi(F_1, \ldots, F_n)]]$$

   and

   $$[[\vec{y_i}]] \times [[\Delta_i']] \xrightarrow{[[v_i]]} [[A]]$$

   where $\Delta_i' = \Delta_i - \vec{y_i}$. Thus, observing that

   $$[[\Phi]] = [[\vec{y_1}]] + \ldots + [[y_n]]$$

we define

$$[[App_{\Phi}[\vec{y_1},\ldots,\vec{y_n}] < v, v_1, \ldots, v_n >: A[\Delta, \Delta'_1, \ldots, \Delta'_n]]]$$

as the morfism represented by the following diagram.

$$[[\Delta]] \times ([[\Delta'_1]] \times \ldots \times [[\Delta'_n]])$$

$$\Big\downarrow < [[v]], I_{([[\Delta'_1]] \times \ldots \times [[\Delta'_n]])} >$$

$$[[\Phi]] \times ([[\Delta'_1]] \times \ldots \times [[\Delta'_n]])$$

$$\Big\downarrow DISTRB$$

$$([[\vec{y_1}]] \times ([[\Delta'_1]] \times \ldots \times [[\Delta'_n]])) + \ldots + ([[\vec{y_n}]] \times ([[\Delta'_1]] \times \ldots \times [[\Delta'_n]]))$$

$$\Big\downarrow [[[v_1]] \circ < I_{[[\vec{y_1}]]}, \pi_{[[\Delta'_1]]} >, \ldots, [[v_n]] \circ < I_{[[\vec{y_n}]]}, \pi_{[[\Delta'_n]]} >]$$

$$[[A]]$$

4.3 The Adequacy of the Categorical Semantics

In this subsection we show that our semantics provides a categorical model for λ_R, that is : *Any cartesian closed category with finite coproducts is a model for* λ_R.

In order to obtain the above mentioned result we firstly need to show that denotations are preserved by the reduction rules. This is the content of the following theorem.

Theorem 4.1 *Let* $t : Y[\Delta]$ *and* $t' : Y[\Delta]$ *be* λ_R*-terms. If* $t \rhd^* t'$, *then we have that* $[[t : Y[\Delta]]] = [[t' : Y[\Delta]]]$.

In order to prove this result we need to define the concept of substitution in semantical terms.

Definition 4.1 *Let*

$$t([\Upsilon] < x, v_1, \ldots, v_n >) : Y[\Delta, x : T_1^{\dagger}, \ldots, T_n^{\dagger}/Z]$$

and $d : Z[y_1 : T_1, \ldots, y_n : T_n, \Gamma]$ *be* λ_R*-terms. Their denotations are*

$$[[\Delta]] \times [[Z]]^{([[T_1]] \times \ldots \times [[T_n]])} \xrightarrow{[[t]]} [[Y]]$$

and

$$[[\Gamma]] \times ([[T_1]] \times \ldots \times [[T_n]]) \xrightarrow{[[d]]} [[Z]]$$

Clearly, from the definition of $[[.]]$ *we have that* $[[t]]$ *is a morfism determined by a diagram that includes the following subdiagram.*

$$\begin{array}{ccc}
[[Z]]^{([[T_1]] \times \ldots \times [[T_n]])} \times ([[\Theta_1]] \times \ldots \times [[\Theta_n]]) & & \\
\Big\downarrow {\scriptstyle < I, < g_1, \ldots, g_n >>} & & \searrow {\scriptstyle [[[\Upsilon] < x, v_1, \ldots, v_n >]]} \\
[[Z]]^{([[T_1]] \times \ldots \times [[T_n]])} \times ([[T_1]] \times \ldots \times [[T_n]]) & \xrightarrow{ev} & [[Z]]
\end{array}$$

Thus, $[[t(x \leftarrow d) : Y[\Gamma, \Delta]]]$ *is the morfism represented by the original diagram when replacing,* conveniently, *the above subdiagram by the following diagram :*

$$\begin{array}{c}
[[\Theta_1]] \times \ldots \times [[\Theta_n]] \\
\Big\downarrow {\scriptstyle < g_1, \ldots, g_n >} \\
[[\Gamma]] \times [[T_1]] \times \ldots \times [[T_n]] \\
\Big\downarrow {\scriptstyle [[d]]} \\
[[Z]]
\end{array}$$

We would like to explain that the word "*conveniently*" means a lightful modification to be performed in the original diagram in order to connect (we apologize for the graph-theoretical terms) the object $[[\Delta]] \times [[\Gamma]]$ with the object $[[\Gamma]] \times [[T_1]] \times \ldots \times [[T_n]]$ preserving the rest of the structure of the diagram. It is clear that this modification can be performed by using product morfisms.

Note that a particular case of this definition takes place when substituting a 0-level type, i.e., when the $x : Z$ with $[[Z]]$ not being an exponential. In this case the substitution is the composition $[[t]] \circ < I_{[[\Delta]]}, [[d]] >$ itself.

Let's observe the following about a general redex as the below one :

$$App_\Phi[\vec{y_1}, \ldots, \vec{y_n}] < \Lambda_\Phi[\Theta_1 \mid \ldots \mid \Theta_k] < t_1, \ldots, t_k >, v_1, \ldots, v_n >: A[\Gamma, \Delta'_1, \ldots, \Delta'_n]$$

- The denotation of a Λ_Φ term, which is a morfism having an injection as its final action, and
- the denotation of the App_Φ term , which essentialy is a composition of a morfism with codomain Φ with the co-product morfism $([.,.])$ representing the alternative morfisms into the type of the App_Φ term.

Thus, if the App_Φ-term is a redex then its denotation is a diagram as the one described by Lemma I. This is shown by the following diagram:

$$([[\Gamma_1]] \times \ldots \times [[\Gamma_k]]) \times ([[\Delta'_1]] \times \ldots \times [[\Delta'_n]])$$

$$\Big\downarrow << \widehat{[[t_1]]}, \times \ldots \times, \widehat{[[t_n]]} >, I >$$

$$([[Y_1]]^{[[\Theta_1]]} \times \ldots \times [[Y_n]]^{[[\Theta_n]]}) \times ([[\Delta'_1]] \times \ldots \times [[\Delta'_n]])$$

$$\Big\downarrow < i_{R_i}, I >$$

$$[[\Phi]] \times ([[\Delta'_1]] \times \ldots \times [[\Delta'_n]])$$

$$\Big\downarrow DISTRB$$

$$([[R_1]] \times ([[\Delta'_1]] \times \ldots \times [[\Delta'_n]])) + \ldots + ([[R_n]] \times ([[\Delta'_1]] \times \ldots \times [[\Delta'_n]]))$$

$$\Big\downarrow [[[v_1]] \circ < I_{[[R_1]]}, \pi_{[[\Delta'_1]]} >, \ldots, [[v_n]] \circ < I_{[[R_n]]}, \pi_{[[\Delta'_n]]} >]$$

$$[[A]]$$

If we now apply the Lemma to the diagram above, we obtain following diagram representing the denotation of the $App_\Phi - \lambda_R$ term:

$$([[\Gamma_1]] \times \ldots \times [[\Gamma_k]]) \times ([[\Delta'_1]] \times \ldots \times [[\Delta'_n]])$$

$$\Big\downarrow << \widehat{[[t_1]]}, \times \ldots \times, \widehat{[[t_n]]} >, \pi_{[[\Delta'_i]]} >$$

$$[[R_i]] \times ([[\Delta'_i]]) \xrightarrow{[[v_i]]} A$$

where, obviously, $R_i = [[Y_1]]^{[[\Theta_1]]} \times \ldots \times [[Y_n]]^{[[\Theta_n]]}$ is the denotation of the i-th introduction rule for Φ.

It remains to show that the morfism shown above is equal to the denotation of the contractum of the redex. This is obtained by an application of Lemma 4.1 below However, it is worth noting that for the sake of clarity the lemma is atated and proved for the case when R_i is not a product of exponentials but only a exponential. The general case is easily obtained form the proof shown in the sequel, as the reader will note.

Before we go to the lemma, we need to understand what the susbstitution condition is, in the context of the categorical semantics. The reader should remember the fact observed in the previous subsection, which is a consequence of the definition of the App_Φ term in relation to all of its Λ_Φ terms. In categorical terms we can take it as stating that for each subdiagram representing the application of a rule discharged by the App_Φ application, and hence with an exponential domain, we have a morfism with codomain being this exponential which takes part in the denotation of the Λ_Φ term occuring in the redex. The later is indeed a morfism constructed with the natural isomorfism used when providing denotation for Λ_Φ terms.

Lemma 4.1 *Let $t_1(y) : F[\Delta, y : T^\dagger/H]$ and $t_2 : C[\Gamma, z : T \rightarrow H/F]$ satisfying the substitution condition required by the operational reduction rule. Their denotations are*

$$[[\Delta]] \times [[H]]^{[[T]]} \stackrel{[[t_1(y)]]}{\longrightarrow} [[F]]$$

$$[[\Gamma]] \times [[F]]^{[[H]]^{[[T]]}} \stackrel{[[t_2]]}{\longrightarrow} [[C]]$$

respectively ,and hence

$$[[\Delta]] \stackrel{[[\widehat{t_1(y)}]]}{\longrightarrow} [[F]]^{[[H]]^{[[T]]}}$$

Then

$$[[\Delta]] \times [[\Gamma]] \stackrel{<[[\widehat{t_1(y)}]], I_{[[\Gamma]]}>}{\longrightarrow} [[F]]^{[[H]]^{[[T]]}} \times [[\Gamma]]$$

$$\Big\downarrow [[t_2]]$$

$$[[C]]$$

equals to $[[t_2(z \leftarrow t_1(y)^+)]]$

Proof The proof proceeds by induction on the complexity of $T \rightarrow H/F$.

Basis step. T is empty. Detailing the terms we have:

$$t_1(y) : F[\Delta, y : H]$$

and

$$t_2(z) : C[\Gamma, z : H/F]$$

By the definition of terms we have that, if z does really occur in t_2, the diagram representing the morfism, including t_2, required by the lemma has the following structure, where we would like to point out the subdiagram representing one application of $z : H/F$ (of the form $< z, d >$) and ommiting the parallel applications of morfisms (constituents of product morfisms) that have nothing to do with the present analysis.

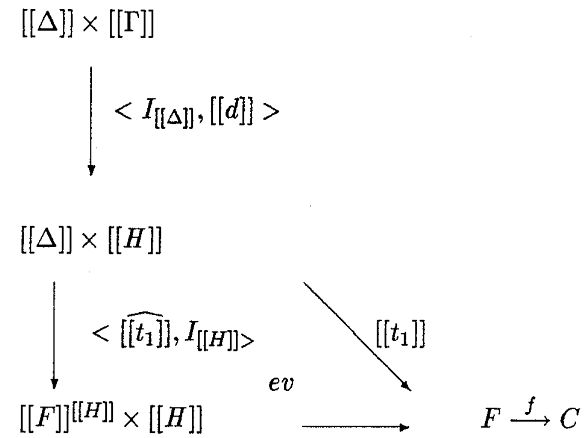
$$\begin{array}{lll}
[[\Delta]] \times [[\Gamma]] & & \\
\downarrow < \widehat{[[t_1]]}, I_{[[\Gamma]]} > & & \\
[[F]]^{[[H]]} \times [[\Gamma]] & \xrightarrow{<I_{[[F]]^{[[H]]}}, [[d]]>} & [[F]]^{[[H]]} \times [[H]] \\
 & & \downarrow ev \\
 & & F \xrightarrow{f} C
\end{array}$$

In the above diagram f is the morfism that follows the application of $z : H/F$ in the denotation of t_2.

We can see that

$$< I_{[[F]]^{[[H]]}}, [[d]] > \circ < \widehat{[[t_1]]}, I_{[[\Gamma]]} >=< \widehat{[[t_1]]}, I_{[[H]]} > \circ < I_{[[\Delta]]}, [[d]] >$$

Thus, the morfism represented by the above diagram equals the morfism represented by the following diagram :



Where we have the existence of $[[t_1]]$, as shown in the diagram, as a consequence of the definition of exponential and of $\widehat{[[t_1]]}$ itself. Finally we should observe that the diagram also represents the morfism :

$$f \circ [[t_1]] \circ < I_{[[\Delta]]}, [[d]] >$$

which equals to $[[t_2(z \leftarrow t_1(y \leftarrow d))]]$, as previously defined, just in the case there is only the application of $z : H/F$ shown in the diagram. The general case is treated in a similar way. Thus, we have proved the base step.

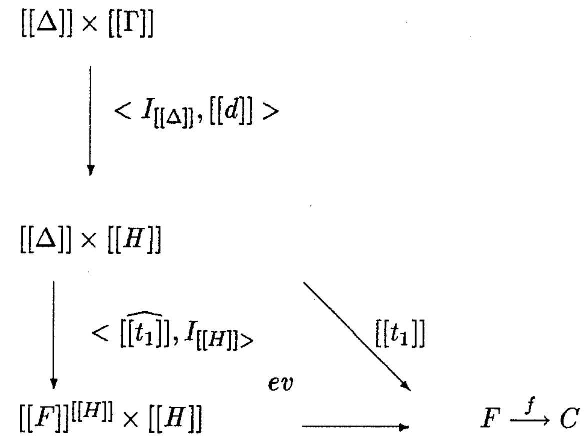
Inductive step. Firstly, we write down some terms and their respective denotations. We use Γ_H to denote the subcontext (subset) of Γ from which the term d_H, the premisse of the indicated z application, depends on :

$$[[\Gamma_H]] \times [[T]] \xrightarrow{[[d_H]]} [[H]]$$

We can see that

$$< I_{[[F]]^{[[H]]}}, [[d]] > \circ < \widehat{[[t_1]]}, I_{[[\Gamma]]} > = < \widehat{[[t_1]]}, I_{[[H]]} > \circ < I_{[[\Delta]]}, [[d]] >$$

Thus, the morfism represented by the above diagram equals the morfism represented by the following diagram :



Where we have the existence of $[[t_1]]$, as shown in the diagram, as a consequence of the definition of exponential and of $\widehat{[[t_1]]}$ itself. Finally we should observe that the diagram also represents the morfism :

$$f \circ [[t_1]] \circ < I_{[[\Delta]]}, [[d]] >$$

which equals to $[[t_2(z \leftarrow t_1(y \leftarrow d))]]$, as previously defined, just in the case there is only the application of $z : H/F$ shown in the diagram. The general case is treated in a similar way. Thus, we have proved the base step.

Inductive step. Firstly, we write down some terms and their respective denotations. We use Γ_H to denote the subcontext (subset) of Γ from which the term d_H, the premisse of the indicated z application, depends on :

$$[[\Gamma_H]] \times [[T]] \xrightarrow{[[d_H]]} [[H]]$$

$$[[\Delta]] \times [[H]]^{[[T]]} \stackrel{[[t_1(y)]]}{\longrightarrow} [[F]]$$

$$[[\Gamma]] \times [[F]]^{[[H]]^{[[T]]}} \stackrel{[[t_2(z)]]}{\longrightarrow} [[C]]$$

Thus :

$$[[\Delta]] \stackrel{[[\widehat{t_1(y)}]]}{\longrightarrow} [[F]]^{[[H]]^{[[T]]}}$$

and

$$[[\Gamma_H]] \stackrel{[[\widehat{d_H}]]}{\longrightarrow} [[H]]^{[}[T]]$$

As we did in the basis step we have the following diagram, already detailed with the applications of the rules, as required by the hypothesis of the lemma. Observe that $t_2(z) \equiv t_1([x] < z, d_H >)$ (x is, of course, of type T). For the sake of clarity we will consider $\Gamma = \Gamma_H$.

$$\begin{array}{lcl}
[[\Delta]] \times [[\Gamma]] & & \\
\quad \Big\downarrow \; < [[\widehat{t_1(y)}]], I_{[[\Gamma]]} > & & \\
[[F]]^{[[H]]^{[[T]]}} \times [[\Gamma]] & \stackrel{< I_{[[F]]^{[[H]]^{[[T]]}}}, [[\widehat{d_H}]] >}{\longrightarrow} & [[F]]^{[[H]]^{[[T]]}} \times [[H]]^{[[T]]} \\
 & & \quad \Big\downarrow \; ev \\
 & & F \stackrel{f}{\longrightarrow} C
\end{array}$$

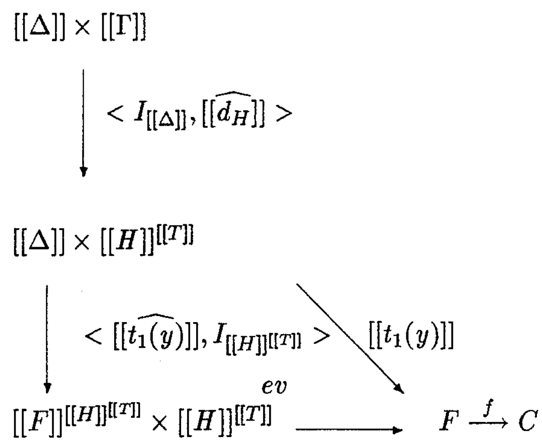
Here we also have that:

$$< I_{[[F]]^{[[H]]^{[[T]]}}}, [[\widehat{d_H}]] > \circ < [[\widehat{t_1}]], I_{[[\Gamma]]} > = < [[\widehat{t_1}]], I_{[[H]]^{[[T]]}} > \circ < I_{[[\Delta]]}, [[d_H]] >$$

and hence by the universal property of the exponential, we have that

$$ev \circ < [[\widehat{t_1(y)}]], I_{[[H]]^{[[T]]}} >= [[t_1(y)]]$$

The following diagram shows this last step :



We can note that the subdiagram concerning the composition of $[[t_1(y)]]$ with f represents, by the definition of substitution, the morfism $[[t_2(z \leftarrow t_1(y))]]$. By inductive hypothesis, we have that the diagram above, which represents :

$$[[t_2(z \leftarrow t_1(y))]] \circ < I_{[[\Delta]]}, [[\widehat{d_H}]] >$$

equals to $[[t_2(z \leftarrow t_1(y \leftarrow d_H^{+}))]]$ which is the same of :

$$[[t_2(z \leftarrow t_1^{+})]]$$

We also note that the same reasoning could be applied when we have more then one occurence of the rule z in t_2, since the applications are independent of each other.

Q.E.D.

It remains to show that the structural reduction rule also preserves denotation. This is obtained by applying lemma II to the situation described below which represents the respective denotations of the structural redex and its contractum.

We firstly consider the following denotations :

$$\begin{aligned} [[\Theta']] &= [[\Theta'_0]] \times \ldots \times [[\Theta'_n]] \\ [[\Theta]] &= [[\Theta']] \times [[\Theta_1]] \times \ldots \times [[\Theta_k]] \\ [[\Delta]] &= [[T_1]] \times \ldots \times [[T_n]] \\ [[\Gamma]] &= [[V_1]] \times \ldots \times [[V_k]] \\ [[\Psi]] &= [[T_1]] + \ldots + [[T_n]] \\ [[\Phi]] &= [[V_1]] + \ldots + [[V_k]] \end{aligned}$$

and the following redex with its respective contractum :

$$\mathbf{App}_\Phi[\Gamma] < \mathbf{App}_\Psi[\Delta] < t, t_1, \ldots, t_n >, v_1, \ldots, v_k >: A[\Theta]$$

$$\rhd$$

$$\mathbf{App}_\Psi[\Delta] < t, \mathbf{App}_\Phi[\Gamma] < t_1, v_1, \ldots, v_k >, \cdots, \mathbf{App}_\Phi[\Gamma] < t_n, v_1, \ldots, v_k >>: A[\Theta]$$

under the assumptions that :

$$t : \Psi[\Theta'_0], \qquad t_i : \Phi[\Theta'_i, T_i], \qquad \textit{and} \qquad v_i : A[\Theta_i, V_i]$$

are λ_R-terms.

Using the same general notation ($DISTRB$) for the morfisms which take part in the definition of the denotation of an **App**-term, we have the following diagram including the denotation of the terms concerned.

$$[[\Theta'_0]] \times \prod([[\Theta'_i]]) \times \prod([[\Theta_i]])$$

$$\Big\downarrow \; < [[t]], I >$$

$$[[\Psi]] \times \prod([[\Theta'_i]]) \times \prod([[\Theta_i]])$$

$$\Big\downarrow \; DISTRB$$

$$\coprod([[T_i]] \times (\prod([[\Theta'_i]]) \times \prod([[\Theta_i]]))) \overset{i}{\hookleftarrow} [[T_i]] \times (\prod([[\Theta'_i]]) \times \prod([[\Theta_i]]))$$

$$\Big\downarrow \; [f_1, \ldots, f_n] \qquad \swarrow \; < [[t_i]], I > \circ << I, \pi_{[[\Theta'_i]]} >, I >$$

$$[[\Phi]] \times \prod([[\Theta_i]])$$

$$\Big\downarrow \; DISTRB$$

$$\coprod([[V_j]] \times \prod([[\Theta_i]])) \overset{i}{\hookleftarrow} [[V_j]] \times \prod([[\Theta_i]])$$

$$\Big\downarrow \; [g_1, \ldots, g_k] \qquad \swarrow \; [[v_j]] \circ < I, \pi_{[[\Theta_i]]} >$$

$$[[A]]$$

where, obviously, $g_j = [[v_j]] \circ < I, \pi_{[[\Theta_i]]} >$ and $f_i =< [[t_i]], I > \circ << I, \pi_{[[\Theta'_i]]} >, I >$. The domains of the identity are not specified for the sake of clarity. The reader should have noted that the existence of the denotation of the **App**-term implies the existence of the denotations of t, t_i and v_i, with their respective domains and codomains as specified by the terms themselves.

Finaly, with this application of Lemma II we have proved the Theorem that shows the adequacy of the categorical semantics, allowing us to conclude that any Cartesian Closed Category with finite co-products is a model for λ_R.

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[1] Dag Prawitz. *Natural Deduction.* Stockholm, 1965.

[2] Peter Schroeder-Heister. A natural extension of natural deduction. *Journal of Symbolic Logic*, 1984.