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On Some General Logical Properties Related to Interpolation and Modularity

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Abstract

We examine connections among some logical properties, such as interpolation and cut, underlying the modularity property for interpretations and conservative extensions, which is crucial in the logical approach to formal program and specification development. In this context, a specification is a theory presentation,, implementations and parameter instantiations being defined in terms of interpretations and conservative extensions. To handle these concepts, one must be able to complete (pushout) rectangles, a construction preserving conservativeness by the Modularisation Theorem. A closer analysis of its proof reveals that it relies on some basic logical properties, related to versions of interpolation and the deduction theorem. We examine these properties and some connections among them, aiming at clarifying their roles, and illustrating the use of internalisation techniques. This provides conditions and alternative formulations for modularity and these logical properties of the consequence relation. Interpolation, for instance, can be formulated as conservativeness preservation. We emphasise a connective-independent approach.

Key words: Formal specifications, Logical theories, Axiomatic presentations, Interpolation, Modularity, Cut, Conservative extensions, Interpretations, Internalisation, Program development, Logical approach

Resumo

São examinadas algumas conexões entre propriedades lógicas, como interpolação e corte, subjacentes à propriedade de modularidade para interpretações e extensões, a qual é crucial no enfoque lógico para desenvolvimento formal de programas e especificações. Neste contexto, uma especificação é uma apresentação de uma teoria, implementações e instantiações de parâmetros sendo definidos em termos de interpretações e extensões conservativas. Para tratar estes conceitos,precisa-se completar retângulos (de somas amalgamadas), construção que preserva conservatividade pelo Teorema da Modularização. Uma análise mais detalhada de sua demonstração revela que ela se baseia em algumas propriedades lógicas, relacionadas a versões de interpolação e do teorema da dedução. Estas propriedades, e suas interconexões, são examinadas para clarificar seus papéis e ilustrar o uso de técnicas de internalização. Isto fornece condições e reformulações para modularidade e para essas propriedades lógicas da relação de conseqüêncian. Interpolação, por exemplo, pode ser formulada em termos de preservação de conservatividade. Enfatiza-se um enfoque independente de conectivos.

Palavras chave: Especificações formais, Teorias lógicas, Apresentações axiomáticas, Interpolação, Modularidade, Corte, Extensões conservativas, Interpretações, Internalização, Desenvolvimento de programas, Enfoque lógico.

CONTENTS

1. INTRODUCTION

We examine some connections among modularity properties, concerning preservation of conservativeness, and interpolation-like properties of the consequence relation. The motivations for this investigation stem from two main sources in logic and in software development.

On the one hand, modularity of extensions is connected to some important logical results, such as Robinson's Joint Consistency, Craig's Interpolation and Beth's Definability theorems, known to be related to each other. It is thus of interest to examine more closely these connections.

On the other hand, the generalisation of modularity of extensions to interpretations, is a crucial property in the logical approach to formal program and specification development. In this context, a specification is a theory presentation, and implementations as well as instantiations of parameters are defined in terms of interpretations and conservative extensions. In order to handle these concepts, one must be able to complete a rectangle by means of a pushout construction [Goldblatt '79; Ehrich '82]. The Modularisation Theorem asserts that this construction preserves conservativeness; its proof involves Craig's Interpolation Lemma and internalisation techniques.

A closer analysis of this proof reveals that it actually relies on some basic logical properties, related to versions of interpolation and the deduction theorem. A more detailed examination of these connections is important in establishing conditions for modularity.

Software development can rely on diverse specification formalisms, based on equations, Horn clauses, first-order sentences, etc. So, an analysis not tied down to a particular formalism would be of interest. We are thus led to a connective-independent approach, somewhat akin to Tarski's classical work on the consequence relation [Tarski '30]. We do not, however, keep the language fixed, since we examine conservativeness and interpolation.

We examine these properties and some connections among them, aiming at a clarification of their roles. This provides conditions and alternative formulations for modularity and these logical properties of the consequence relation. For instance, both interpolation and the Deduction Theorem can be formulated in terms of conservativeness preservation.

This paper consists of three parts. In the first one (section 2) we briefly review the logical approach to formal specifications and examine the role played by modularity in formal software development. The second part focus on variations of modularity and interpolation in Classical First-Order Logic: in section 3 we examine joint consistency and modularity of extensions as well as some internalisation techniques related to modularity of interpretations; section 4 concentrates on properties akin to interpolation and cut. The third part (section 5) then investigates some interconnection among these properties in a connective-free context by resorting to relativised consequence relations. The final section briefly comments on some extensions and future developments.

We employ the usual terminology and notation for logical concepts [Shoenfield '67; Enderton '72; Chang + Keisler '73]. We use the notations $\Sigma \vdash \sigma$, or $\sigma \in Cn(\Gamma)$, to state that sentence σ is a *consequence* of the set Σ of sentences.

2. MODULARITY IN SOFTWARE DEVELOPMENT

We shall now briefly examine some motivations for interpretation modularity stemming from software development. Interpretation modularity guarantees the preservation of conservativeness under an "orthogonal" interpretation, which is of importance in formal stepwise development.

Interpretation Modularity is a crucial property in the formal development of specifications and programs in a stepwise manner [Turski + Maibaum '87; Veloso '87]. Its importance arises mainly from its role in two situations, namely in composing implementations and in instantiating parameterised specifications.

2.1 THE LOGICAL APPROACH TO FORMAL SPECIFICATIONS

The logical approach to formal specifications employs the formalism of first-order logic. The motivations for this approach [Maibaum + Veloso '81; Maibaum + Veloso + Sadler '91; Maibaum + Veloso '94] come mainly from two related sources. On the one hand, logical axioms employ language and concepts akin to program verification [Manna '74]; on the other hand, the logical formalism accommodates 'liberal' specifications, which provide flexibility for specifying what one wishes without forcing over-specification [Maibaum + Veloso '81; Turski + Maibaum '87]. In addition, this logical approach has been instrumental in extending some of these ideas to problem solving [Veloso + Veloso '81; Veloso '88, '91] and to formal algorithm design [Smith + Lowry '90].

In the logical approach, a *specification* is a *theory presentation*, i. e. a pair $\langle L,\Sigma\rangle$, consisting of a language L, and a set Σ of sentences of L (its *axioms*). The content of a specification $S=\langle L,\Sigma\rangle$ is the theory generated by it: the set $Cn(R):=\{\sigma \in Sent(L)/\Sigma \vdash \sigma\}$ of its *theorems*.

Basic logical concepts of importance for specification and program development are (conservative) extensions and interpretations. An extension adds symbols and theorems; it is conservative when it adds no new consequences in the smaller language. An interpretation is a language translation that preserves consequences, in that theorems are translated to theorems.

We shall now briefly review the logical concepts of implementation and parameterisation, as well as the role played by Interpretation Modularity in this context.

2.2 IMPLEMENTATIONS OF LOGICAL SPECIFICATIONS

Let us start by considering implementations. One has a specification M which one wishes to implement on a specification P. For this purpose, one has to provide on top of P some support for the abstract concepts of M. One account of what is involved, in terms of 'liberal' specifications presented by axioms, is as follows.

One extends the concrete specification P by adding symbols to correspond to the abstract ones in M, perhaps together with some auxiliary symbols. Since one does not wish to disturb the given concrete specification P, this extension Q should not impose any new constraints on P. This can be formulated by requiring the extension $Q \supseteq P$ to be conservative.

One also wishes to correlate the abstract symbols in M to corresponding ones in Q. But, the properties of M are important, for instance, in guaranteeing the correctness of an abstract program supported by M. Thus, in translating from M to Q, one wishes to preserve the properties of M as given by its axioms. This can be formulated by requiring the translation $i{:}M \rightarrow Q$ to be an interpretation.

We thus arrive at the concept of an *implementation* of M on P as an interpretation i of M into a conservative extension Q (usually called a *mediating specification*) of P [Maibaum + Veloso '81; Maibaum + Veloso + Sadler '91]. Such 'implementation triangle' is sometimes called a 'canonical step' [Turski + Maibaum '87].

In stepwise development it is highly desirable to be able to compose refinement steps in a natural way. Consider two consecutive implementation steps: a first implementation of M on P (with mediating specification Q) and a second implementation of P on T (with mediating specification R). One would wish to compose these two implementations, in an easy and natural manner, so as to obtain a composite implementation of M directly on T (see figure 1).

An immediate question that arises is: what would the mediating specification be? This is where the property of Interpretation Modularity comes into play. For, it will allow one to obtain such a mediating specification S, extending R conservatively, together with an interpretation k of Q into S. In other words, it will enable one to complete the rectangle, thereby obtaining a composite implementation of M directly on T, as illustrated in figure 1.

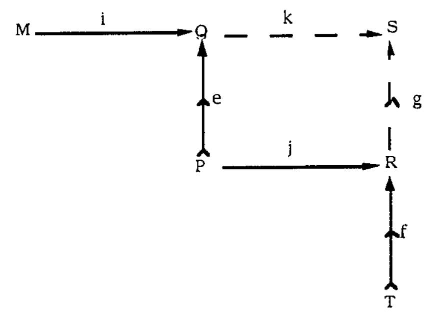

Figure 1: Composing implementation steps

Here it is worthwhile noting that this composition mirrors what a programmer would do by simply putting together the corresponding modules. This is possible because we do not require that the given mediating symbols be eliminated in constructing the composite one.

2.3 PARAMETERISED LOGICAL SPECIFICATIONS

The structuring of a specification into a context and parameter has been found to be particularly useful. The idea is that the context can be plugged into different situations by appropriate choices of values (instances) for the parameters. Such structured specifications are called parameterised specifications [Ehrich '82]. Thus a parameterised specification, such as SEQ[DATA], provides means for obtaining other specifications, such as SEQ[NAT], and so forth.

The simple tools of conservative extensions and interpretations provide a quite straightforward account of parameterisation, which regards parameterised specifications as 'normal' specifications, rather than as (partial) functions on models or on specifications, which is a usual approach.

A specification Q is said to be *parameterised* by a sub-specification P (called *parameter*) whenever Q is a conservative extension of P. Now, in instantiating the parameter we wish its properties, as expressed by its theorems, to be preserved. So, by a *parameter instantiation* we mean an interpretation $p{:}P \rightarrow R$ [Maibaum + Veloso + Sadler '91]. We thus have a situation similar to the one encountered in composing implementations (see figure 2).

Once again the property of Interpretation Modularity comes into play. It will enable us to complete the rectangle, thereby yielding the resulting instantiated specification S, as illustrated in figure 2. Here it is worthwhile noting that the construction of this instantiated specification mimics what our intuition suggests. Moreover, the instantiated specification S is still a conservative extension of the actual argument R.

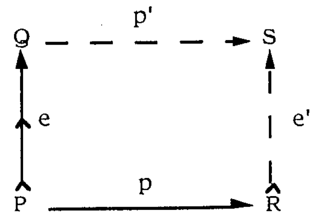

Figure 2: Completing a rectangle with a specification

3. MODULARITY IN CLASSICAL FIRST-ORDER LOGIC

In broad terms, modularity concerns the preservation of some logical properties of presentations under certain constructions. We shall now introduce three variations of modularity, concerning interpretations, extensions and joint consistency, and examine them in the context of Classical First-Order Logic (CFOL, for short). This will clarify some connections among them and illustrate the application of some internalisation techniques as well. Interpretation modularity is motivated by stepwise software development (as explained in section 2) and extension modularity deals with unions of presentations, a special case of which is joint consistency (as in the well-known Robinson's Joint Consistency Theorem).

We consider a *language* as characterised by its *alphabet* of extra-logical (predicate and function) symbols together with syntactical declarations. A *specification* is a *theory presentation*, i. e. a pair $S=\langle L,\Sigma\rangle$, consisting of a language L, and a set Σ of sentences of L (its *axioms*).

We say that I is a *sub-language* of J (denoted by $I\subseteq J$) when J can be obtained from I by adding some symbols (and declarations). Now, consider specifications $P=\langle I,\Delta\rangle$ and $Q=\langle J,\Gamma\rangle$. We say that Q is an *extension* of P (denoted by $P\subseteq Q$) iff $I\subseteq J$ and every consequence of Δ is a consequence of Γ. We call P and Q *equivalent* (denoted by $P\equiv Q$) iff they are extensions of each other. In case $I\subseteq J$, we say that Q *conserves* P (denoted by $P<Q$) iff, for every sentence σ of L, $\Delta\vdash\sigma$ whenever $\Gamma\vdash\sigma$. A *conservative extension* $P\leq Q$ is an extension $P\subseteq Q$ such that Q conserves P.

By a *translation* t from source language I to target language K we mean a syntax-preserving language morphism mapping each symbol of I to a corresponding symbol in K of the same kind; so each formula φ of I is translated to a formula $t(\varphi)$ of K. An *interpretation* from $P=\langle I,\Delta\rangle$ to $R=\langle K,\Theta\rangle$ is a translation of the underlying languages that translates every consequence of Δ to a consequence of Θ. An interpretation t:P→R is called *faithful* iff for every sentence σ of I, Δ⊢ σ iff Θ⊢ t(σ) [Shoenfield '67; Enderton '72].

3.1 VARIATIONS OF MODULARITY

We shall now introduce three variations of modularity, concerning interpretations, extensions and joint consistency. The first version is motivated by software development, as explained in section 2, and the third one comes from the well-known Robinson's Joint Consistency Theorem in CFOL concerning unions of presentations, extension modularity being somewhat intermediate between these two.

Let us start by examining interpretation modularity. It concerns a (pushout) construction, which completes a rectangle of presentation interpretations (see figure 3). As explained in section 2, this is an important issue in implementation and instantiation of specifications.

Consider the following situation. We have presentations P=⟨I,Δ⟩, Q=⟨J,Γ⟩ and R=⟨K,Θ⟩, such that Q is an extension of P, and an interpretation f:P→R. We wish to construct a presentation S=⟨L,Σ⟩ and a translation g:J→L, such that S extends R and g interprets Q into S.

$$\begin{array}{ccc} \overbrace{\langle I, \Delta\rangle}^{P} & \xrightarrow{\ f\ } & \overbrace{\langle K, \Theta\rangle}^{R} \\ \cap & & \cap \\ \underbrace{\langle J, \Gamma\rangle}_{Q} & \xrightarrow[\ g\ ]{} & \underbrace{\langle L, \Theta \cup g(\Gamma)\rangle}_{S} \end{array}$$

Figure 3: Pushout Construction for Presentation Interpretations

For this purpose, we first construct the pushout rectangle of underlying languages and translations and then use it to define the axiomatisation Σ for S, as follows:

1. Form language L by adding to K the symbols in A:=J-I (together with their declarations).
2. Now, extend translation f:I→K to a translation g from J to L by using the identity on the symbols in A=J-I.
3. Finally, set Σ:=Θ∪g(Γ), to obtain S:=⟨L,Σ⟩.

(Notice that it would suffice to translate the new axioms in Γ-Cn(Δ). Thus, if Q is finitely axiomatisable over P, then so is R over S.)

Interpretation modularity asserts that this pushout construction preserves conservativeness.

Property IM: *Interpretation Modularity*

Given presentations $P=\langle I,\Delta\rangle$, $Q=\langle J,\Gamma\rangle$ and $R=\langle K,\Theta\rangle$, such that Q is an extension of P, and an interpretation $f:P\to R$, the pushout of translations gives a translation $g:J\to L$; consider presentation $S:=\langle L,\Sigma\rangle$, with $\Sigma:=\Theta\cup g(\Gamma)$. If $Q=\langle J,\Gamma\rangle$ is a conservative extension of $P=\langle I,\Delta\rangle$, then $S=\langle L,\Sigma\rangle$ is a conservative extension of $R=\langle K,\Theta\rangle$.

Clearly extensions are special interpretations. So, from the logical viewpoint it is quite natural to consider the specialisation of the preceding considerations to extensions. We then have the following property, asserting that union preserves conservativeness.

Property EM: *Extension Modularity*

Consider presentations $Q=\langle J,\Gamma\rangle$ and $R=\langle K,\Theta\rangle$, both extending $P=\langle I,\Delta\rangle$. Assume that languages J and K share only the symbols in I, i. e. $J\cap K=I$, and consider the union presentation $S:=\langle J\cup K,\Gamma\cup\Theta\rangle$. If $Q=\langle J,\Gamma\rangle$ is a conservative extension of $P=\langle I,\Delta\rangle$, then $S=\langle L,\Sigma\rangle$ is a conservative extension of $R=\langle K,\Theta\rangle$.

So, Modularity of Extensions asserts that conservativeness is preserved by this union construction; it guarantees a relationship between the resulting presentations provided that the given presentations have some relationship.

A variation of this property is preservation of consistency, as in the well-known Robinson's Joint Consistency Theorem. It asserts that consistency is preserved under union over a maximally consistent presentation; it guarantees a property of the union presentation provided that the given presentations have some property.

Property JC: *Joint Consistency*

Consider presentations $Q=\langle J,\Gamma\rangle$ and $R=\langle K,\Theta\rangle$, both extending $P=\langle I,\Delta\rangle$. Assume that languages J and K share only the symbols in I, i. e. $J\cap K=I$, and consider the union presentation $S:=\langle J\cup K,\Gamma\cup\Theta\rangle$. If $P=\langle I,\Delta\rangle$ is complete and consistent (maximally consistent) and both $Q=\langle J,\Gamma\rangle$ and $R=\langle K,\Theta\rangle$ are consistent, then the union $S:=\langle J\cup K,\Gamma\cup\Theta\rangle$ is consistent.

If we compare the properties of Joint Consistency and Extension Modularity, we see that the latter appears to be more flexible than the former, in that it replaces consistent extensions of a complete presentation by a conservative extension of a (not necessarily complete) presentation. We thus appear to have a simple hierarchy of modularity properties: **IM** $\Rightarrow$ **EM** $\Rightarrow$ **JC**. We shall examine it more closely in the sequel.

Now, Robinson's Joint Consistency Theorem is known to be related to Craig's Interpolation Theorem. In view of the above connection, it is interesting to investigate in more detail the relationships between interpolation and modularity properties.

3.2 JOINT CONSISTENCY AND EXTENSION MODULARITY

We start with some simple characterisations for conservativeness of extensions and for conservative extensions of maximally consistent presentations.

A motivation for conservativeness is preserving consistency. Indeed: if $\mathcal{Q}$ conserves P and P is consistent, then so is $\mathcal{Q}$.

The following result gives some simple characterisations for conservativeness. Notice that characterisation in item (e), in terms of elementary equivalence, is a simplified version of the one usually formulated in terms of elementary extension [Shoenfield '67; p. 95, exercise 9].

Lemma: *Characterisations of Conservativeness*

Given presentations $P=\langle I,\Delta\rangle$ and $\mathcal{Q}=\langle J,\Gamma\rangle$ with $I\subseteq J$, the following are equivalent.

a) $\mathcal{Q}$ conserves P ($P<\mathcal{Q}$).

b) For any set Θ of sentences of I, $\Delta\cup\Theta<\Gamma\cup\Theta$.

c) For any set Θ of sentences of I, if $\Delta\cup\Theta$ is consistent then so is $\Gamma\cup\Theta$.

d) For any maximally consistent presentation $\langle I,T\rangle$ with $T\supseteq\Delta$, $\Gamma\cup T$ is consistent.

e) For any model $\mathfrak{M}\in Mod(P)$ there exists $\mathfrak{N}\in Mod(\mathcal{Q})$ such that $\mathfrak{N}\vDash Th(\mathfrak{M})$.

Proof.

($a\Rightarrow b$) Consider an arbitrary sentence σ of I such that $\Gamma\cup\Theta\vdash\sigma$. Then, by compactness $\Gamma\cup\{\varphi\}\vdash\sigma$, for some finite conjunction φ of sentences of Θ; so $\Gamma\vdash(\varphi\rightarrow\sigma)$. Thus, by (a), $\Delta\vdash(\varphi\rightarrow\sigma)$; whence $\Delta\cup\Theta\vdash\sigma$.

($b\Rightarrow c$) By the preceding remark.

($c\Rightarrow d$) Consider a maximally consistent set T of sentences of I such that $T\supseteq\Delta$. Since $\Delta\cup T=T$ is consistent, we have by (c), $\Gamma\cup T$ consistent.

($d\Rightarrow e$) Consider a model $\mathfrak{M}\in Mod(P)$. Then, we have a maximally consistent set of sentences of I $Th(\mathfrak{M})\supseteq\Delta$. So, by (d), $\Gamma\cup Th(\mathfrak{M})$ is consistent, whence by completeness, we have some model $\mathfrak{N}\vDash\Gamma\cup Th(\mathfrak{M})$.

($e \Rightarrow a$) Consider a sentence σ of I such that $\Delta \nvDash \sigma$. Then, completeness gives a model $\mathfrak{M} \in \mathrm{Mod}(P)$ such that $\mathfrak{M} \nvDash \sigma$, whence $\neg\sigma \in \mathrm{Th}(\mathfrak{M})$. By (d), we have some model $\mathfrak{N} \in \mathrm{Mod}(Q)$ such that $\mathfrak{N} \vDash \mathrm{Th}(\mathfrak{M})$. So $\mathfrak{N} \nvDash \sigma$, whence $\Gamma \nvDash \sigma$.

QED

A simple consequence of these characterisations is that the conservative extensions of a maximally consistent presentation are the consistent ones.

Corollary: *Conservative Extensions and Maximal Consistency*

Consider presentations $P=\langle I,\Delta\rangle$ and $Q=\langle J,\Gamma\rangle$, with Q extending P. If P is maximally consistent and Q is consistent, then $P \leq Q$.

Proof.

Considering a maximally consistent presentation $\langle I,T\rangle$ with $T \supseteq \Delta$, since P is maximally consistent we have $T=\Delta$, whence $\Gamma=\Gamma \cup T$ is consistent. Thus, by the lemma item (d), $P<Q$.

QED

We are now ready to examine Joint Consistency and Extension Modularity in Classical First-Order Logic. That CFOL has the property **JC** is clear, because of Robinson's Joint Consistency Theorem [Chang + Keisler '73, p. 88; Shoenfield '67, p. 79], which motivates its formulation.

To show Extension Modularity, we use the above characterisation (e) in conjunction with **JC**, as follows. Consider presentations $P=\langle I,\Delta\rangle$, $Q=\langle J,\Gamma\rangle$ and $R=\langle K,\Theta\rangle$, such that $P \subseteq Q$, $P \subseteq R$ with $J \cap K=I$; we wish to show that $R<Q \cup R$ whenever $P<Q$. (Notice that if R is inconsistent then clearly $R \leq Q \cup R$; so we may assume R consistent.) Given a model $\mathfrak{R} \in \mathrm{Mod}(R)$, its reduct $\mathfrak{P}$ to language I is a model of P with $\mathrm{Th}(\mathfrak{P}) \subseteq \mathrm{Th}(\mathfrak{R})$; so, since $P<Q$ the previous lemma gives a model $\mathfrak{Q} \in \mathrm{Mod}(Q)$ such that $\mathrm{Th}(\mathfrak{P}) \subseteq \mathrm{Th}(\mathfrak{Q})$. We then have a maximally consistent presentation $\langle J \cap K, \mathrm{Th}(\mathfrak{P})\rangle$ with two consistent extensions $\langle J,\mathrm{Th}(\mathfrak{Q})\rangle$ and $\langle K,\mathrm{Th}(\mathfrak{R})\rangle$. By **JC**, the union $\langle J \cup K,\mathrm{Th}(\mathfrak{Q}) \cup \mathrm{Th}(\mathfrak{R})\rangle$ is consistent, and has a model $\mathfrak{S}$. Then, $\mathfrak{S} \in \mathrm{Mod}(Q)$ with $\mathfrak{S} \vDash \mathrm{Th}(\mathfrak{R})$. By our lemma $R<Q \cup R$.

Notice that the above argument shows that **JC** $\Rightarrow$ **EM** in CFOL. To see that **EM** $\Rightarrow$ **JC** one can use the above corollary, as follows. Consider presentations $P=\langle I,\Delta\rangle$, $Q=\langle J,\Gamma\rangle$ and $R=\langle K,\Theta\rangle$, such that $P \subseteq Q$, $P \subseteq R$ with $J \cap K=I$, as above. We wish to show that $Q \cup R$ is consistent, if Q and R are so and P is maximally consistent. By the corollary we have $P \leq Q$. So, by **EM**, $R \leq Q \cup R$. Thus, by the remark, $Q \cup R$ is consistent.

3.3 INTERNALISATION VIA TRANSLATION DIAGRAM

We now wish to show that CFOL has the property of Interpretation Modularity. We are going to reduce it to Extension Modularity by resorting to presentations in larger languages that code the information in the language translations. So, we first examine the internalisation technique of translation diagram [Veloso '92; Veloso '93].

Consider a translation t from source language I to target language K. We form the disjoint union language I+K, and construct the set $\Delta[t]:=\{a\leftrightarrow t(a)/a\in I\}$ of sentences of I+K asserting the equivalence between a symbol of I and its translation under t. We thus have a presentation: the *diagram presentation* $D[t]:=\langle I+K,\Delta[t]\rangle$.

More precisely, the *translation diagram* $\Delta[t]$ of a translation $t{:}I\to K$ consists of the following sentences of the disjoint-union language I+K:

for each pair of symbols a of I and b of K, such that t(a)=b, $\Delta[t]$ has

$\forall x_1,\dots,x_m[a(x_1,\dots,x_m)\leftrightarrow b(x_1,\dots,x_m)]$, if a is an m-ary predicate symbol;

$\forall y\forall x_1,\dots,x_n[y\approx a(x_1,\dots,x_n)\leftrightarrow y\approx b(x_1,\dots,x_n)]$, if a is an n-ary function symbol.

The next result gives some fundamental properties of translation diagrams, which corroborate that they indeed code the information given in the language translation.

Lemma: *Properties of Translation Diagram*

Consider a translation $t{:}I\to K$ with diagram presentation $D[t]=\langle I+K,\Delta[t]\rangle$.

a) For every formula φ of I, $\Delta[t]\vdash[\varphi\leftrightarrow t(\varphi)]$.

b) Given a presentation $R=\langle K,\Theta\rangle$, form the diagram extension $D[t]+R:=\langle I+K,\Delta[t]\cup\Theta\rangle$. Then

(i) $D[t]+R:=\langle I+K,\Delta[t]\cup\Theta\rangle$ is a conservative extension of $R=\langle K,\Theta\rangle$;

(ii) given a presentation $P=\langle I,\Delta\rangle$, t interprets P into R iff $P\subseteq D[t]+R$.

Proof.

a) By induction on the structure of formula φ.

b) Consider $R=\langle K,\Theta\rangle$ and $D[t]+R:=\langle I+K,\Delta[t]\cup\Theta\rangle$.

(i) Every symbol a of I is introduced into I+K by means of a (single) defining axiom $a\leftrightarrow t(a)$ of $\Delta[t]$. So we have an extension by definitions, which is conservative.

(ii) Consider a theorem σ of P. By (a) and (i), we have $\Delta[t]\cup\Theta\vdash[\sigma\leftrightarrow t(\sigma)]$ with $R\leq D[t]+R$. Hence, $\Theta\vdash t(\sigma)$ iff $\Delta[t]\cup\Theta\vdash\sigma$.

QED

3.4 REDUCTION OF INTERPRETATION TO EXTENSION MODULARITY

We are now going to use diagram internalisation to reduce Interpretation Modularity to Extension Modularity.

Let us first recall the pushout construction given in 3.1. We have presentations $P=\langle I,\Delta\rangle$, $Q=\langle J,\Gamma\rangle$ and $R=\langle K,\Theta\rangle$, such that Q is an extension of P, and an interpretation $f:P\rightarrow R$. We wish to construct a presentation $S=\langle L,\Sigma\rangle$ and translation $g:J\rightarrow L$, such that S extends R and $g:Q\rightarrow S$.

Notice that we may assume that the only symbols, if any, shared by languages J, of Q, and K, of R, are those of I, i. e. $J\cap K\subseteq I$.

We first complete the rectangle of language translations by a (pushout) construction (see figure 4).

Let A consist of the new symbols added to I to form J: A:= J-I.

1. Form language L by adding to K the symbols in A (together with their declarations).
2. Now, extend translation f from I to K to a translation g from J to L by using the identity on the symbols in A.

$$J\left\{\begin{array}{ccc} A & \xrightarrow{1} & A \\ \hline I & \xrightarrow[f]{} & K\end{array}\right\} L$$

Figure 4: Pushout Construction for Language Translations: construction of language L and translation g.

An important property of this construction, which makes it a pullback as well, is that the only symbol that g maps into K are those of $I: g^{-1}(K)=I$.

We now complete the rectangle of interpretations as in figure 3.

1. Complete the rectangle of language translations by the above (pushout) construction, to obtain language L and translation $g:J\rightarrow L$.
2. Use translation g to translate the set Γ of axioms of Q to $g[\Gamma]$ in L.
3. Finally, set $\Sigma:=\Theta\cup g(\Gamma)$, to obtain $S:=\langle L,\Sigma\rangle$.

We are now ready to reduce Interpretation Modularity to Extension Modularity by means of diagram internalisation. Consider, as above, presentations $P=\langle I,\Delta\rangle$, $Q=\langle J,\Gamma\rangle$ and $R=\langle K,\Theta\rangle$, such that Q is an extension of P, and an interpretation $f:P\rightarrow R$. Construct the pushout presentation $S=\langle L,\Sigma\rangle$, which clearly extends R, and translation $g:J\rightarrow L$, which interprets Q into S. Assuming $P\leq Q$, we wish to show that $R\leq S$. For this purpose, we proceed as follows (see figure 5).

$$\begin{array}{l} \overbrace{\langle I,\Delta\rangle}^{P} \subseteq \langle I+K,\Delta[f]\cup\Theta\rangle \quad > \quad \overbrace{\langle K,\Theta\rangle}^{R} \\ \quad \Lambda \\ \underbrace{\langle J,\Gamma\rangle}_{Q} \subseteq \langle J+K,\Gamma\cup\Delta[f]\cup\Theta\rangle \equiv \underbrace{\langle J+K,\Gamma\cup\Delta[g]\cup\Theta\rangle}_{T} \supseteq \underbrace{\langle L,\Theta\cup g(\Gamma)\rangle}_{S} \end{array}$$

Figure 5: Reduction of Interpretation to Extension Modularity by means of diagram internalisation.

1. Form the diagram extension $D[f]+R:=\langle I+K,\Delta[f]\cup\Theta\rangle$.
2. By the lemma on Properties of Translation Diagram item b, we have that $R=\langle K,\Theta\rangle\leq\langle I+K,\Delta[f]\cup\Theta\rangle\supseteq\langle I,\Delta\rangle=P$.
3. By the Pushout Construction for Language Translations, we have, since $J\cap K\subseteq I$ and $A:= J-I$, that $J\cap(I+K)=(J\cap I)\cup(J\cap K)=I\cup(J\cap K)=I$ and $J\cup(I+K)=(I\cup A)\cup(I\cup K)=I\cup(K+A)=I\cup L\supseteq L$.
4. In view of 2 and 3, **EM** yields $\langle I+K,\Delta[f]\cup\Theta\rangle\leq\langle J+K,\Gamma\cup\Delta[f]\cup\Theta\rangle$.
5. From 2 and 4, we have $R=\langle K,\Theta\rangle\leq\langle J+K,\Gamma\cup\Delta[f]\cup\Theta\rangle$.
6. By the Pushout Construction for Language Translations, g extends f by the identity. So, $\Delta[f]\vdash\Delta[g]$, whence $\langle J+K,\Gamma\cup\Delta[f]\cup\Theta\rangle\equiv\langle J+K,\Gamma\cup\Delta[g]\cup\Theta\rangle$.
7. By the lemma on Properties of Translation Diagram item b(ii), since $g:Q\to S$, $g(\Gamma)\subseteq Cn(\Gamma\cup\Delta[g])$.
8. By construction $\Sigma:=\Theta\cup g(\Gamma)$; so, in view of 7, $\Sigma=\Theta\cup g(\Gamma)\subseteq Cn(\Gamma\cup\Delta[g]\cup\Theta)$.
9. We thus have a presentation $T:=\langle J+K,\Gamma\cup\Delta[g]\cup\Theta\rangle$, such that $\langle K,\Theta\rangle=R\leq T$ (by 5 and 6) and $\langle L,\Sigma\rangle=S\subseteq T$ (by 3 and 8).
10. Therefore $R\leq S$, as claimed.

    Indeed, given a sentence σ of K, if $\Sigma\vdash\sigma$, then $\Gamma\cup\Delta[g]\cup\Theta\vdash\sigma$, whence $\Theta\vdash\sigma$.

Thus, since CFOL has Extension Modularity, it also has Interpretation Modularity.

3.5 VARIATIONS OF INTERNALISATION: KERNEL AND QUOTIENT

We have just seen how to reduce Interpretation Modularity to Extension Modularity by means of diagram internalisation. This internalisation technique relies on the following two familiar properties of the biconditional connective $\leftrightarrow$:

Detachment: $\{\varphi\leftrightarrow\psi\}\cup\{\varphi\}\vdash\psi$ and $\{\varphi\leftrightarrow\psi\}\cup\{\psi\}\vdash\varphi$.

Substitutivity: $\{\varphi\leftrightarrow\psi\}\vdash\theta\leftrightarrow\theta'$, whenever θ' is obtained from θ by replacing occurrences of φ by ψ.

The idea underlying internalisation of a translation is coding, into sentences in an appropriate language, (part of) the information in the translation [Veloso '92; Veloso '93; Veloso + Maibaum '95]. In the case of diagram, we use the disjoint union language to code the information in the translation. A simpler variation uses only the source language to code only (part of) this information.

Consider a translation t from source language I to target language K. We construct the set $\Lambda[t]:=\{a\leftrightarrow a'/t(a)=t(a')\}$ of sentences of I asserting the equivalence of symbols of I that have the same translation under t. (Notice that each such sentence is mapped by t to a valid sentence of K.) We thus have a presentation: the *kernel presentation* $N[t]:=\langle I,\Lambda[t]\rangle$.

More precisely, the *internalised kernel* $\Lambda[t]$ of a translation $t:I\rightarrow K$ consists of the following sentences of the source language I:

for each pair of symbols a and a' of I, such that t(a)=t(a'), $\Lambda[t]$ has

$\forall x_1,\ldots,x_m[a(x_1,\ldots,x_m)\leftrightarrow a'(x_1,\ldots,x_m)]$, if a is an m-ary predicate symbol;

$\forall y\forall x_1,\ldots,x_n\ [y\approx a(x_1,\ldots,x_n)\leftrightarrow y\approx a'(x_1,\ldots,x_n)]$, if a is an n-ary function symbol.

The characteristic property of the internalised kernel is

$\Lambda[t]\vdash[t(\varphi)\leftrightarrow t(\psi)]$, whenever $t(\varphi)=t(\psi)$.

These two internalisation techniques are conservatively connected:

$N[t]:=\langle I,\Lambda[t]\rangle\leq\langle I+K,\Delta[t]\rangle:=D[t]$.

This connection can be established by considering a function $t':t(I)\rightarrow I$ such that the composite t' followed by t gives the identity on the image t(I). This argument relies on choice of representatives, which suggests another approach and the idea of quotient presentations [Poubel + Veloso '93].

Consider a translation t from source language I to target language K. The (*external*) *kernel* of *t* is the equivalence relation $\approx_t$ on the alphabet of I relating symbols a and a' such that t(a)=t(a'). We can use it to construct a *quotient language* $I/\approx_t$, which factorises $t:I\rightarrow K$ into a natural (surjective) projection translation $t^{\sim}:I\rightarrow I/\approx_t$ followed by (injective) translation $t^{\wedge}:I/\approx_t\rightarrow K$. Now, given a presentation $P=\langle I,\Delta\rangle$, we can form the quotient presentation $P/\approx_t:=\langle I/\approx_t,t^{\sim}(\Delta)\rangle$, which is isomorphic to the image $t(P)=\langle t(I),t(\Delta)\rangle$.

We then have the following connection between these techniques:

$t(\Delta)\vdash t(\sigma)$ iff $t^{\sim}(\Delta)\vdash t^{\sim}(\sigma)$ iff $\Delta\cup\Lambda[t]\vdash\sigma$.

As corollary, we have faithful interpretations, where $P\cup N[t]:=\langle I,\Delta\cup\Lambda[t]\rangle$:

$t:P\cup N[t]\rightarrow t(P)$ and $t^{\sim}:P\cup N[t]\rightarrow P/\approx_t$

These ideas suggest yet another reduction of Interpretation Modularity to Extension Modularity, namely by means of internalised kernel (or quotient presentation). We proceed as follows (see figure 6).

$$\begin{array}{ccccccc} P & \subseteq & & P\cup N[f] & \xrightarrow{f} & f(P)\subseteq & R \\ \wedge & & & \wedge & & \wedge & \cap \\ Q & \subseteq & Q\cup N[f] & \equiv Q\cup N[g] & \xrightarrow[g]{} & g(Q)\subseteq & R\cup g(Q)=S \end{array}$$

Figure 6: Reducing Interpretation to Extension Modularity by means of internalised kernel.

We start by replacing P and Q, respectively, by stronger presentations $P\cup N[f]:=\langle I,\Delta\cup\Lambda[f]\rangle$ and $Q\cup N[f]:=\langle J,\Gamma\cup\Lambda[f]\rangle$. By the lemma characterising conservativeness item (b) since $\Lambda[f]$ consists of sentences of I, we have $P\cup N[f]:\langle I,\Delta\cup\Lambda[f]\rangle\leq\langle J,\Gamma\cup\Lambda[f]\rangle=Q\cup N[f]$. As before, since g extends f by the identity, we have $\Lambda[f]\vdash\Lambda[g]$; whence $\langle J,\Gamma\cup\Lambda[f]\rangle\equiv\langle J,\Gamma\cup\Delta[g]\rangle$. Thus, $P\cup N[f]=\langle I,\Delta\cup\Lambda[f]\rangle\leq\langle J,\Gamma\cup\Lambda[g]\rangle=Q\cup N[g]$.

Now, consider the image presentations $f(P):=\langle f(I),f(\Delta)\rangle$ and $g(Q):=\langle g(J),g(\Gamma)\rangle$; and notice that f interprets $\langle I,\Delta\cup\Lambda[f]\rangle$ into f(P) and g interprets $\langle J,\Gamma\cup\Lambda[g]\rangle$ into g(Q). Furthermore, by the preceding connection, interpretation $g:Q\cup N[g]\rightarrow g(Q)$ is faithful. Hence $f(P)\leq g(Q)$. (Indeed, if $g(\Gamma)\vdash f(\sigma)$ with σ a sentence of I, then $\Gamma\cup\Lambda[g]\vdash\sigma$, whence $\Delta\cup\Lambda[f]\vdash\sigma$, and so $f(\Delta)\vdash f(\sigma)$.)

Finally, since $S=R\cup g(Q)$ and $K\cap g(J)=f(I)$, by Property **EM**, we obtain the desired conclusion $R\leq S$.

4. Interpolation in Classical First-Order Logic

We shall now examine some properties of first-order logic related to interpolation and cut

An important property of CFOL is the so-called Craig's Interpolation, which appears in a few versions in the literature [Shoenfield '67; Chang + Keisler '73]. Such interpolation properties enable the decomposition of derivations involving formulae in distinct languages by interpolating formulae with the common extra-logical symbols.

A simple version of *Craig's Interpolation Lemma* is as follows [Chang + Keisler '73, p. 84]: given sentences σ of language J and τ of language K, if $\sigma\vdash\tau$, then there exists an *interpolant sentence* ρ of language $J\cap K$, such that $\sigma\vdash\rho$ and $\rho\vdash\tau$. Model-theoretically, if $Mod(\sigma)\subseteq Mod(\tau)$ with $\sigma\in Sent(J)$ and $\tau\in Sent(K)$, then $Mod(\sigma)\subseteq Mod(\rho)\subseteq Mod(\tau)$, for some $\rho\in Sent(J\cap K)$. A variant of Craig's Interpolation Lemma is the so-called *Split Interpolation* version [Rodenburg + van Glabbeek '88]. This is close to the following usual formulation of *Craig's Interpolation Lemma* [Shoenfield '67, p. 80]. Consider specifications $Q=\langle J,\Gamma\rangle$ and $R=\langle K,\Theta\rangle$ with union $Q\cup R$. If $\Gamma\cup\Theta\vdash(\phi\rightarrow\psi)$, for formulae $\phi\in$ Frml(J) and $\psi\in$ Frml(K), then there exists an *interpolant formula* $\theta\in$ Frml(J$\cap$K), such that $\Gamma\vdash(\phi\rightarrow\theta)$ and $\Theta\vdash(\theta\rightarrow\psi)$.

In CFOL, these versions of interpolation are interderivable, and equivalent to Robinson's Joint Consistency Theorem, because of the Compactness and Deduction Theorems.

4.1 SIMPLE INTERPOLATION

We first examine a property of simple interpolation sentence for presentations. To derive it from the above simple version of Craig's Interpolation Lemma, we shall resort to the following property of CFOL. Notice that this (connective-free) formulation of simple compactness can be viewed as an interpolation property.

SK: Given $\Gamma\subseteq$Sent(L), if $\Gamma\vdash\tau$ then there exists a sentence σ of L such that $\Gamma\vdash\sigma$ and $\sigma\vdash\tau$.

The simple interpolation sentence property is as follows.

Lemma *Simple Interpolation Sentence*

Consider $\Gamma\subseteq$Sent(J). For every sentence τ of language K such that $\tau\in$Cn(Γ), there exists an interpolating sentence ρ of J$\cap$K, such that $\rho\in$Cn(Γ) and $\tau\in$Cn($\{\rho\}$).

Proof.

By Simple Compactness SK, there exists a sentence $\sigma\in$Sent(J), such that $\Gamma\vdash\sigma$ and $\sigma\vdash\tau$. Thus, by the above simple version of Craig's Interpolation Lemma, there exists an interpolant sentence ρ of J$\cap$K, such that $\sigma\vdash\rho$ and $\rho\vdash\tau$. So $\Gamma\vdash\rho$, because $\Gamma\vdash\sigma$ and $\sigma\vdash\rho$.

QED

What is crucial for establishing Extension Modularity is the existence of interpolant presentations, rather than interpolant sentences. To handle interpolant presentations, it is convenient to use an extended notation for consequence: for sets Γ and Θ of sentences, we use $\Gamma\vdash\Theta$ to mean that $\Gamma\vdash\varphi$ for every $\varphi\in\Theta$, i. e. $\Theta\subseteq$Cn(Γ).

We now extend the preceding property of simple interpolation sentence to a global version for presentations.

Proposition *Simple Interpolation Presentation*

Given presentations $Q=\langle J,\Gamma\rangle$ and $R=\langle K,\Theta\rangle$, if $\Theta\subseteq Cn(\Gamma)$, then there exists an *interpolant presentation* $M=\langle J\cap K,\Psi\rangle$ such that $\Psi\subseteq Cn(\Gamma)$ and $\Theta\subseteq Cn(\Psi)$.

Proof.

Consider a sentence $\tau\in\Theta$. Then $\tau\in Sent(K)$ and by assumption $\Gamma\vdash\tau$. So, by the above lemma on simple interpolation sentence, we have an interpolating sentence ρ_τ of $J\cap K$, such that $\Gamma\vdash\rho_\tau$ and $\rho_\tau\vdash\tau$. We now set

$\Psi:=\{\rho_\tau\in Sent(J\cap K)/\tau\in\Theta\}$. Then

(i) $\Gamma\vdash\Psi$, since, for each $\rho_\tau\in\Psi$, $\Gamma\vdash\rho_\tau$; and

(ii) $\Theta\vdash\Psi$, since for each $\tau\in\Theta$, $\rho_\tau\in\Psi$ and $\rho_\tau\vdash\tau$.

QED

4.2 Interpolation-like Properties Related to Cut

We shall also employ another property of the consequence relation that is related to interpolation. By way of introduction, recall the so-called *Deduction Theorem* ($\rightarrow$-introduction) for first-order logic sentences: if $\Gamma\cup\{\sigma\}\vdash\tau$ then $\Gamma\vdash(\sigma\rightarrow\tau)$. Notice that if σ and τ are sentences of language L, then so is $(\sigma\rightarrow\tau)$. Also, by Modus Ponens ($\rightarrow$-elimination), $\{(\sigma\rightarrow\tau),\sigma\}\vdash\tau$. This suggests hiding the connective $\rightarrow$ into an interpolating sentence by formulating a connective-free version of the Deduction Theorem: "if $\Gamma\cup\{\sigma\}\vdash\tau$ with $\sigma,\tau\in Sent(L)$, then there exists a sentence χ of L such that $\Gamma\vdash\chi$ and $\{\chi,\sigma\}\vdash\tau$". Now, the converse of this connective-free formulation is simply: if $\Gamma\vdash\chi$ and $\{\chi\}\cup\{\sigma\}\vdash\tau$ then $\Gamma\cup\{\sigma\}\vdash\tau$. This converse is a special case of the familiar *Cut Rule*: if $\Gamma\vdash\xi$ and $\{\xi\}\cup\Sigma\vdash\tau$ then $\Gamma\cup\Sigma\vdash\tau$. We now wish to extend both the Cut Rule and the connective-free Deduction Theorem to sets of sentences.

The extension of the Cut Rule to sets of sentences does not present any major difficulty.

Lemma *Global Cut Property* (GC)

Consider sets of sentences: $\Sigma,\Theta\subseteq Sent(K)$, and Γ and Δ of sub-language $I\subseteq K$. If $\Delta\subseteq Cn(\Gamma)$ and $\Theta\subseteq Cn(\Delta\cup\Sigma)$, then $\Theta\subseteq Cn(\Gamma\cup\Sigma)$.

Proof.

Consider a sentence $\varphi\in\Theta$. By assumption $\Delta\cup\Sigma\vdash\varphi$. So, by the Compactness Theorem, there exists a finite conjunction δ_φ of sentences of Δ such that $\{\delta_\varphi\}\cup\Sigma\vdash\theta$. Since we assume $\Gamma\vdash\Delta$, we have $\Gamma\vdash\delta_\varphi$. Thus, by the familiar Cut Rule, $\Gamma\cup\Sigma\vdash\theta$.

QED

We also wish to extend the connective-free Deduction Theorem to presentations. We first establish a connective-free Global Deduction Theorem. For this purpose, we shall resort to the following property of CFOL. Notice that this (connective-free) formulation of distributed compactness can be regarded as an interpolation property.

DK: Given $\Gamma \subseteq Sent(J)$ and $\Theta \subseteq Sent(K)$, for every $\sigma \in Sent(K)$ such that $\sigma \in Cn(\Gamma \cup \Theta)$, there exists $\varphi \in Sent(K)$ such that $\varphi \in Cn(\Gamma)$ and $\sigma \in Cn(\Gamma \cup \{\varphi\})$.

Lemma *Global Deduction Theorem*

Consider presentations $Q = \langle J, \Gamma \rangle$ and $R = \langle K, \Theta \rangle$. For every sentence σ of language K such that $\sigma \in Cn(\Gamma \cup \Theta)$, there exists a sentence χ of K such $\chi \in Cn(\Gamma)$ and $\sigma \in Cn(\{\chi\} \cup \Theta)$.

Proof.

By Distributed Compactness DK, there exists $\varphi \in Sent(K)$, such that $\Theta \vdash \varphi$ and $\Gamma \cup \{\varphi\} \vdash \sigma$. Thus, by the above connective-free Deduction Theorem, there exists a sentence χ of K, such that $\Gamma \vdash \chi$ and $\{\chi\} \cup \{\varphi\} \vdash \sigma$. Hence, by the familiar Cut Rule, we have $\{\chi\} \cup \Theta \vdash \sigma$.

QED

We now wish to extend the Global Deduction Theorem to presentations. We will have a kind of converse to the Cut Rule, with some simple information on languages, which originates from our connective-free formulation.

Proposition *Converse-Cut Presentation*

Consider presentations $Q = \langle J, \Gamma \rangle$ and $R = \langle K, \Theta \rangle$. For every $\Sigma \subseteq Sent(K)$ such that $\Sigma \subseteq Cn(\Gamma \cup \Theta)$, there exists a *converse-cut presentation* $C = \langle K, \Omega \rangle$ such that $\Omega \subseteq Cn(\Gamma)$ and $\Sigma \subseteq Cn(\Omega \cup \Theta)$.

Proof.

Consider a sentence $\sigma \in \Sigma$. Then $\sigma \in Sent(K)$ and by assumption $\Gamma \cup \Theta \vdash \sigma$. So, by the above Global Deduction Theorem, we have a sentence $\chi_\sigma \in Sent(K)$, such that $\Gamma \vdash \chi_\sigma$ and $\{\chi_\sigma\} \cup \Theta \vdash \sigma$. Now, set $\Omega := \{\chi_\sigma \in Sent(K) / \sigma \in \Sigma\}$. Then

(i) $\Gamma \vdash \Omega$, since, for each $\chi_\sigma \in \Omega$, $\Gamma \vdash \chi_\sigma$; and

(ii) $\Omega \cup \Theta \vdash \Sigma$, since for each $\sigma \in \Sigma$, $\chi_\sigma \in \Omega$ and $\{\chi_\sigma\} \cup \Theta \vdash \sigma$.

QED

This result is a kind of converse to the Cut Rule, with some simple information on languages. It is this information on languages that prevents one from taking $C = \langle K, \Omega \rangle$ trivially as $Q = \langle J, \Gamma \rangle$.

5. Modularity and Interpolation

In the preceding sections we have seen that Classical First-Order Logic has the three variations of modularity, namely Joint Consistency, Extension and Interpretation, as well as some interpolation-like properties. Also, it is apparent that the crucial step in each case concerned establishing conservativeness.

Since Robinson's Joint Consistency Theorem and Craig's Interpolation Theorem for CFOL are known to be tightly connected, we now start taking a closer look at relationships among interpolation and modularity properties. This analysis will be carried out in a connective-independent manner, in the spirit of Tarski's classical work on the consequence relation [Tarski '30]. We do not, however, keep the language fixed, since we focus on conservativeness and interpolation. So we shall consider consequence relations relativised to languages.

5.1 Relativised Consequence Relation

Since we wish to keep track of the languages involved, we now relativise the consequence relation to a language, regarded as a parameter. The idea is as follows. In general the formulation of a logical calculus, by means of axioms, natural deduction, sequents, etc., consists of logical axioms and rules of inference, which refer to formulae. We may regard the language, characterised by its extra-logical symbols, as a parameter of the formulation; e. g., by extending the language, we have more formulae, and so more logical axioms and rules of inference.

Given sets of sentences Γ and $\{\sigma\}$ of language L, by $\Gamma \vdash_L \sigma$ - also denoted $\sigma \in Cn_L(\Gamma)$ - we mean that σ is a *consequence* of Γ *within* L, in the sense that there exists a (not necessarily finite) derivation of σ from Γ using logical axioms and rules of inference involving only formulae of L. In addition, for a set Σ of sentences of L, we use $\Gamma \vdash_L \Sigma$ to mean that $\Gamma \vdash_L \sigma$ for <u>every</u> $\sigma \in \Sigma$, i. e. $\Sigma \subseteq Cn_L(\Gamma)$.

The previous concepts relative to extensions of presentations $P=\langle I,\Delta\rangle$ and $Q=\langle J,\Gamma\rangle$ are now expressed as follows.

$Q=\langle J,\Gamma\rangle$ *extends* $P=\langle I,\Delta\rangle$ ($P\subseteq Q$) iff $I\subseteq K$ and $\Gamma \vdash_J \sigma$ whenever $\Delta \vdash_I \sigma$;

Q *converses* P ($P<Q$) iff $I\subseteq K$ and $\Delta \vdash_I \sigma$ whenever $\Gamma \vdash_J \sigma$ with $\sigma \in Sent(I)$;

Q *extends conservatively* P ($P\leq Q$) iff $Cn_I(\Delta)=Cn_J(\Gamma)\cap Sent(I)$.

Since we shall be concentrating on presentations, we shall not need compactness. We shall, however, need some simple properties of our relativised consequence relations, which amount to relativisations of the Global Cut Property, which CFOL was seen to have in section 4.

5.2 VERSIONS OF MODULARITY

As mentioned, the crucial issue concerning modularity is preservation of conservativeness. So, we formulate the appropriate version of Extension Modularity as follows.

Property PM: *Presentation Modularity* (addition of new symbols)

Consider presentations $P=\langle I,\Delta\rangle$ and $Q=\langle J,\Gamma\rangle$ such that Q converses P ($P<Q$). Then, for any presentation $R=\langle K,\Theta\rangle$ such that $K\cap J\subseteq I$, $Q\cup R:=\langle J\cup K,\Gamma\cup\Theta\rangle$ converses $P\cup R:=\langle I\cup K,\Delta\cup\Theta\rangle$.

One can view an extension as being constructed by a two-step procedure: first add new symbols, then add new axioms. Likewise, we can go from a presentation P to a union presentation $P\cup R$ in two steps Two simple properties, which turn out to be special cases of Presentation Modularity, refer to these two steps.

The first property concerns language modularity: it guarantees that the addition of new symbols preserves conservativeness.

Property LM: *Language Modularity* (addition of new symbols)

Consider presentations $P=\langle I,\Delta\rangle$ and $Q=\langle J,\Gamma\rangle$ such that $P<Q$. Then, for any language K, with $K\cap J\subseteq I$, $\langle I\cup K,\Delta\rangle<\langle J\cup K,\Gamma\rangle$.

Property **LM** should not be confused with the perhaps more familiar property that the addition of new symbols produces a conservative extension; what Property **LM** asserts is that such addition preserves conservativeness.

It is not difficult to see that Property **LM** is a special case of Presentation Modularity **PM**: consider the presentation $R:=\langle K,\varnothing\rangle$.

The second property, axiom modularity, asserts that the addition of sentences of the smaller language preserves conservativeness.

Property AM: *Axiom Modularity* (addition of axioms)

Consider presentations $P=\langle I,\Delta\rangle$ and $Q=\langle J,\Gamma\rangle$ such that $P<Q$. Then, for any set Θ of sentences of language I, $\langle I,\Delta\cup\Theta\rangle\leq J,\Gamma\cup\Theta\rangle$.

It is easy to see that Property **AM** is a special case of Presentation Modularity **PM**: it suffices to consider the presentation $R:=\langle I,\Theta\rangle$.

Hence, both simple properties **LM** and **AM** are indeed necessary conditions for Presentation Modularity. (As a corollary, since CFOL has property **PM**, it has both properties **LM** and **AM**.)

Moreover, **LM** and **AM** are sufficient conditions for **PM**, which can be seen as follows. Consider presentations $P=\langle I,\Delta\rangle$, $Q=\langle J,\Gamma\rangle$ and $R=\langle K,\Theta\rangle$, such that $P<Q$ and $K\cap J\subseteq I$. Then, by **LM**, we have $\langle I\cup K,\Delta\rangle<\langle J\cup K,\Gamma\rangle$, whence, since $\Theta\subseteq Sent(I\cup K)$, **AM** yields $\langle I\cup K,\Delta\cup\Theta\rangle<\langle J\cup K,\Gamma\cup\Theta\rangle$, as required by **PM**.

Therefore, we have the connections: **LM** & **AM** $\Leftrightarrow$ **PM**

5.3 VERSIONS OF INTERPOLATION

In section 4 we have examined some interpolation-like properties of CFOL. We shall now examine some interconnections among them in the more general connective-free context with relativised consequence $\vdash_L$.

A property generalising the Deduction Theorem concerns the existence of converse-cut presentations. We can formulate it as an interpolation-like property of the relativised consequence relations as follows.

Property CC: *Converse-Cut Presentation*

Consider presentations $Q=\langle J,\Gamma\rangle$ and $R=\langle K,\Theta\rangle$, and let $L:=J\cup K$. For every $\Sigma\subseteq Sent(K)$ such that $\Gamma\cup\Theta\vdash_L\Sigma$, there exists a *converse-cut presentation* $C=\langle K,\Omega\rangle$ such that $\Gamma\vdash_L\Omega$ and $\Omega\cup\Theta\vdash_K\Sigma$.

Similarly, we can formulate the simple interpolation property as follows.

Property SI: *Simple Interpolating Presentation*

Consider presentations $Q=\langle J,\Gamma\rangle$ and $R=\langle K,\Sigma\rangle$, and let $L:=J\cup K$. If $\Gamma\vdash_L\Sigma$, then there exists a *simple interpolating presentation* $M=\langle K\cap J,\Psi\rangle$ such that $\Gamma\vdash_J\Psi$ and $\Psi\vdash_K\Sigma$.

In a similar spirit, we can formulate a version of split interpolation.

Property DI: *Distributed Interpolating Presentation*

Consider presentations $Q=\langle J,\Gamma\rangle$ and $R=\langle K,\Theta\rangle$, and let $L:=J\cup K$. For every $\Sigma\subseteq Sent(K)$ such that $\Gamma\cup\Theta\vdash_L\Sigma$, there exists a *distributed interpolating presentation* $T=\langle K\cap J,\Xi\rangle$ such that $\Gamma\vdash_J\Xi$ and $\Xi\cup\Theta\vdash_K\Sigma$.

It is easy to see that, as expected, **SI** is a special case of **DI**: it suffices to take presentation $R:=\langle K,\varnothing\rangle$. Also, **CC** follows from **DI**, since $J\cap K\subseteq K$, by resorting to a special case of the Global Cut property, which CFOL exhibits, concerning a single language. Call it Language Extension:

Property **LE**: If $\Gamma\vdash_I\Delta$ and $I\subseteq K$, then $\Gamma\vdash_K\Delta$.

For the converse we need another special case of the Global Cut property, concerning a single language. Call it Theorem Cut:

Property **TC**: If $\Gamma\vdash_L\Delta$ and $\Delta\cup\Theta\vdash_L\Sigma$, then $\Gamma\cup\Delta\vdash_L\Sigma$.

To see that, in the presence of **TC**, **DI** follows from **CC** and **SI** we may proceed as follows. Assume that $\Gamma\cup\Theta\vdash_L\Sigma$ where $L:=J\cup K$, $\Gamma\subseteq Sent(J)$ and

$\Sigma\cup\Theta\subseteq$Sent(K). Then, by **CC** we have $\Omega\subseteq$Sent(K) such that $\Gamma\vdash_L\Omega$ and $\Omega\cup\Theta\vdash_K\Sigma$. Now, **SI** applied to the former gives $\Psi\subseteq$Sent(K$\cap$J) such that $\Gamma\vdash_J\Psi$ and $\Psi\vdash_K\Omega$. Thus, by **TC**, $\Psi\cup\Theta\vdash_K\Sigma$, as required by **DI**.

Therefore, we have the connections:

$$\text{DI}\ \begin{array}{l}\longrightarrow \text{SI} \\ \qquad\ \ \& \\ \xrightarrow{\text{LE}} \text{CC}\end{array}\Bigg\}\xrightarrow{\text{TC}} \text{DI}$$

5.4 CONNECTIONS BETWEEN MODULARITY AND INTERPOLATION

We can now relate our modularity and interpolating-like properties by resorting to some simple properties of the relativised consequence $\vdash_L$.

We first show that our interpolation properties are sufficient conditions for the versions of modularity, in the presence of properties of the relativised consequence relations, which amount to relativisations of the Global Cut Property. The latter property, which CFOL was seen to have in section 4, can be restated as follows.

Property **GC**: If $\Gamma\vdash_I\Delta$ and $\Delta\cup\Sigma\vdash_K\tau$ with I$\subseteq$K, then $\Gamma\cup\Sigma\vdash_K\tau$.

To see that **DI** $\Rightarrow$ **PM** in the presence of the Global Cut Property **GC**, we consider presentations P=$\langle$I,$\Delta\rangle$, Q=$\langle$J,$\Gamma\rangle$ and R=$\langle$K,$\Theta\rangle$, such that P<Q and K$\cap$J$\subseteq$I. Since we wish to show that P$\cup$R<Q$\cup$R, we let L:=J$\cup$K and consider $\tau\in$Sent(I$\cup$K) such that $\Gamma\cup\Theta\vdash_{J\cup K}\tau$. Then $\Gamma\cup\Theta\vdash_L\{\tau\}$ with $\Theta\cup\{\tau\}\subseteq$Sent(I$\cup$K); so, since J$\cup$(I$\cup$K)=L and J$\cap$(I$\cup$K)=I, **DI** gives $\Xi\subseteq$Sent(I) such that $\Gamma\vdash_J\Xi$ and $\Xi\cup\Theta\vdash_K\{\tau\}$. Since P<Q the former yields $\Delta\vdash_I\Xi$, whence **GC** gives $\Delta\cup\Theta\vdash_{I\cup K}\tau$.

To derive **LM** from **SI**, we resort to a special case of **GC** (with $\Sigma=\varnothing$), call it Language Cut:

Property **LC**: If $\Gamma\vdash_I\Delta$ and $\Delta\vdash_K\tau$ with I$\subseteq$K, then $\Gamma\vdash_K\tau$.

Now, to see that **SI** $\Rightarrow$ **LM** in the presence of Property **LC**, we consider presentations P=$\langle$I,$\Delta\rangle$, Q=$\langle$J,$\Gamma\rangle$ and a language K, such that K$\cap$J$\subseteq$I. To establish $\langle$I$\cup$K,$\Delta\rangle$<$\langle$J$\cup$K,$\Gamma\rangle$, we let L:=J$\cup$K and consider $\tau\in$Sent(I$\cup$K) such that $\Gamma\vdash_{J\cup K}\tau$. Then $\Gamma\vdash_L\{\tau\}$ with $\{\tau\}\subseteq$Sent(I$\cup$K); so, since J$\cup$(I$\cup$K)=L and J$\cap$(I$\cup$K)=I, **SI** gives $\Psi\subseteq$Sent(I) such that $\Gamma\vdash_J\Psi$ and $\Psi\vdash_K\{\tau\}$. Since P<Q the former yields $\Delta\vdash_I\Psi$, whence **LC** gives $\Delta\vdash_{I\cup K}\tau$.

Now, to go from **CC** to **AM**, we resort to another special case of **GC** (when the languages are the same), call it Theorem Cut:

Property **TC**: If $\Gamma\vdash_L\Delta$ and $\Delta\cup\Sigma\vdash_L\tau$, then $\Gamma\cup\Sigma\vdash_L\tau$.

To see that **CC** $\Rightarrow$ **AM** in the presence of Property **TC**, we consider presentations $P=\langle I,\Delta\rangle$, $Q=\langle J,\Gamma\rangle$ and $\Theta\subseteq Sent(I)$. To show $\langle I,\Delta\cup\Theta\rangle<\langle J,\Gamma\cup\Theta\rangle$, we consider $\sigma\in Sent(I)$ such that $\Gamma\cup\Theta\vdash_J\sigma$. Then $\Gamma\cup\Theta\vdash_J\{\sigma\}$ with $\Theta\cup\{\sigma\}\subseteq Sent(I)$; so, since $J\cup I=J$, **CC** gives $\Omega\subseteq Sent(I)$ such that $\Gamma\vdash_J\Omega$ and $\Omega\cup\Theta\vdash_I\{\sigma\}$. Since $P<Q$ the former yields $\Delta\vdash_I\Omega$, whence **TC** gives $\Delta\cup\Theta\vdash_I\sigma$.

We shall now show that, conversely, our interpolating-like properties are necessary conditions for the versions of modularity, in the presence of a simple property of $\vdash_L$, which is present in CFOL and many others, namely:

Property **Rf**: If $\Gamma\subseteq\Delta$ then $\Delta\vdash_L\Gamma$.

Reflexivity allows us to establish the usual property of the restriction construction [Shoenfield '67, p. 95, exercise 9]. Given a presentation $S=\langle L,\Sigma\rangle$, by its *restriction* to sub-language $I\subseteq L$ we mean the presentation ${}_IS:=\langle I,{}_ICn_L(\Sigma)\rangle$, where ${}_ICn_L(\Sigma):=\{\chi\in Sent(I)/\Sigma\vdash_L\chi\}$. Clearly ${}_ICn_L(\Sigma)\subseteq Cn_L(\Sigma)$; and reflexivity entails the conservativeness ${}_IS<S$.

To show that **LM** $\Rightarrow$ **SI** in the presence of Reflexivity, consider $Q=\langle J,\Gamma\rangle$ and $R=\langle K,\Sigma\rangle$, and let $L:=J\cup K$ and $I:=K\cap J$. Assuming $\Gamma\vdash_L\Sigma$, we seek a simple interpolating presentation $M=\langle I,\Psi\rangle$. Take M as the restriction ${}_IQ$ to sub-language $I\subseteq L$. By the above remark we have $\Gamma\vdash_J\Psi$ and $\langle I,\Psi\rangle={}_IQ<Q=\langle J,\Gamma\rangle$. Now, since $I\cup K=K$, **LM** applied to the latter yields $\langle K,\Psi\rangle<\langle L,\Gamma\rangle$. Hence $\Psi\vdash_K\Sigma$.

Now, to see that **AM** $\Rightarrow$ **CC** in the presence of **Rf**, consider $Q=\langle J,\Gamma\rangle$ and $R=\langle K,\Theta\rangle$, and let $L:=J\cup K$. Assuming $\Gamma\cup\Theta\vdash_L\Sigma$ with $\Sigma\subseteq Sent(K)$, we seek a converse-cut presentation $C=\langle K,\Omega\rangle$. Consider presentation $S=\langle L,\Gamma\rangle$ and take C as the restriction ${}_KS$ to sub-language $K\subseteq L$. By the above remark we have $\Gamma\vdash_L\Omega$ and $\langle K,\Omega\rangle={}_KS<S=\langle L,\Gamma\rangle$. Now, an application of **AM** to the latter gives $\langle K,\Omega\cup\Theta\rangle<\langle L,\Gamma\cup\Theta\rangle$; whence $\Omega\cup\Theta\vdash_K\Sigma$.

Similarly, to show that **PM** $\Rightarrow$ **DI** in the presence of Reflexivity, consider $Q=\langle J,\Gamma\rangle$ and $R=\langle K,\Theta\rangle$, and let $L:=J\cup K$ and $I:=K\cap J$. Assuming $\Gamma\cup\Theta\vdash_L\Sigma$ with $\Sigma\subseteq Sent(K)$, we seek a distributed interpolating presentation $T:=\langle I,\Xi\rangle$. Take T as the restriction ${}_IQ$ to sub-language $I\subseteq J$. By the above remark we have $\Gamma\vdash_J\Xi$ and $\langle I,\Xi\rangle={}_IQ<Q=\langle J,\Gamma\rangle$. Now, since $I\cup K=K$, **PM** applied to the latter gives $\langle K,\Xi\cup\Theta\rangle<\langle L,\Gamma\cup\Theta\rangle$. Hence $\Xi\cup\Theta\vdash_K\Sigma$.

Therefore, we have the following connections between interpolation and corresponding modularity properties:

$$\mathrm{SI} \underset{\mathbf{Rf}}{\overset{\mathrm{LC}}{\rightleftarrows}} \mathrm{LM} \qquad \mathrm{CC} \underset{\mathbf{Rf}}{\overset{\mathrm{TC}}{\rightleftarrows}} \mathrm{AM} \qquad \mathrm{DI} \underset{\mathbf{Rf}}{\overset{\mathrm{GC}}{\rightleftarrows}} \mathrm{PM}$$

Thus, our interpolating-like properties can be regarded as equivalent - in the presence of some simple properties of $\vdash_L$ - to modularity properties, which concern conservativeness preservation.

5.5 CONDITIONS FOR MODULARITY AND INTERPOLATION

We have seen that, given some simple properties of the relativised consequence relation, our three versions of modularity, namely **LM**, **AM** and **PM**, which guarantee preservation of conservativeness, are equivalent to our three versions of interpolation, namely **SI**, **CC** and **DI**, which guarantee decompositions of derivations in appropriate languages.

We shall now examine more closely the properties we require of our relativised consequence relations, aiming at establishing some general conditions for modularity and interpolation.

Let us first note a property of our notation: if $\Gamma\vdash_L\Sigma$ and $\Gamma\vdash_L\Theta$ then $\Gamma\vdash_L\Sigma\cup\Theta$.

One usually assumes [Tarski '30] transitivity and reflexivity, which refer to a single language:

Transitivity **Tr**: If $\Gamma\vdash_L\Delta$ and $\Delta\vdash_L\tau$, then $\Gamma\vdash_L\tau$.

Reflexivity **Rf**: If $\gamma\in\Gamma$, then $\Gamma\vdash_L\gamma$.

To cope with change of language, we also assume

Language Extension **LE**: If $\Sigma\vdash_I\sigma$ and $I\subseteq J$, then $\Sigma\vdash_J\sigma$.

Notice that a consequence of transitivity and reflexivity is monotonicity **Mon**: if $\Gamma\vdash_L\sigma$ and $\Gamma\subseteq\Delta$ then if $\Delta\vdash_L\sigma$.

We consider these three properties for the following two reasons:

- CFOL exhibits them;
- they are reasonable in view of our idea of language as parameter;
- the Global Cut Property **GC** follows from them.

Indeed, we have the following connections among them:

$$\mathrm{GC} \Rightarrow \left\{ \begin{array}{l} \left.\begin{array}{lcl} \mathrm{LC} & & \mathbf{Rf} \\ \mathrm{TC} & \Rightarrow & \mathrm{Tr} \end{array}\right\} \Rightarrow \mathrm{Mon} \\ \qquad\quad\;\; \mathrm{LE} \end{array} \right\} \Rightarrow \mathrm{GC}$$

They also have some interesting interpretations in terms of presentations.

Transitivity means that $\langle L,\Delta\rangle\subseteq\langle L,\Gamma\rangle$ whenever $\Gamma\vdash_L\Delta$.

Monotonicity means that $\langle L,\Delta\rangle\subseteq\langle L,\Gamma\rangle$ whenever $\Delta\subseteq\Gamma$.

Language Extension means that $\langle I,\Sigma\rangle \subseteq \langle J,\Sigma\rangle$ whenever $I \subseteq J$.

We can now summarise our analysis of conditions for modularity and interpolation in figure 7.

$$\begin{array}{ccccc} PM & \Leftrightarrow & (LM & \& & AM) \\ \Updownarrow & & \Updownarrow & & \Updownarrow \\ DI & \Leftrightarrow & (SI & \& & CC) \end{array}$$

Figure 7: Conditions for Modularity and Interpolation.

Theorem: *Conditions for Modularity and Interpolation*

Consider a relativised consequence relation $\vdash_L$ with the properties Transitivity **Tr**, Reflexivity **Rf** and Language Extension **LE**.

a) $\vdash_L$ has Language Modularity **LM** iff it has Simple Interpolation **SI**.

b) $\vdash_L$ has Axiom Modularity **AM** iff it has Converse Cut **CC**.

c) $\vdash_L$ has Presentation Modularity **PM** iff it has Distributed Interpolation **DI**.

d) $\vdash_L$ has Presentation Modularity **PM** iff it has both Language Modularity **LM** and Axiom Modularity **AM**.

e) $\vdash_L$ has Distributed Interpolation **DI** iff it has both Simple Interpolation **SI** and Converse Cut **CC**.

6. CONCLUSIONS

We have investigated some connections among modularity properties, concerning preservation of conservativeness, and interpolation-like properties of the consequence relation, aiming at clarifying their roles.

The motivations for this investigation stem from two main sources in logic and in software development. From the logical side, we have known connections between modularity-like results (such as Robinson's Joint Consistency) and Craig's Interpolation theorem. From the standpoint of formal approach to program and specification development, modularity of interpretations is a crucial property in composing implementations and in instantiating parameterised specifications. Also, the plurality of formalisms prevalent in software development suggests the desirability of a connective-independent analysis, somewhat akin to Tarski's classical work on the consequence relation [Tarski '30], without, however, keeping the language fixed.

A crucial idea in connecting modularity of extensions to that of interpretations is resorting internalisation techniques, which code (part of) the information of language translations. We have examined two internalisation constructions: kernel using sentences of the source language, and (graphic) diagram employing sentences of the disjoint-union language. Let us now comment on some possible extensions of these techniques, including the many-sorted case, which is important for software specification and development.

Sometimes one considers interpretations mapping symbols to formulae, which define the translations in the target presentation. Our constructions are not affected by this extension, because it can be reduced to an extension by definitions.

Another version considers interpretations with relativisation predicates. This can be handled as in the many-sorted case. In the many-sorted case, one sometimes consider a translation t mapping a sort s of source language I to a sequence $s_1,..,s_k$ of sorts, together with a relativisation predicate r) of target language K [Turski + Maibaum '87]. In this case, the construction of the translation diagram $\Delta[t]$ is adapted to sentences stating that s is the subsort, defined by the relativisation predicate r, of the product of sorts $s_1,..,s_k$ [Veloso '93]. This can be done by adding axioms characterising the introduction of product sorts and subsorts in a definition-like manner [Meré + Veloso '92, '94].

We have examined these properties and some connections among them, aiming at a clarification of their role. This investigation has provided general conditions and alternative formulations for modularity and these logical properties of the consequence relation.

Interpolation properties can be formulated, perhaps more intuitively, as preservation of conservativeness. Also, some principles, akin to the Deduction Theorem, can be given global, connective-independent formulations resembling interpolation, and thus equivalent to conservativeness preservation.

Our results hinge on some simple, albeit language-dependent, properties of the consequence relation, extending those investigated by [Tarski '30] to the cases of extensions and interpretations. We do not, however, require any finiteness property like compactness. (Indeed, compactness was relied upon only in arguing that our global formulations encompass the familiar ones in Classical First-Order Logic.) We also suggested that compactness, and related finiteness properties, can be regarded as versions of interpolation. This may pave the way for examining more closely connections between our global versions and finitary formulations as well as versions with interpolant sentences.

This investigation has favoured a connective-independent viewpoint, so it is natural to conceive extending it to a framework in the spirit of abstract model theory or of categorical logic. Our very notation $\Gamma \vdash \Theta$ for $\Theta \subseteq Cn(\Gamma)$ suggests considering categories whose objects are such sets of sentences, morphisms being (sets of) derivations, or appropriate equivalence classes. Preliminary efforts in the direction of institutions are underway.

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