PUC

ISSN 0103-9741

Monografias em Ciência da Computação
nº 09/93

Implication Graph and Quadratic 0-1 Optimization

Brigitte Jaumard
Celso Carneiro Ribeiro

Departamento de Informática

PONTIFÍCIA UNIVERSIDADE CATÓLICA DO RIO DE JANEIRO
RUA MARQUÊS DE SÃO VICENTE, 225 - CEP 22453-900
RIO DE JANEIRO - BRASIL

PUC RIO - DEPARTAMENTO DE INFORMÁTICA ISSN 0103-9741

Monografias em Ciência da Computação, Nº 09/93
Editor: Carlos J. P. Lucena Junho, 1993

Implication Graph and Quadratic 0-1 Optimization*

Brigitte Jaumard**
Celso Carneiro Ribeiro

* This work has been sponsored by the Secretaria de Ciência e Tecnologia da Presidência da República Federativa do Brasil.

** École Polytechnique de Montréal, Department of Applied Mathematics, Case Postale 6079, Succursale A, Montréal, Québec H3C 3A7, Canadá

In charge of publications:
Rosane Teles Lins Castilho
Assessoria de Biblioteca, Documentação e Informação
PUC Rio — Departamento de Informática
Rua Marquês de São Vicente, 225 — Gávea
22453-900 — Rio de Janeiro, RJ
Brasil
Tel. +55-21-529 9386 Telex +55-21-31048 Fax +55-21-511 5645
E-mail: rosane@inf.puc-rio.br
techrep@inf.puc-rio.br (for publications only)

Implication Graph and Quadratic 0-1 Optimization[1]

Brigitte Jaumard

École Polytechnique de Montréal
Department of Applied Mathematics
Case Postale 6079, Succursale A
Montréal, Québec H3C 3A7
Canadá
E-mail:brigitt@crt.umontreal.ca

Celso Carneiro Ribeiro

Catholic University of Rio de Janeiro
Department of Computer Science
Rua Marquês de São Vicente 225
Rio de Janeiro 22452
Brasil
E-mail: celso@inf.puc-rio.br

PUCRioInf-MCC09/93

December 1992

Abstract:

We present in this paper a new approach for the optimization of a quadratic function in 0-1 variables, based on the solution of a sequence of nested set covering problems. Each constraint of the set covering problems is associated with a circuit in the implication graph derived from the span of the quadratic function.

Keywords: Quadratic optimization, 0-1 variables, implication graph, circuits, set covering.

Resumo:

Apresenta-se neste artigo um novo enfoque para a otimização de uma função quadrática em variáveis 0-1, baseado na solução de uma seqüência de problemas de recobrimento aninhados. Cada restrição do problema de particionamento está associada a um circuito do grafo de implicação derivado do gerador da função quadrática.

Palavras-chaves: Otimização quadrática, variáveis 0-1, grafo de implicação, circuitos, problema de recobrimento.

[1]This work was done as part of the joint research project between the Department of Computer Science of the Catholic University of Rio de Janeiro and the Department of Applied Mathematics of the *École Polytechnique de Montréal*, in the framework of the cooperation agreement between the National Council for Scientific and Technological Development (Brazil) and the Natural Sciences and Engineering Research Council (Canada), research grant CNPq 910215/91-0.

1 Introduction

We consider the problem of optimizing a quadratic function in 0-1 variables:

$$\begin{cases} \text{minimize} & f(x) = \sum_{j=1}^{n} \sum_{\ell=j}^{n} q_{j\ell} x_j x_\ell \\ \text{subject to:} & x = (x_1, x_2, \ldots, x_n) \in \{0,1\}^n, \end{cases}$$

Several approaches have been proposed for this problem, such as branch-and-bound [10, 12], cutting planes [4], linearization techniques [1], and reduction to concave programming [3], among others. We propose in this paper an exact algorithm based on the solution of a sequence of nested set covering problems defined on the implication graph of the span of $f(x)$.

The paper is organized as follows. In the next section, we first introduce some notation and basic results. A decomposition technique [5] based on the construction of the implication graph is described. We show how to identify persistency properties of $f(x)$ directly on the implication graph. Following, we show in Section 3 that the minimization of $f(x)$ is equivalent to the elimination of some circuits in the implication graph and that the latter problem can be formulated as a set covering problem. Two exact algorithms associated with different implementation strategies to explore the implication graph are given. Some concluding remarks are discussed in the last section.

2 Reduction and Decomposition

We first recall in this section how an arbitrary quadratic function $f(x)$ can be transformed into a posiform $\phi(x,\overline{x})$. Following, we associate with $\phi(x,\overline{x})$ an implication graph. This leads to a decomposition scheme of the original problem into smaller ones associated with the strongly connected components of the implication graph. We conclude the section by proposing a new procedure for variable fixation, in which the implication graph is used to detect persistent variables.

2.1 Properties and Definitions

We denote by $X = \{x_1, \ldots, x_n\}$ the set of binary variables and by $\overline{X} = \{\overline{x}_1, \ldots, \overline{x}_n\}$ the set of complemented variables (i.e., $\overline{x}_j = 1 - x_j$). A literal is any element of $X \cup \overline{X}$. Every quadratic function $f(x), x \in \{0,1\}^n$, has a unique polynomial representation in the variables $x_1, \ldots, x_n$, called its *canonical form*. However, $f(x)$ can always be rewritten by complementing some of the variables as

$$f(x) = \phi(x,\overline{x}) + C(\phi),$$

with

$$\phi(x,\overline{x}) = \sum_{i=1}^{p} q_i T_i(x,\overline{x})$$

and

$$T_i(x,\overline{x}) = \prod_{j \in N_i} x_j^{\alpha_{j,i}},$$

where, for any $i \in \{1,\dots,p\}$, q_i is a positive real number, N_i is the index subset of the two literals in the i^{th} monomial $T_i(x,\overline{x})$, and $C(\phi)$ is a constant which depends on the construction of $\phi(x,\overline{x})$. All the monomials of $\phi(x,\overline{x})$ are supposed to be distinct. For any $j \in \{1,\dots,n\}$ and $i \in \{1,\dots,p\}$, $x_j^{\alpha_{j,i}} = x_j$ if $\alpha_{j,i} = 1$ and $x_j^{\alpha_{j,i}} = \bar{x}_j = 1 - x_j$ if $\alpha_{j,i} = 0$. Let us denote $\bar{\alpha}_{j,i} = 1 - \alpha_{j,i}$. Such a function $\phi(x,\overline{x})$ is called a *posiform* associated with the function $f(x)$ and is not unique in general. The boolean function

$$\hat{\phi}(x,\overline{x}) = T_1(x,\overline{x}) \vee T_2(x,\overline{x}) \vee \dots \vee T_p(x,\overline{x})$$

is called the *boolean span* of $f(x)$ (the monomials are here considered as boolean-valued, with the convention *false* (0) and *true* (1)). We recall that (i) $C(\phi)$ is a lower bound of $f(x)$, and (ii) the minimum f^* of $f(x)$ is equal to $C(\phi)$ if and only if the quadratic boolean equation $\hat{\phi}(x,\overline{x}) = 0$ is consistent [11]. Consistency of a quadratic boolean equation can be checked in $O(p)$ time (see e.g. [2]).

Given a set $\mathcal{T} = \{T_1(x,\overline{x}),\dots,T_n(x,\overline{x})\}$ of products of literals (clauses) defined on $X \cup \overline{X}$ and a weighting function $q : \mathcal{T} \rightarrow \mathcal{Z}$ which associates with each clause $T_i \in \mathcal{T}$ a weight q_i, the *(literal) weighted maximum satisfiability problem* [8] consists in finding the smallest value of h such that there exists $x \in \{0,1\}^n$ satisfying $\sum_{i \in I} q_i \leq h$, where $I = \{i \in \{1,\dots,n\} \mid T_i(x,\overline{x}) = 1\}$ (i.e., we minimize the sum of the weights of clauses which are satisfied). Then:

Theorem 1: (Simeone [11])
Given a quadratic 0-1 function $f(x) = \phi(x,\overline{x}) + C(\phi)$ and its boolean span $\hat{\phi}(x,\overline{x})$, let ϕ^* be the optimal value of the associated weighted maximum satisfiability problem. Then, the minimum of $f(x)$ is $f^* = \phi^* + C(\phi)$. ∎

Hence, the determination of the minimum of $f(x)$ can be reduced to the solution of the associated weighted maximum satisfiability problem.

2.2 Implication Graph and Decomposition

We consider the implication graph $G = (X \cup \overline{X}, E)$ associated with the boolean span $\hat{\phi}(x,\overline{x})$ of the quadratic 0-1 function $f(x)$, introduced by Aspvall, Plass and Tarjan [2]. The set of nodes is $X \cup \overline{X}$ and the set of arcs is such that, with each monomial $T_i(x,\overline{x}) = x_j^{\alpha_{j,i}} x_\ell^{\alpha_{\ell,i}}$ of $f(x)$ are associated two arcs $(x_j^{\alpha_{j,i}}, x_\ell^{\bar{\alpha}_{\ell,i}})$ and $(x_\ell^{\alpha_{\ell,i}}, x_j^{\bar{\alpha}_{j,i}})$ of E. (In order to simplify the notation and the presentation, we introduced a slight abuse of notation, in the sense that x_i and $\overline{x_i}$ denote both 0-1 variables and vertices of the implication graph. Accordingly, X and $\overline{X}$ denote both sets of literals and subsets of the vertex set of the implication graph.) The following result holds:

Theorem 2:
Let $f(x)$ be a quadratic 0-1 function written as a posiform $\phi(x,\overline{x}) + C(\phi)$ and let f^* be its minimum. Then, $f^* = C(\phi)$ if and only if the implication graph G of $f(x)$ has no strongly connected component containing both x_j and $\bar{x}_j$, for some $j \in \{1,\dots,n\}$.
Proof: From Theorem 1, $f^* = C(\phi)$ is equivalent to $\phi^* = 0$. Then, the boolean equation $\phi(x,\overline{x})$ is consistent. The result follows from Aspvall, Plass and Tarjan [2]. ∎

Let $C_1,\dots,C_q$ be the sets of vertices of each strongly connected component of G containing both a variable and its complement. These components may be determined in linear time by the depth-first search algorithm of Tarjan [13]. Let $G_\ell = (C_\ell, U_\ell)$, $\ell \in \{1,\ldots,q\}$, denote the subgraph of G induced by C_ℓ. With each G_ℓ we associate the 0-1 function

$$f_\ell(x) = \sum_{i|T_i(x,\overline{x}) \in M_\ell} q_i T_i(x,\overline{x}),$$

where M_ℓ is the set of monomials of $f(x)$ whose (two) associated arcs in the implication graph belong to U_ℓ. It is clear that for all $(i,j) \in \{1,\ldots,q\}^2$, $i \neq j$, $M_i \cap M_j = \emptyset$. The following result holds:

Theorem 3: (Billionnet and Jaumard [5])
Let $f(x)$ be a quadratic 0-1 function written as a posiform. Let $\mathcal{C} = \{C_1,\ldots,C_q\}$ be the set of strongly connected components of the associated implication graph containing both a vertex and its complement. Then, $f^* = \sum_{\ell=1}^{q} f_\ell^*$, where f_ℓ^* is the minimum of $f_\ell(x)$ over $\{0,1\}^n$. ∎

From the decomposition result of Theorem 3, we can from now on restrict ourselves to the case of the minimization of a quadratic 0-1 function such that its implication graph has a unique strongly connected component containing both a vertex and its complement.

2.3 Persistency and Variable Fixation

Given an optimization problem, a variable is persistent if it has the same value in all optimal solutions. Hammer, Hansen and Simeone [9] have shown that persistent variables for the quadratic 0-1 minimization problem can be detected through computations based on roof duality. Their algorithm was later improved by Boros and Hammer [7]. More recently, Sutter [12] proposed an extension of their approach, which in general allows the fixation of more variables, based on the extraction of an appropriate constant from $f(x)$ through the solution of the Lagrangian dual followed by the recursive application of the technique proposed by Hammer, Hansen and Simeone to the resulting posiform. More recently, Billionnet and Sutter [6] proposed a linear time algorithm for determining persistent variables from a best roof of the quadratic function. We propose in the following an alternative procedure for variable fixation, in which the implication graph is used to detect persistent variables.

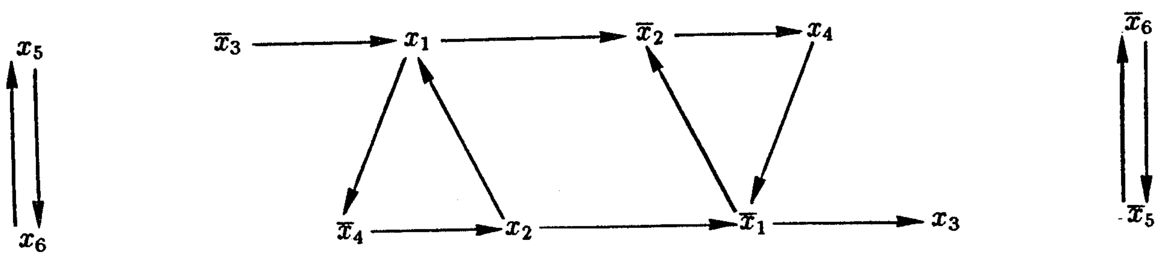
We consider the set $\mathcal{C}'$ formed by all strongly connected components of the implication graph G which does not contain simultaneously any vertex $x_i \in X$ and its complement $\overline{x}_i$. From the construction of the implication graph, if some literals are in one of such strongly connected components, their complements appear together in another strongly connected component (see [2]). Let $\mathcal{C}' = \{C'_1,\ldots,C'_r,\overline{C}'_1,\ldots,\overline{C}'_r\}$ where, for any literal $v \in X \cup \overline{X}$ such that $v \in C'_j$ (resp. $v \in \overline{C}'_j$) for some $j \in \{1,\ldots,r\}$, then $\overline{v} \in \overline{C}'_j$ (resp. $\overline{v} \in C'_j$). We define the *reduced graph* $G_R = (X_R, E_R)$, where $X_R = \{C'_1,\ldots,C'_p,\overline{C}'_1,\ldots,\overline{C}'_p\}$ and $E_R = \{(U,V) \in X_R \times X_R \mid \exists (u,v) \in E$ such that $u \in U$ and $v \in V\}$.

Theorem 4:
Let $v \in (X \cup \overline{X}) \cap C'_j$ for some $j = 1,\ldots,r$. Then:
(a) $v = 0$ is a persistent variable if there exists a path in G_R from C'_j to $\overline{C}'_j$,
(b) $v = 1$ is a persistent variable if there exists a path in G_R from $\overline{C}'_j$ to C'_j.
Proof: We first recall that there are two arcs $(x_j^{\alpha_{j,i}}, x_\ell^{\overline{\alpha}_{\ell,i}})$ and $(x_\ell^{\alpha_{\ell,i}}, x_j^{\overline{\alpha}_{j,i}})$ in the implication graph associated with each monomial $T_i(x,\overline{x}) = x_j^{\alpha_{j,i}} x_\ell^{\alpha_{\ell,i}}$. These two arcs may also be interpreted as

conditional relations $x_j^{\alpha_{j,i}} \leq x_\ell^{\delta_{\ell,i}}$ and $x_\ell^{\alpha_{\ell,i}} \leq x_j^{\delta_{j,i}}$. Then, a path p' in G from $v \in (X \cup \overline{X}) \cap C'_j$ to $\overline{v} \in (X \cup \overline{X}) \cap \overline{C'_j}$ means that $v \leq \overline{v}$, i.e., $v = 0$. From the definition of the reduced graph, to each such path p' in G there exists an unique corresponding path in G_R. This completes the proof of (a) above. The same proof stands for case (b). ∎

The computation of the transitive closure of the reduced graph G_R is a practical way to exploit the results of Theorem 4 above, resulting in an $O(|X_R|^3)$ algorithm based on the implication graph to detect a subset of the persistent variables. In the general case, Theorem 4 does not allow the identification of all persistent variables. An appropriate choice of the posiform may help to increase the number of variables fixed due to persistency properties.

As an example, consider the posiform $f(x) = x_1x_2 + x_2\overline{x}_1 + \overline{x}_1\overline{x}_3 + x_4x_1 + x_2\overline{x}_3 + \overline{x}_2\overline{x}_4 + x_5\overline{x}_6 + x_6\overline{x}_5$, whose implication graph and its strongly connected components are given in Figure 1. We have $C'_1 = \{x_5, x_6\}$, $C'_2 = \{\overline{x}_3\}$, $C'_3 = \{x_1, x_2, \overline{x}_4\}$, $\overline{C'_1} = \{\overline{x}_5, \overline{x}_6\}$, $\overline{C'_2} = \{x_3\}$, and $\overline{C'_3} = \{\overline{x}_1, \overline{x}_2, x_4\}$. Since there is a path $C'_2 \to C'_3 \to \overline{C'_3} \to \overline{C'_2}$, the following fixations can be performed: $\overline{x}_3 = x_1 = x_2 = \overline{x}_4 = 0$.



$C'_1 = \{x_5, x_6\}$ $C'_2 = \{\overline{x}_3\}$ $C'_3 = \{x_1, x_2, \overline{x}_4\}$ $\overline{C'_3} = \{\overline{x}_1, \overline{x}_2, x_4\}$ $\overline{C'_2} = \{x_3\}$ $\overline{C'_1} = \{\overline{x}_5, \overline{x}_6\}$

Figure 1: Implication graph and its strongly connected components.

3 A Solution Approach Based on the Implication Graph

We assume througout this session that $f(x)$ is written as a posiform. The following decomposition scheme based on Theorem 3 will be applied to the minimization of $f(x)$:

Step 1. Construct the implication graph associated with the span $\hat{\phi}(x, \overline{x})$ of $f(x)$.

Step 2. Let $C' = \{C'_1, \ldots, C'_r, \overline{C'_1}, \ldots, \overline{C'_r}\}$ be the set of strongly connected components of the implication graph which does not contain simultaneously a vertex and its complement. Then, for each literal $v \in (X \cup \overline{X}) \cap C'_j$ for some $j = 1, \ldots, r$, fix $v = 0$ (resp. $v = 1$) if there exists a path from C'_j to $\overline{C'_j}$ (resp. from $\overline{C'_j}$ to C'_j) in the reduced graph $G_R = (X_R, E_R)$.

Step 3. Let $C = \{C_1, \ldots, C_q\}$ be the set of strongly connected components of the implication graph containing both a vertex and its complement.

Step 4. For each $\ell = 1, \ldots, q$, obtain the minimum f_ℓ^* of $f_\ell(x)$ using algorithms GQUAD1 or GQUAD2 given below.

Step 5. Compute $f^* = \sum_{\ell=1}^{q} f_\ell^*$.

From step 4 above it follows that the minimization of $f(x)$ amounts to the solution of q smaller problems. Each of these problems corresponds to the minimization of $f_\ell(x)$ for some $\ell = 1, \ldots, q$. By construction, the implication graph associated with the span $\hat{\phi}_\ell(x, \overline{x})$ of $f_\ell(x)$ has only one strongly connected component containing both a vertex and its complement. Based on the decomposition scheme above, we may assume without loss of generality that the implication graph G has only one strongly connected component containing both a vertex and its complement.

We know from Theorem 2 that the minimum of the quadratic function $f(x)$ written as a posiform is $f^* = C(\phi)$ if and only if the boolean equation $\phi(x, \overline{x}) = 0$ is consistent, i.e., if its implication graph does not have any strongly connected component containing both a vertex and its complement. From Theorem 1 we know that if the boolean equation $\hat{\phi}(x, \overline{x}) = 0$ is not consistent, then finding the minimum of $f(x)$ is equivalent to solving a weighted maximum satisfiability problem.

Now, let $y_i = 1$ if the monomial $T_i(x, \overline{x})$ is equal to one in the solution of the weighted maximum satisfiability problem; $y_i = 0$ otherwise. For any circuit γ of the implication graph G, let $a_i^\gamma = 1$ if at least one of the arcs associated with the monomial $T_i(x, \overline{x})$ belongs to γ, $a_i^\gamma = 0$ otherwise. Hence, the minimum of $f(x)$ can be obtained as the optimal solution of the following set covering problem:

$$(SC) \begin{cases} \text{minimize} & \sum_{i=1}^{p} q_i y_i \\ \text{subject to:} & \sum_{i=1}^{p} a_i^\gamma y_i \geq 1 \quad \text{for every circuit } \gamma \text{ of } G \\ & y \in \{0,1\}^p. \end{cases}$$

Then, the following result holds:

Theorem 5:
Associate with each arc of the implication graph G the weight q_i of the corresponding monomial of the posiform $\phi(x, \overline{x})$. Then, the associated weighted maximum satisfiability problem is equivalent to the set covering problem (SC), i.e., to the determination of a minimum weighted set of arcs which should be removed from G in order that the resulting graph has no circuit containing both a vertex and its complement. ∎

The number of constraints of (SC) may be very large, i.e., almost as large as the number of circuits in G. Instead of solving (SC) directly, we define a row generation scheme and we propose in the sequel two algorithms based on the solution of a sequence of nested set covering problems, each of which incorporating new circuit elimination constraints. Both of them determine a minimum weighted set of arcs of the implication graph of $f(x)$ which eliminate all of its circuits once they are removed.

3.1 Algorithm GQUAD1

This algorithm consists in the search for new circuit elimination constraints, which are incorporated to those obtained during the previous iterations. At each iteration, a new set covering problem is solved and its optimal solution is a minimum weighted subset of arcs such that its elimination breaks all circuits already enumerated in the implication graph. If the circuit elimination constraints already generated are not sufficient to eliminate all circuits containing both a vertex and its complement, new circuit elimination constraints are identified and appended to those previously enumerated, and a new, extended set covering problem is solved. The algorithm stops when the resulting graph has no circuits containing both a vertex and its complement. Convergence is easily proved, since in the worst case all circuits of the implication graph of the boolean equation $\hat{\phi}(x,\overline{x}) = 0$ associated with $f(x)$ will be enumerated.

Step 0 (Initialization):

0.1. Let $G^0 = G = (X \cup \overline{X}, E)$ be the implication graph associated with the span $\hat{\phi}(x,\overline{x})$ of $f(x)$.

0.2. Determine a subset of circuits of G^0, each of which containing a pair $\{x_i, \overline{x_i}\}$ of vertices, for some $x_i \in X$.

0.3. Construct the corresponding set of constraints $A^1 y \geq 1$ and let (SC^1) be the initial set covering problem:

$$(SC^1)\begin{cases} \text{minimize} & \sum_{i=1}^{p} q_i y_i \\ \text{subject to:} & A^1 y \geq 1 \\ & y \in \{0,1\}^p. \end{cases}$$

0.4. Set $k \leftarrow 1$.

Step 1 (New solution):

1.1. Solve (SC^k) and let $y_i^{(k)}$ be its optimal solution. Let $I_k = \{i \in \{1,\ldots,p\} : y_i^{(k)} = 1\}$.

1.2. Let $G^k = (X \cup \overline{X}, E_k)$, where $E_k = E \setminus \{(x_j^{\alpha_{j,i}}, x_\ell^{\bar{\alpha}_{\ell,i}}),\ (x_\ell^{\alpha_{\ell,i}}, x_j^{\bar{\alpha}_{j,i}}) \in E \mid i \in I_k\}$.

1.3. If G^k has at least one strongly connected component containing both a vertex and its complement, then go to step 2. Otherwise, stop: any solution of the consistent boolean equation $\hat{\phi}^k(x,\overline{x}) = 0$, obtained from $\hat{\phi}(x,\overline{x})$ by removing all monomials $T_i(x,\overline{x})$ such that $i \in I_k$, leads to the minimum of $f(x)$.

Step 2 (Determination of new circuits):

2.1. For each strongly connected component of G^k containing a pair $\{x_i, \overline{x}_i\}$ of vertices, for some $x_i \in X$, do:

2.1.1. Determine a subset of the circuits of the strongly connected component, each of which containing both a vertex and its complement.

2.1.2. Append the corresponding constraints to the constraint matrix of the current (SC^k) set covering problem.

2.2. Let A_{k+1} be the new constraint matrix and (SC^{k+1}) the associated set covering problem:

$$(SC^k)\begin{cases} \text{minimize} & \sum_{i=1}^{p} q_i y_i \\ \text{subject to:} & A^{k+1}y \geq 1 \\ & y \in \{0,1\}^p. \end{cases}$$

2.3. Set $k \leftarrow k+1$ and return to step 1.

The finiteness of algorithm GQUAD1 follows from the fact that, in the worst case, all circuits of G will be enumerated. In this case, the last problem to be solved is the full set covering problem (SC). Correctness of the algorithm then follows from Theorem 5.

Several strategies could be used for identifying circuits in the implication graph G^k at steps 0.2 and 2.1.1. The first one consists in finding short circuits in terms of the cost of the monomials corresponding to their arcs. One way to proceed is: (i) select a vertex v appearing together with its complement in the same strongly connected component of G; (ii) find a shortest path in G from v to $\bar{v}$ and another one from $\bar{v}$ to v, with the lengths of the arcs of G corresponding to the costs of the corresponding monomials; (iii) combine these two paths to form a circuit passing through v; and (iv) repeat these steps to find (at least) one circuit passing through each vertex appearing together with its complement in the same strongly connected component of G. A second strategy would consist in finding short circuits in terms of the number of monomials corresponding to their arcs. A similar procedure could be defined by replacing (ii) above by the computation of shortest paths with respect to the number of arcs.

As an example, we take $f(x) = \phi(x,\bar{x}) = 8x_1x_3 + 6x_1\bar{x}_3 + 3\bar{x}_1x_3 + 11\bar{x}_1\bar{x}_3 + 2x_4x_5 + 11x_4\bar{x}_5 + 3\bar{x}_4x_5 + 16\bar{x}_4\bar{x}_5 + 2x_1x_2 + 0.5\bar{x}_2x_3 + \bar{x}_3\bar{x}_5$, whose associated implication graph $G^0 = G$ is given in Figure 2.

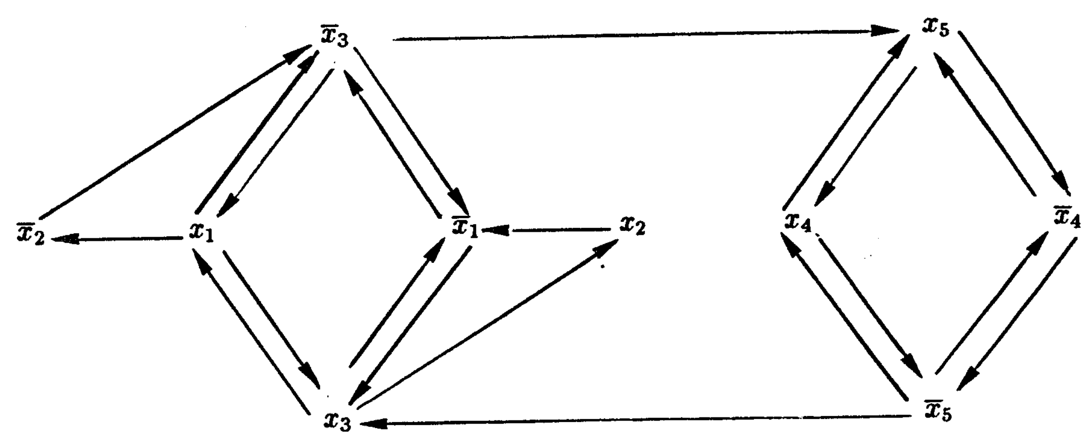


Figure 2: Implication graph $G^0 = G$.

Two circuits are identified at step 0 by the first strategy:

- Circuit γ^1, containing x_1 and $\bar{x}_1$: path $x_1 \to \bar{x}_2 \to \bar{x}_3 \to \bar{x}_1$ (length 8.5) followed by path $\bar{x}_1 \to \bar{x}_3 \to x_5 \to x_4 \to \bar{x}_5 \to x_3 \to x_1$ (length 13), also containing literals x_3, $\bar{x}_3$, x_5, and $\bar{x}_5$; and

- Circuit γ^2, containing x_2 and $\overline{x}_2$: path $x_2 \to \overline{x}_1 \to \overline{x}_3 \to x_5 \to x_4 \to \overline{x}_5 \to x_3 \to x_1 \to \overline{x}_2$ (length 17) followed by path $\overline{x}_2 \to \overline{x}_3 \to x_5 \to x_4 \to \overline{x}_5 \to x_3 \to x_2$ (length 8), also containing literals x_4 and $\overline{x}_4$.

The binary variables $y_i \in \{0,1\}$, $i = 1,\ldots,11$, are indexed in the same order in which the corresponding monomials of $f(x)$ appear. We obtain the first set covering problem:

$$(SC^1)\begin{cases} \text{minimize } f(x) = \\ 8y_1 + 6y_2 + 3y_3 + 11y_4 + 2y_5 + 11y_6 + 3y_7 + 16y_8 + 2y_9 + 0.5y_{10} + y_{11} \\ \text{subject to:} \\ \quad y_2 + y_3 + y_5 + y_7 + y_9 + y_{10} + y_{11} \geq 1 \\ \quad y_3 + y_5 + y_7 + y_9 + y_{10} + y_{11} \geq 1 \\ y_i \in \{0,1\},\ i = 1,\ldots,11, \end{cases}$$

whose optimal solution is $y_{10}^{(1)} = 1$; $y_i^{(1)} = 0$, $i \neq 10$. The boolean equation $\hat{\phi}^1(x,\overline{x}) = 0$ obtained from $\hat{\phi}(x,\overline{x})$ by removing the monomial $T_{10}(x,\overline{x})$ is not consistent. The new implication graph G^1 contains all arcs of G^0 except $(\overline{x}_2, \overline{x}_3)$ and (x_3, x_2), which are those associated with the monomial $T_{10}(x,\overline{x})$; see Figure 3. Two new circuits containing both a vertex and its complement are now identified in G^1 at step 2.1.1:

- Circuit γ^3, containing x_1 and $\overline{x}_1$: path $x_1 \to \overline{x}_3 \to \overline{x}_1$ (length 14) followed by path $\overline{x}_1 \to \overline{x}_3 \to x_5 \to x_4 \to \overline{x}_5 \to x_3 \to x_1$ (length 18), also containing variables x_3, $\overline{x}_3$, x_5, and $\overline{x}_5$.
- Circuit γ^4, containg x_4 and $\overline{x}_4$: path $x_4 \to x_5 \to \overline{x}_4$ (length 13) followed by path $\overline{x}_4 \to \overline{x}_5 \to x_4$ (length 19), also containing variables x_5 and $\overline{x}_5$.

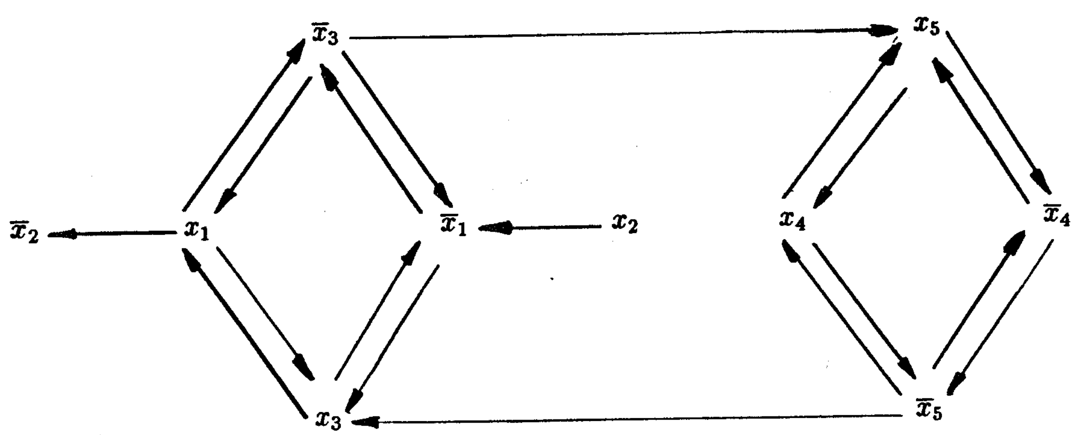


Figure 3: Implication graph G^1.

Two new constraints associated to circuits γ^3 and γ^4 are appended to (SC^1), defining the new set covering problem (SC^2):

$$
(SC^2)\begin{cases}
\text{minimize } f(x) = \\
8y_1 + 6y_2 + 3y_3 + 11y_4 + 2y_5 + 11y_6 + 3y_7 + 16y_8 + 2y_9 + 0.5y_{10} + y_{11} \\
\text{subject to:} \\
\quad y_2 + y_3 + y_5 + y_7 + y_9 + y_{10} + y_{11} \geq 1 \\
\quad y_3 + y_5 + y_7 + y_9 + y_{10} + y_{11} \geq 1 \\
\quad y_1 + y_2 + y_3 + y_5 + y_7 + y_{11} \geq 1 \\
\quad y_5 + y_6 + y_7 + y_8 \geq 1 \\
y_i \in \{0,1\},\ i = 1,\ldots,11,
\end{cases}
$$

The optimal solution of (SC^2) is $y_5^{(2)} = 1$; $y_i^{(2)} = 0$, $i \neq 5$. The new implication graph G^2 contains all arcs of G^0 except $(x_4, \overline{x}_5)$ and $(x_5, \overline{x}_4)$, which are those associated with the monomial $T_5(x, \overline{x})$; see Figure 4. The following circuit containing both a vertex and its complement is identified in the implication graph G^2 at step 2.1.1:

- Circuit γ^5, containing x_1 and $\overline{x}_1$: path $x_1 \to \overline{x}_2 \to \overline{x}_3 \to \overline{x}_1$ (length 8.5) followed by path $\overline{x}_1 \to x_3 \to x_1$ (length 14), also containing variables x_3 and $\overline{x}_3$.

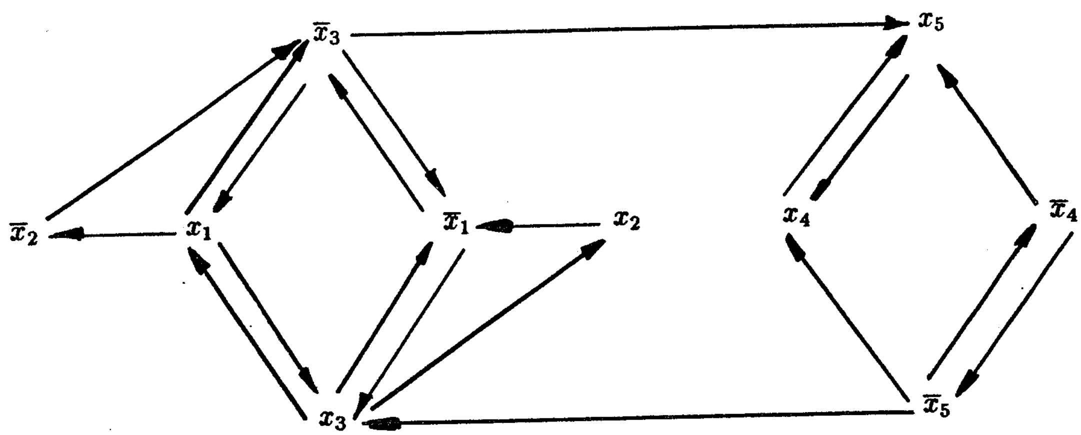


Figure 4: Implication graph G^2.

A new constraint $y_2 + y_3 + y_4 + y_9 + y_{10} \geq 1$ (associated with γ^5) is appended to (SC^2), generating the new set covering problem (SC^3). The optimal solution of (SC^3) is $y_5^{(3)} = y_{10}^{(3)} = 1$; $y_i^{(3)} = 0$, $i = 1, 2, 3, 4, 6, 7, 8, 9, 11$. A new circuit containing both a vertex and its complement is found in the implication graph G^3; see Figure 5:

- Circuit γ^6, containing x_1 and $\overline{x}_1$: path $x_1 \to x_3 \to \overline{x}_1$ (length 14) followed by path $\overline{x}_1 \to \overline{x}_3 \to x_1$ (length 14), also containing x_3 and $\overline{x}_3$.

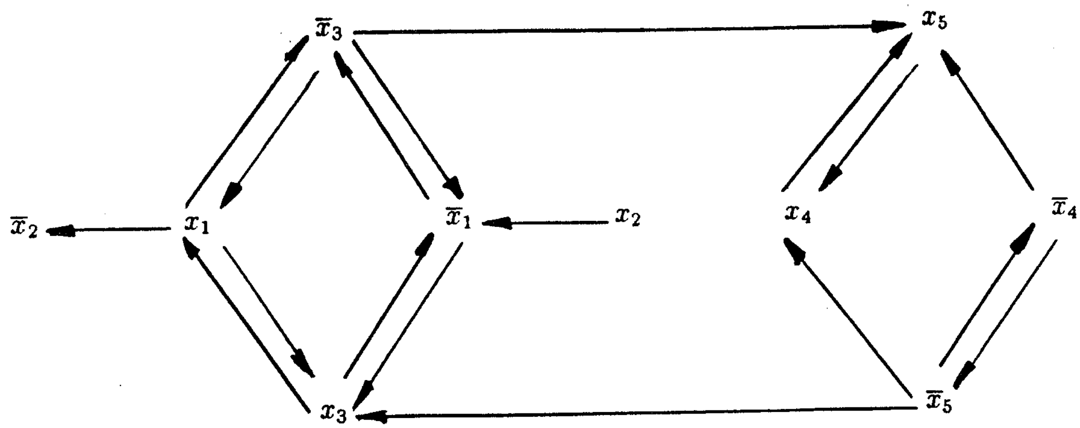


Figure 5: Implication graph G^3.

The new constraint $y_1+y_2+y_3+y_4 \geq 1$ (associated with γ^6) is appended to (SC^3), generating the new set covering problem (SC^4):

$$
(SC^4)\left\{\begin{array}{lllllllllllll}
\multicolumn{13}{l}{\text{minimize } f(x) =} \\
8y_1 & +6y_2 & +3y_3 & +11y_4 & +2y_5 & +11y_6 & +3y_7 & +16y_8 & +2y_9 & +0.5y_{10} & +y_{11} & & \\
\multicolumn{13}{l}{\text{subject to:}} \\
 & y_2 & +y_3 & & +y_5 & & +y_7 & & +y_9 & +y_{10} & +y_{11} & \geq 1 \\
 & & y_3 & & +y_5 & & +y_7 & & +y_9 & +y_{10} & +y_{11} & \geq 1 \\
y_1 & +y_2 & +y_3 & & +y_5 & & +y_7 & & & & +y_{11} & \geq 1 \\
 & & & & y_5 & +y_6 & +y_7 & +y_8 & & & & \geq 1 \\
 & y_2 & +y_3 & +y_4 & & & & & +y_9 & +y_{10} & & \geq 1 \\
y_1 & +y_2 & +y_3 & +y_4 & & & & & & & & \geq 1 \\
\multicolumn{13}{l}{y_i \in \{0,1\},\ i = 1,\ldots,11.}
\end{array}\right.
$$

The optimal solution of (SC^4) is $y_3^{(4)} = y_5^{(4)} = 1$; $y_i^{(4)} = 0$, $i = 1,2,4,6,7,8,9,10,11$. The algorithm stops, since the new implication graph G^4 does not have any circuit containing both a vertex and its complement; see Figure 6. Using Theorem 4, we then have $x_1^* = 0$, $x_2^* = x_3^* = x_4^* = x_5^* = x_6^* = 1$.

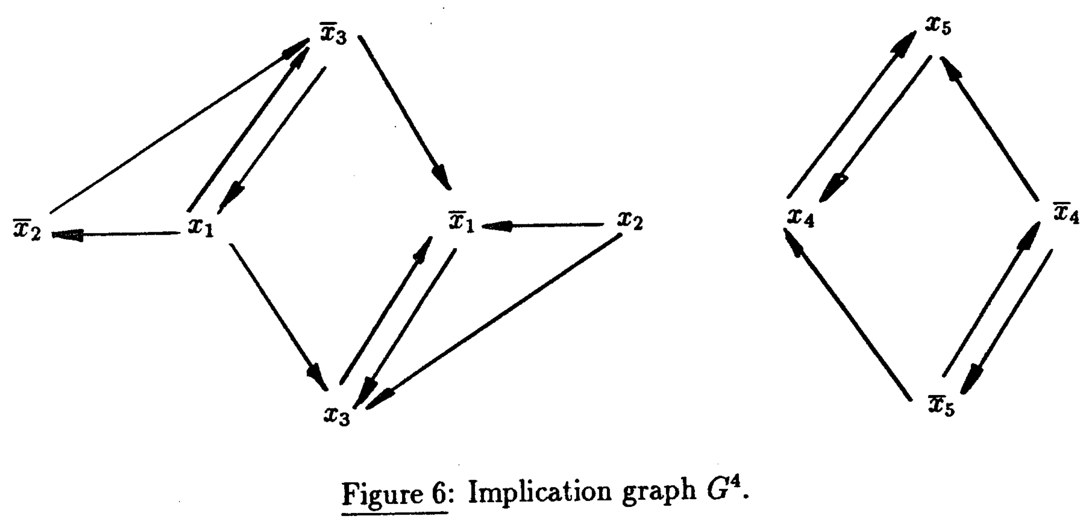


Figure 6: Implication graph G^4.

3.2 Algorithm GQUAD2

The algorithm described here follows a backtrack strategy. At a given level of the search tree, circuit elimination contraints associated with the current implication graph are generated and a set covering problem is solved. If the solution of the set covering problem breaks all circuits containing both a vertex and its complement, then the current subproblem has been solved and the circuit elimination constraints associated with it are appended to those of the subproblem in the immediately higher level of the search tree. Otherwise, each strongly connected component (containing both a variable and its complement) of the resulting graph generates a smaller subproblem at the immediately lower level. A depth-first-search strategy is used to explore the search tree and, as long as the algorithm does not backtrack, the set covering problems associated with the new subproblems gets smaller (in terms of the number of variables and constraints).

Step 0 (Initialization):

0.1. Let $G^0 = G = (X \cup \overline{X}, E)$ be the implication graph associated with the span $\hat{\phi}(x, \overline{x})$ of $f(x)$. Set $components(0) \leftarrow 1$ and $scan(0) \leftarrow 1$.

0.2. Set $k \leftarrow 0$.

Step 1 (Branching):

1.1. Let C_k be the node set of a non-scanned strongly connected component of G^k containing both a vertex and its complement.

1.2. Let $G' = (X' \cup \overline{X}', E')$ be the subgraph of G^k generated by C_k, with $X' \subseteq X$ and $\overline{X}' \subseteq \overline{X}$, and set $scan(k) \leftarrow scan(k) - 1$.

1.3. Set $k \leftarrow k + 1$.

1.4. Determine a subset of the circuits of G', each of wich containing a pair $\{x_i, \overline{x_i}\}$ of vertices, for some $x_i \in X'$ (the same strategies described in Section 3.1 for Algorithm GQUAD1 may be used). Let A^k be the corresponding constraint matrix.

Step 2 (Solving the subproblem):

2.1. Let (SC^k) be the current set covering problem associated with level k:

$$(SC^k)\begin{cases} \text{minimize} & \sum_{i=1}^{p} q_i y_i \\ \text{subject to:} & A^k y \geq 1 \\ & y \in \{0,1\}^p. \end{cases}$$

2.2. Solve (SC^k) and let $y_i^{(k)}$ be its optimal solution. Let $I_k = \{i \in \{1, \ldots, p\} : y_i^{(k)} = 1\}$.

2.3. Let $G^k = (X \cup \overline{X}, E_k)$, where $E_k = E' \setminus \{(x_j^{\alpha_{j,i}}, x_\ell^{\bar{\alpha}_{\ell,i}}),\ (x_\ell^{\alpha_{\ell,i}}, x_j^{\bar{\alpha}_{j,i}}) \in E' \mid i \in I_k\}$.

2.4. Let $components(k)$ be the number of strongly connected components of G^k containing both a variable x_i and its complement $\overline{x}_i$ for some $x_i \in X'$ and set $scan(k) \leftarrow components(k)$.

2.5. If $components(k) \neq 0$, then go back to Step 1.

Step 3 (Backtracking and termination):

3.1. If $k = 1$ then stop.

3.2. Append the constraint matrix A^k to A^{k-1}.

3.3. Set $k \leftarrow k - 1$.

3.4. If $scan(k) \neq 0$, then go back to Step 1.

3.5. Otherwise, go back to Step 2.

We consider the same example given for the first algorithm. The first set covering problem solved in the first level of the search tree (step 2.2) is the same problem (SC^1) solved by algorithm GQUAD1. Its optimal solution is $y_{10}^{(1)} = 1$; $y_i^{(1)} = 0$, $i \neq 10$. The resulting graph (step 2.3) is the implication graph G^1 already shown in Figure 3, which has two strongly connected components containing both a vertex and its complement: $\{x_1, \overline{x}_1, x_3, \overline{x}_3\}$ and $\{x_4, \overline{x}_4, x_5, \overline{x}_5\}$.

The first set covering problem (SC^2) solved at the second level of the search tree is that associated with the circuit $x_1 \rightarrow \overline{x}_3 \rightarrow \overline{x}_1 \rightarrow x_3$ of the first strongly connected component of G^1 containing both a vertex and its complement, whose optimal solution is $y_3^{(2)} = 1$; $y_i^{(2)} = 0$, $i \neq 3$.

$$
(SC^2)\begin{cases}
\text{minimize } f(x) = \\
8y_1 \quad +6y_2 \quad +3y_3 \quad +11y_4 \\
\text{subject to:} \\
y_1 \quad +y_2 \quad +y_3 \quad +y_4 \geq 1 \\
y_i \in \{0,1\},\ i = 1,\ldots,4.
\end{cases}
$$

Since the graph G^2 obtained at step 2.3 does not have any strongly connected component containing both a vertex and its complement, the algorithm backtracks appending constraint $y_1 + y_2 + y_3 + y_4 \geq 1$ to those already appearing in the first problem (SC^1).

A new set covering problem (SC^2) is solved at the second level of the search tree, associated with the circuit $x_4 \rightarrow \overline{x}_5 \rightarrow \overline{x}_4 \rightarrow x_5$ of the second strongly connected component of G^1 containing both a vertex and its complement. Its optimal solution is $y_5^{(2)} = 1$; $y_i^{(2)} = 0$, $i \neq 5$.

$$
(SC^2)\begin{cases}
\text{minimize } f(x) = \\
2y_5 \quad +11y_6 \quad +3y_7 \quad +16y_8 \\
\text{subject to:} \\
y_5 \quad +y_6 \quad +y_7 \quad +y_8 \geq 1 \\
y_i \in \{0,1\},\ i = 1,\ldots,4,
\end{cases}
$$

Once again, the graph G^2 obtained at step 2.3 does not have any strongly connected component containing both a vertex and its complement. Then, the algorithm backtracks appending constraint $y_5 + y_6 + y_7 + y_8 \geq 1$ to those already appearing in the first problem (SC^1). At this point, another set covering problem (SC^1) has to be solved (step 2.2) at the first level of the search tree. The constraints of this problem are the same of the first problem (SC^1), plus those originated from the two problems (SC^2) solved at the second level of the search tree:

$$
(SC^1)\left\{\begin{array}{llllllllllll}
\multicolumn{12}{l}{\text{minimize } f(x) =} \\
8y_1 & +6y_2 & +3y_3 & +11y_4 & +2y_5 & +11y_6 & +3y_7 & +16y_8 & +2y_9 & +0.5y_{10} & +y_{11} & \\
\multicolumn{12}{l}{\text{subject to:}} \\
 & y_2 & +y_3 & & +y_5 & & +y_7 & & +y_9 & +y_{10} & +y_{11} & \geq 1 \\
 & & y_3 & & +y_5 & & +y_7 & & +y_9 & +y_{10} & +y_{11} & \geq 1 \\
y_1 & +y_2 & +y_3 & +y_4 & & & & & & & & \geq 1 \\
 & & & & y_5 & +y_6 & +y_7 & +y_8 & & & & \geq 1 \\
\multicolumn{12}{l}{y_i \in \{0,1\},\ i = 1,\ldots,11,}
\end{array}\right.
$$

The optimal solution of the new set covering problem (SC^1) is $y_3^{(2)} = y_5^{(2)} = 1$; $y_i^{(2)} = 0$, $i = 1,2,4,6,7,8,9,10,11$. The algorithm stops, since the new resulting graph G^2 does not have any circuit containing both a vertex and its complement. Again, the optimal solution is $x_1^* = 0$, $x_2^* = x_3^* = x_4^* = x_5^* = x_6^* = 1$.

It can be noticed that algorithm GQUAD2 is based on a more agressive strategy, successively breaking circuits of graphs which get smaller until the algorithm does not backtrack. When the algorithm backtracks, all constraints enumerated in the current node of the search tree are appended to those already appearing in the set covering problem solved at the previous level.

4 Concluding Remarks

The algorithms presented in the previous section may be viewed as row generation schemes, i.e., a cutting plane method. The first algorithm is a pure cutting plane method, while the second one is a combination of branch-and-cut and backtrack strategies. The number of variables appearing in the set covering problems is at most equal to the number of terms of the quadratic 0-1 function to be optimized. In turn, the latter is equal to the number of variables appearing in the linearization techniques used in other cutting plane methods (see e.g. Barahona, Jünger and Reinelt [4]). Further research should be pursued, with the goal of investigating the relationship between the underlying polytope and the cuts used by algorithms GQUAD1 and GQUAD2, i.e., the circuits which contain both a vertex and its complement.

It should be noticed that the set covering problems appearing at each iteration of both algorithms do not have to be solved exactly every time. Indeed, they should be approximately solved by some efficient heuristic procedure. Only if the approximate solution leads to an implication graph without circuits containing both a vertex and its complement, then in that case an exact algorithm should be used to check the optimality of the heuristic solution.

We also remark that, in practice, many constraints will be redundant. Hence, in order to avoid solving large redundant set covering problems, it should be useful to consider constraint elimination rules. Redundancy may also be reduced by the use of other strategies for finding circuits in the implication graph containing both a vertex and its complement.

References

[1] W. P. ADAMS and H. D. SHERALI, "A Tight Linearization and an Algorithm for Zero-One Quadratic Programming Problems", *Management Science* 10 (1986), 1274–1290.

[2] A. ASPVALL, M. F. PLASS and R. E. TARJAN, "A Linear Time Algorithm for Testing the Truth of Certain Quantified Boolean Formulas", *Information Processing Letters* 8 (1979), 121–123 (Erratum, *Information Processing Letters* 14 (1982), 195).

[3] A. BAGCHI and B. KALANTARI, "An Algorithm for Quadratic Zero-One Programs", *Naval Research Logistics Research Quarterly* 37 (1990), 527–538.

[4] F. BARAHONA, M. JÜNGER and G. REINELT, "Experiments in Quadratic 0-1 Programming", *Mathematical Programming* 44 (1989), 127–137.

[5] A. BILLIONNET and B. JAUMARD, "A Decomposition Method for Minimizing Quadratic Pseudo-Boolean Functions", *Operations Research Letters* 8 (1989), 161–163.

[6] A. BILLIONNET and A. SUTTER, "Persistency in Quadratic 0-1 Optimization", *Mathematical Programming* 54 (1992), 115–119.

[7] E. BOROS and P. L. HAMMER, "A Max-Flow Approach to Improved Roof Duality in Quadratic 0-1 Minimization", RUTCOR Research Report #15-89, Rutgers University, 1989.

[8] P. M. CAMERINI and F. MAFFIOLI, "Weighted Satisfiability Problems and some Implications", in *Lecture Notes in Control and Information Sciences, Proceedings of the 9th IFIP Conference on Optimization Techniques (Part 2)*, Warsaw, September 1979, Springer Verlag, Berlin, 170–175.

[9] P. L. HAMMER, P. HANSEN and B. SIMEONE, "Roof Duality, Complementation and Persistency in Quadratic 0-1 Optimization", *Mathematical Programming* 28 (1984), 121–155.

[10] P. M. PARDALOS and G. P. Rodgers, "Computational Aspects of a Branch-and-Bound Algorithm for Quadratic Zero-One Programming", *Computing* 45 (1990), 131–144.

[11] B. SIMEONE, *Quadratic 0-1 Programming, Boolean Functions and Graphs*, Doctoral Dissertation, University of Waterloo, 1979.

[12] A. SUTTER, *Programmation non-linéaire en variables 0-1 et application à des problèmes de placement de tâches dans les systèmes distribués*, D.Sc. Dissertation, Conservatoire National des Arts et Métiers, Paris, 1989.

[13] R. E. TARJAN, "Depth-First Search and Linear Graph Algorithms", *SIAM Journal on Computing* 1 (1972), 146–160.