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From Extensions to Interpretations: Pushout Consistency, Modularity and Interpolation

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FROM EXTENSIONS TO INTERPRETATIONS:
Pushout Consistency, Modularity and Interpolation

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Abstract. We generalise three known results concerning (conservative) extensions to (faithful) interpretations. These results are Extension Modularity (a special case of the Modularisation Theorem for logical specifications) and two familiar logical theorems, namely Robinson's Joint Consistency and Craig-Robinson Interpolation. Their generalisations involve a pushout construction, in lieu of union, and their proofs rely on internalisation techniques, including a novel one, which reduce - to a large extent - interpretations to extensions.

Key words: Formal specifications, interpretations, translations, extensions, consistency, modularity, interpolation, conservative extension, faithful interpretation, internalisation, abstract data types, pushout, coequaliser, software engineering.

Resumo. Três resultados conhecidos sobre extensões (conservativas) são generalizados a interpretações (fiéis). Estes resultados são Modularidade de Extensões (um caso particular do Teorema da Modularização para especificações lógicas) e dois teoremas lógicos, a saber Consistência Conjunta de Robinson e Interpolação de Craig-Robinson. Suas generalizações envolvem uma construção de soma amalgamada (pushout), ao invés de união, e suas demonstrações se baseiam em técnicas de internalização, incluindo uma nova, as quais reduzem - em boa parte - interpretações a extensões.

Palavras chave: Especificações formais, interpretações, traduções, extensões, consistência, modularidade, interpolação, extensão conservativa, interpretação fiel, tipos abstratos de dados, soma amalgamada (pushout), coequalizador, engenharia de software.

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1. INTRODUCTION

We generalise three known results concerning extensions to interpretations. These results are two familiar logical theorems, namely Robinson's Joint Consistency and Craig-Robinson Interpolation, and Extension Modularity (a special case of the Modularisation Theorem for logical specifications). Their generalisations involve a pushout construction, in lieu of union, and their proofs rely on internalisation techniques, including a novel one.

Now, Craig Interpolation Lemma and Robinson's Joint Consistency Theorem are known to be tightly connected; both concern extensions and unions of theories. The Modularisation Theorem is a fundamental result for the logical approach to formal specifications and it involves both extensions and interpretations. It is quite natural to consider the generalisation of these results obtained by replacing (conservative) extensions by (faithful) interpretations. We shall examine these generalised versions and their proofs.

1.1 Motivation

The Modularisation Theorem is a fundamental result for the logical approach to formal specifications [Maibaum & Veloso '81; Maibaum et al. '84; Veloso et al. '85]. It provides the basis for composition of implementation steps and for instantiation of parameterised specifications [Maibaum et al. '91; Maibaum & Veloso '95]. Its proofs involve (some version of) Craig Interpolation Lemma as well as internalisation techniques [Veloso '92, '93; Maibaum & Veloso '95].

Now, Craig Interpolation Lemma is known to have tight connections with Robinson's Joint Consistency Theorem [Barwise '77; Chang & Keisler, '73; Shoenfield '67]. The latter is a tool for guaranteeing consistency: it asserts that consistency is preserved under union over a maximally consistent theory; it guarantees a property - consistency - of the union theory provided that the given extensions have the property of being consistent. On the other hand, the Modularisation Theorem is a tool for conservativeness preservation. A special case of it, Modularity of Extensions, is similar to Robinson's Joint Consistency Theorem: it asserts that conservativeness is preserved by over a theory; it guarantees a relationship - conservativeness - between the resulting extensions provided that the given extensions have the relationship of being conservative [Veloso '95].

But, whereas both Craig Interpolation Lemma and Robinson's Joint Consistency Theorem concern extensions, the Modularisation Theorem involves both extensions and interpretations. The latter deals with a pushout rectangle of extensions and interpretations. A natural generalisation of the Modularisation Theorem arises by replacing (conservative) extensions by (faithful) interpretations. The union construction underlying Robinson's Joint Consistency and Craig-Robinson Interpolation is a special pushout: when the interpretations turn out to be extensions [Arbib & Mannes '75; Goldblatt '79]. It is thus natural to consider similar generalisations for them: by replacing extensions by interpretations.

We thus arrive at generalisations based on a pushout rectangle of underlying language translations. We shall establish these three generalisations by relying on some simple properties of faithful interpretations as well as on some internalisation techniques. These internalisation techniques will enable coding the construction of the pushout into sentences of appropriate languages.

These generalised versions concern theories over languages in a pushout rectangle of language translations. They are roughly as follows. Pushout Consistency asserts that the pushout of consistent interpretations over a maximally consistent theory is consistent. Pushout Modularity guarantees that the pushout construction preserves faithfulness. Pushout Interpolation provides interpolating sentences decomposing derivations of certain sentences in the pushout theory into derivations in the given theories.

We shall generally use standard logical terminology and notation. We will briefly review some concepts and notation before stating the generalised versions of the results.

1.2 Preliminaries

We employ the usual terminology and notation for logical concepts [Enderton, '72; van Dalen '89; Shoenfield '67; Chang & Keisler, '73]. We use the notations $\Gamma \models \sigma$, or $\sigma \in Cn(\Gamma)$, to state that sentence σ is a (logical) *consequence* of the set Γ of sentences, i. e. $Mod(\Gamma) \subseteq Mod(\sigma)$.

We consider a *language* as characterised by its *alphabet* of extra-logical (predicate and function) symbols together with syntactical declarations. A *presentation* is a pair $S = \langle L, \Gamma \rangle$ consisting of a language L and a set Γ of sentences of L (its *axioms*), generating its *theory* $Cn(S) := \{\sigma \in Snt(L) / \Gamma \models \sigma\}$.

We say that I is a *sub-language* of J (denoted by $I \subseteq J$) when J can be obtained from I by adding some symbols (and declarations). Now, consider presentations $P = \langle I, \Gamma \rangle$ and $Q = \langle J, \Sigma \rangle$. We say that Q is an *extension* of P (denoted by $P \subseteq Q$) iff $I \subseteq J$ and every consequence of Γ is a consequence of Σ (we also say that P is a *sub-presentation* of Q). A *conservative extension* $P \leq Q$ is an extension $P \subseteq Q$ such that, for every sentence σ of I, $\Gamma \models \sigma$ iff $\Sigma \models \sigma$. We call P and Q *equivalent* (denoted by $P \equiv Q$) iff they are extensions of each other.

By a *translation* t from source language I to target language K we mean a syntax-preserving language morphism (denoted by $t: I \to K$) mapping each symbol of I to a corresponding symbol in K of the same kind. This mapping induces translation of formulae: each formula $\varphi \in Frml(I)$ is translated to a formula $t(\varphi)$ of K.

Now, an *interpretation* from $P=\langle I,\Gamma\rangle$ to $R=\langle K,\Theta\rangle$ is a translation of the underlying languages that translates every consequence of Γ to a consequence of Θ; we also say that translation $t:I\rightarrow K$ *interprets* P into R (denoted by $t:P\rightarrow R$). The analogue of conservativeness for translations is faithfulness: an interpretation $t:P\rightarrow R$ is called *faithful* iff for every sentence σ of I, $\Gamma\vDash\sigma$ iff $\Theta\vDash t(\sigma)$.

Extensions are special interpretations for sub-languages. Given $P=\langle I,\Gamma\rangle$ and $R=\langle K,\Theta\rangle$, if $I\subseteq J$ then we have an insertion $j:I\rightarrow K$. Thus, $P\subseteq R$ iff the insertion $j:I\rightarrow K$ interprets P into R, and extension $P\subseteq R$ is conservative iff $j:I\rightarrow K$ is faithful.

We recall the Interpretation Theorem [Shoenfield '67, p. 62; Turski & Maibaum '87, p. 85]. It asserts that translation $t:I\rightarrow K$ interprets $P=\langle I,\Gamma\rangle$ into $R=\langle K,\Theta\rangle$ iff t translates every axiom $\gamma\in\Gamma$ to a theorem $t(\gamma)\in Cn(\Theta)$. Also recall that extensions by definitions of (predicate and function) symbols are always conservative [Enderton, '72; van Dalen '89; Shoenfield '67; Veloso & Maibaum '94].

1.3 Overview

Let us now state the basic results which we wish to generalise, namely Craig Interpolation, Robinson's Joint Consistency and the Modularisation Theorem.

Craig Interpolation appears in a few versions in the literature [Shoenfield '67; Chang and Keisler '73], which turn out to be interderivable in Classical First-Order Logic. One of these versions is sometimes called split interpolation [Rodenburg and van Glabbeek '88]. This is similar to the following usual formulation of the Craig-Robinson Interpolation Theorem [Shoenfield '67, p. 80]: "Given presentations $Q=\langle J,\Sigma\rangle$ and $R=\langle K,\Theta\rangle$, for each sentence $\sigma\rightarrow\tau$, with σ in language J and τ in K, such that $\Sigma\cup\Theta\vDash\sigma\rightarrow\tau$, there exists an interpolant sentence ρ of the intersection language $J\cap K$, such that $\Sigma\vDash\sigma\rightarrow\rho$ and $\Theta\vDash\rho\rightarrow\tau$".

Robinson's Joint Consistency Theorem (RJC) concerns unions of consistent presentations over a maximally consistent presentation. It asserts [Chang and Keisler '73, p. 88; Shoenfield '67, p. 79]: "Given consistent presentations $Q=\langle J,\Sigma\rangle$ and $R=\langle K,\Theta\rangle$, both extending a maximally consistent $P=\langle I,\Gamma\rangle$, with language I being the intersection of J and K, then the union presentation $\langle J\cup K,\Sigma\cup\Theta\rangle$ is consistent as well".

The Modularisation Construction deals with the situation where one has presentations $P=\langle I,\Gamma\rangle$, $Q=\langle J,\Sigma\rangle$ and $R=\langle K,\Theta\rangle$, with Q extending P and an interpretation $f:P\rightarrow R$. It completes a rectangle of interpretations by amalgamated sum, thereby producing an interpretation g of Q into the (pushout) presentation $S=\langle L,g(\Sigma)\cup\Theta\rangle$. The Modularisation Theorem guarantees that this Modularisation Construction preserves conservativeness: "if Q is a conservative extension of P, then S is a conservative extension of R" [Veloso '92, '93; Maibaum & Veloso '95].

A natural generalisation of the Modularisation Theorem arises by replacing (conservative) extensions by (faithful) interpretations. One can similarly generalise the related results Robinson's Joint Consistency and Craig-Robinson Interpolation. By considering a pushout rectangle of language translations $e:I\rightarrow J$ and $f:I\rightarrow K$, yielding $g:J\rightarrow L$ and $h:K\rightarrow L$ [Ehrich '82; Ehrig & Mahr '85], we arrive at the following formulations.

PoC (*Pushout Consistency*):

If maximally consistent presentation $P=\langle I,\Gamma\rangle$ is interpreted by e into consistent $Q=\langle J,\Sigma\rangle$ and interpreted by f into consistent $R=\langle K,\Theta\rangle$, then the (pushout) presentation $\langle L,g(\Sigma)\cup h(\Theta)\rangle$ is consistent as well.

PoM (*Pushout Modularity*):

If presentation $P=\langle I,\Gamma\rangle$ is interpreted by f into presentation $R=\langle K,\Theta\rangle$ and faithfully interpreted by e into presentation $Q=\langle J,\Sigma\rangle$, then the interpretation h of presentation $R=\langle K,\Theta\rangle$ into the (pushout) presentation $\langle L,g(\Sigma)\cup h(\Theta)\rangle$ is faithful as well.

PoI (*Pushout Interpolation*):

For presentations $Q=\langle J,\Sigma\rangle$ and $R=\langle K,\Theta\rangle$, given sentences σ of language J and τ of language K such that sentence $g(\sigma)\rightarrow h(\tau)$ of L is consequence of $g(\Sigma)\cup h(\Theta)$, there exists a po-interpolant sentence ρ of language I, such that $\Sigma\models\sigma\rightarrow e(\rho)$ and $\Theta\models f(\rho)\rightarrow\tau$.

We shall establish these three generalisations. The plan is as follows: by relying on some simple properties of faithful interpretations,

1. PoC will follow from the usual RJC, by internalisation techniques;
2. From PoC we derive POM, by means of pre-image;
3. From PoM we derive POI, by pre-image and compactness.

(It would not be difficult to derive PoM from PoI and PoC from PoM.)

This plan involves some auxiliary constructions and properties. The internalisation techniques will enable coding information about the translations into sentences of appropriate languages.

The structure of this paper is as follows. In the next section we present some internalisation constructions and simple properties of (faithful) interpretations, leaving the details for section 4. In section 3 we establish our main results: Pushout Consistency, Modularity and Interpolation. In section 4 we examine more closely the auxiliary constructions and properties providing proofs; we consider pre-image and faithfulness in 4.1 and in 4.2 we turn to our internalisation techniques: diagram, kernel and coequaliser internalisations.

2. Auxiliary Constructions and Properties

We shall use some auxiliary constructions and properties concerning internalisation and (faithful) interpretations. We will now briefly outline them, leaving the details for later (see section 4).

2.1 Internalisations and their Properties

We wish to reduce translations to extensions by coding (part of) the translation information into sentences of an appropriate language. Two kinds of such internalisations of translation are diagram and kernel internalisation [Veloso '92, '93; Maibaum & Veloso '95], and a third one will internalise pairs of translations with a common source. Kernel internalisation relies on the source language, whereas the other two construct an appropriate coproduct language.

Given a translation $t:I \to K$, we form language $L[t]$ as the coproduct $I+K$. We can then 'internalise' the translation by coding its information into matching sentences $(i \leftrightarrow t(i))$ of $L[t]$ expressing that each source symbol i is equivalent to its translation $t(i)$. By the *diagram axiomatisation* of translation $t:I \to K$ we mean the set $\Delta[t]$ of such matching sentences $(i \leftrightarrow t(i))$ of language $L[t]=I+K$ [Veloso '93, '95].

Some basic properties of diagram internalisation are given in the next result, showing that the diagram axiomatisation of a translation indeed codes the information in it.

Proposition *Diagram Internalisation* (Di)

Consider a translation $t:I \to K$ with diagram axiomatisation $\Delta[t]$, and a presentation $R=\langle K,\Theta\rangle$ with diagram extension $\Delta[t] \cup R := \langle L[t], \Delta[t] \cup \Theta\rangle$.

a) The diagram extension is a conservative extension: $R \leq \Delta[t] \cup R$.

b) Consider a presentation $P=\langle I,\Gamma\rangle$. Then:
   (i) t interprets P into R iff $\Delta[t] \cup R$ extends P, i. e. $P \sqsubseteq \Delta[t] \cup R$;
   (ii) t interprets P faithfully into R iff $\Delta[t] \cup R$ extends P conservatively, i. e. $P \leq \Delta[t] \cup R$.

A translation $t:I \to K$ maps source to target symbols; as such it defines an equivalence relation $ker(t) := \{\langle i_1,i_2\rangle : t(i_1)=t(i_2)\}$ on the source alphabet. This information can be 'internalised' within the source language by means of the set $\Lambda[t] := \{(i_1 \leftrightarrow i_2) : \langle i_1,i_2\rangle \in ker(t)\}$ of identifying sentences of I, expressing the equivalence of source symbols with the same translation. We call this set $\Lambda[t]$ the (*internalised*) *kernel* of translation $t:I \to K$ [Veloso '92; Maibaum & Veloso '95].

The kernel internalisation of a translation provides part of the information given by the mapping, namely which symbols have the same translation, without providing the translations themselves. But this information is already sufficient to characterise faithfulness into the image, as the next result shows.

Proposition *(Internalised) Kernel* (iK)

Consider a translation $t:I \to K$ with (internalised) kernel $\Lambda[t]$. Given a presentation $P = \langle I, \Gamma \rangle$, interpretation t of $P = \langle I, \Gamma \rangle$ into the image presentation $t(P) := \langle K, t(\Gamma) \rangle$ is faithful iff the (internalised) kernel $\Lambda[t]$ consists of consequences of Γ, i. e. $\Lambda[t] \subseteq Cn(\Gamma)$.

Pushouts can be constructed by means of coproducts and coequalisers [Arbib & Mannes '75; Goldblatt '79]. A combination of the preceding ideas suggests how to internalise the coequaliser: by the equivalence between the two translations of each common source symbol. Given translations $e:I \to J$ and $f:I \to K$ with common source language I, we form language L[e,f] as the coproduct J+K. We can then code the coequaliser by means of sentences $(e(i) \leftrightarrow f(i))$ of L[e,f] expressing the equivalence of the translations of a source symbol. By the *(internalised) coequaliser* of translations $e:I \to J$ and $f:I \to K$ we mean the set $\Omega[e,f]$ of such sentences $(e(i) \leftrightarrow f(i))$ of the coproduct language L[e,f]=J+K.

Some basic properties of (internalised) coequaliser are given in the following results, showing that this construction achieves its aim of coding the pushout construction.

The next lemma connects diagram internalisation with (internalised) coequaliser, showing that the information for the latter is provided by the diagram axiomatisations of its mappings.

Lemma *(Internalised) Coequaliser and Diagram Axiomatisations* (CD)

Consider translations $e:I \to J$ and $f:I \to K$, with respective diagram axiomatisations $\Delta[e]$ and $\Delta[f]$, and (internalised) coequaliser $\Omega[e,f]$. Then the (internalised) coequaliser $\Omega[e,f]$ consists of consequences of $\Delta[e] \cup \Delta[f]$: $\Omega[e,f] \subseteq Cn(\Delta[e] \cup \Delta[f])$.

The next result connects (internalised) coequaliser with the pushout language and indicates that the former codes the information for constructing the latter. In its formulation we employ the usual concept of coproduct mediator [Arbib & Mannes '75; Goldblatt '79].

Proposition *(Internalised) Coequaliser and Pushout Mediator* (CM)

Given a pushout rectangle of language translations $e:I \to J$ and $f:I \to K$, yielding $g:J \to L$ and $h:K \to L$, consider the (internalised) coequaliser $\Omega[e,f]$ of $e:I \to J$ and $f:I \to K$ and the mediator $(g|h):J+K \to L$ of their pushouts $g:J \to L$ and $h:K \to L$.

a) The (internalised) kernel $\Lambda[(g|h)]$ consists of consequences of the (internalised) coequaliser $\Omega[e,f]$: $\Lambda[(g|h)] \subseteq Cn(\Omega[e,f])$.

b) The mediator (g|h) interprets $\langle J+K, \Omega[e,f] \rangle$ faithfully into $\langle L, \varnothing \rangle$.

2.2 Pre-images and faithful interpretations

We shall also employ some simple properties of pre-images and of faithful interpretations.

Given a translation $t:I \to K$ and a presentation $R=\langle K,\Theta \rangle$, by the *pre-image* of R under t we mean the presentation ${}_tR:=\langle I,{}_t\Theta \rangle$, where ${}_t\Theta:=t^{-1}[Cn(\Theta)]$ consists of those sentences σ of language I such that $t(\sigma)\in Cn(\Theta)$.

The pre-image is similar to the restriction construction [Shoenfield '67, p. 95, exercise 9]. The next lemma gives the basic property of pre-image: faithful interpretation into the originating presentation.

Lemma *Pre-image and faithfulness* (Rf)

A translation $t:I \to K$ interprets the pre-image ${}_tR=\langle I,{}_t\Theta \rangle$ of presentation $R=\langle K,\Theta \rangle$ under t faithfully into R.

The following result gives some characterisations of faithfulness in terms of their behaviour with respect to the consistent addition of new axioms. They amount to simple generalisations of corresponding characterisations for conservativeness [Veloso & Maibaum '94].

Lemma *Characterisation of faithfulness* (cF)

Consider a translation $t:I \to K$ as well as presentations $P=\langle I,\Gamma \rangle$ and $R=\langle K,\Theta \rangle$, such that t interprets P into R. Then, the following are equivalent.

a) Interpretation $t:P \to R$ is faithful.

b) For every $\Delta \subseteq Snt(I)$, t interprets $\langle I,\Gamma \cup \Delta \rangle$ faithfully into $\langle K,\Theta \cup t(\Delta) \rangle$.

c) Presentation $\langle K,\Theta \cup t(\Delta) \rangle$ is consistent whenever $\langle I,\Gamma \cup \Delta \rangle$ is so.

d) For any maximally consistent presentation $\langle I,\Xi \rangle$ extending $P=\langle I,\Gamma \rangle$, $\langle K,\Theta \cup t(\Xi) \rangle$ is consistent.

We can now state some simple properties of faithful interpretations concerning consistency and completeness of the presentations involved.

Corollary *Properties of faithful interpretations* (pF)

Consider a translation $t:I \to K$ as well as presentations $P=\langle I,\Gamma \rangle$ and $R=\langle K,\Theta \rangle$, such that t interprets P faithfully into R.

a) Presentation $P=\langle I,\Gamma \rangle$ is consistent iff $R=\langle K,\Theta \rangle$ is so.

b) If $R=\langle K,\Theta \rangle$ is complete then so is $P=\langle I,\Gamma \rangle$.

Conservative extensions are special cases of faithful interpretations. Thus, these results have simple analogues for conservative extensions.

This completes our presentation of the auxiliary constructions and properties we will need. Most of these basic ideas and results are simple. We shall provide some more details about them in section 4.

3. The Main Results: Consistency, Modularity and Interpolation

We now establish the three generalisations mentioned in the introduction. We first generalise Robinson's Joint Consistency Theorem

(RJC) to Pushout Consistency (PoC), then derive from it Pushout Modularity (PoM), which finally will yield Pushout Interpolation (PoI).

To establish the generalisation of Robinson's Joint Consistency to pushouts, we use the following facts: the alphabet of pushout language L is the quotient of the coproduct of the alphabets of J and K modulo the equivalence relation r* generated by $r := \{<j,k> / \text{ for some } i \text{ in } I\ e(i)=j\ \&\ k=f(i)\}$; and the coproduct mediator (g|h) is the natural projection from J+K onto L, whose kernel is r* [Arbib & Mannes '75; Goldblatt '79; Ehrich '82; Ehrig & Mahr '85]. The internalisation techniques will enable us to code such information into sentences of appropriate languages.

Theorem *Pushout Consistency* (PoC)

Consider a pushout rectangle of language translations $e:I \rightarrow J$ and $f:I \rightarrow K$, yielding $g:J \rightarrow L$ and $h:K \rightarrow L$. Assume consistent presentations $P=\langle I,\Gamma\rangle$, $Q=\langle J,\Sigma\rangle$ and $R=\langle K,\Theta\rangle$, such that e interprets $P=\langle I,\Gamma\rangle$ into $Q=\langle J,\Sigma\rangle$ and f interprets $P=\langle I,\Gamma\rangle$ into $R=\langle K,\Theta\rangle$. If presentation $P=\langle I,\Gamma\rangle$ is complete, then the (pushout) presentation $\langle L,g(\Sigma)\cup h(\Theta)\rangle$ is consistent.

Proof

The argument consists of two major phases, both using properties of faithful interpretations. We first use diagram internalisation and the usual Robinson's Joint Consistency Theorem to construct a consistent presentation T (steps 1 and 2). We then use coequaliser internalisation to argue that a sub-presentation of T is faithfully interpreted onto the pushout presentation (steps 3 to 5).

1. Since e interprets $P=\langle I,\Gamma\rangle$ into $Q=\langle J,\Sigma\rangle$, by proposition (Di) on diagram internalisation we have $\langle I,\Gamma\rangle=P\subseteq\Delta[e]\cup Q\geq Q=\langle J,\Sigma\rangle$, where $\Delta[e]\cup Q$ is the diagram extension $\langle L[e],\Delta[e]\cup\Sigma\rangle$. Thus, since $Q=\langle J,\Sigma\rangle$ is consistent, so is its conservative extension $\Delta[e]\cup Q$, by item (a) of the corollary (pF) on properties of faithful interpretations.
   Similarly, since f interprets $P=\langle I,\Gamma\rangle$ into consistent $R=\langle K,\Theta\rangle$, we have a consistent diagram extension $\Delta[f]\cup R=\langle L[f],\Delta[f]\cup\Theta\rangle$ extending $P=\langle I,\Gamma\rangle$.
2. Now, since we may assume J and K to share no symbols, except for those of I, the familiar Robinson's Joint Consistency Theorem yields the consistency of the union presentation $\langle L[e]\cup L[f],\Sigma\cup\Delta[e]\cup\Delta[f]\cup\Theta\rangle$.
3. Notice that $L[e,f]=J+K\subseteq I\cup(J+K)=L[e]\cup L[f]$. So, lemma (CD) on (internalised) coequaliser and diagram axiomatisations yields $\langle L[e,f],\Sigma\cup\Omega[e,f]\cup\Theta\rangle\subseteq\langle L[e]\cup L[f],\Sigma\cup\Delta[e]\cup\Delta[f]\cup\Theta\rangle$, whence $\langle L[e,f],\Sigma\cup\Omega[e,f]\cup\Theta\rangle$ is consistent as well.
4. Now, by proposition (CM) on (internalised) coequaliser and pushout mediator, (g|h) interprets $\langle L[e,f],\Omega[e,f]\rangle$ faithfully into $\langle L,\varnothing\rangle$. Thus, by the lemma (cF) characterising faithfulness, (g|h) interprets consistent $\langle L[e,f],\Sigma\cup\Omega[e,f]\cup\Theta\rangle$ faithfully into $\langle L,(g|h)(\Sigma\cup\Theta)\rangle$.

5 Therefore, by item (a) of the corollary (pF) on properties of faithful interpretations, $\langle L, g(\Sigma) \cup h(\Theta) \rangle = \langle L, (g|h)(\Sigma \cup \Theta) \rangle$ is consistent.

QED

We now establish the generalisation of the Modularisation Theorem to interpretations: pushouts preserve faithfulness. We will derive it from Pushout Consistency PoC by relying on our simple properties of pre-image and of faithful interpretations.

Theorem *Pushout Modularity* (PoM)

Consider a pushout rectangle of language translations $e: I \rightarrow J$ and $f: I \rightarrow K$, yielding $g: J \rightarrow L$ and $h: K \rightarrow L$. Assume presentations $P = \langle I, \Gamma \rangle$, $Q = \langle J, \Sigma \rangle$ and $R = \langle K, \Theta \rangle$, such that e interprets $P = \langle I, \Gamma \rangle$ into $Q = \langle J, \Sigma \rangle$ and f interprets $P = \langle I, \Gamma \rangle$ into $R = \langle K, \Theta \rangle$. If interpretation e of P into Q is faithful, then so is interpretation h of $R = \langle K, \Theta \rangle$ into $\langle L, g(\Sigma) \cup h(\Theta) \rangle$.

Proof

By the lemma (cF) characterising faithfulness, it suffices to show that $\langle L, g(\Sigma) \cup h(\Theta) \cup h(\Xi) \rangle$ is consistent whenever $\langle K, \Xi \rangle$ is a maximally consistent extension of $R = \langle K, \Theta \rangle$. So, consider a maximally consistent extension $\underline{R}$ of R, and form its pre-image ${}_f\underline{R} = \langle I, {}_f\Xi \rangle$ under $f: I \rightarrow K$.

1. By the lemma (Rf) on pre-image and faithfulness, we have a faithful interpretation $f: {}_f\underline{R} \rightarrow \underline{R}$. Thus, by the corollary (pF) on properties of faithful interpretations, ${}_f\underline{R} = \langle I, {}_f\Xi \rangle$ is maximally consistent.
2. Since $\underline{R}$ extends R and $f: P \rightarrow R$, ${}_f\underline{R} = \langle I, {}_f\Xi \rangle$ extends $P = \langle I, \Gamma \rangle$. Thus, ${}_f\underline{R} = \langle I, {}_f\Xi \rangle$ is a maximally consistent extension of $P = \langle I, \Gamma \rangle$.
3. Clearly e interprets ${}_f\underline{R} = \langle I, {}_f\Xi \rangle$ into $\langle J, \Sigma \cup e({}_f\Xi) \rangle$. Since e interprets $P = \langle I, \Gamma \rangle$ faithfully into $Q = \langle J, \Sigma \rangle$, the lemma (cF) characterising faithfulness shows that $\langle J, \Sigma \cup e({}_f\Xi) \rangle$ is consistent.
4. Therefore, Pushout Consistency PoC yields the consistency of $\langle L, g[\Sigma \cup e({}_f\Xi)] \cup h(\Xi) \rangle$, and hence that of the sub-presentation $\langle L, g(\Sigma) \cup h(\Theta) \cup h(\Xi) \rangle \subseteq \langle L, g[\Sigma \cup e({}_f\Xi)] \cup h(\Xi) \rangle$.

QED

We now establish the generalisation of Craig-Robinson Interpolation to pushouts. Pushout Modularity PoC together with simple properties of pre-image will yield a po-interpolant presentation, from which compactness extracts a po-interpolant sentence.

Theorem *Pushout Interpolation* (PoI)

Consider a pushout rectangle of language translations $e: I \rightarrow J$ and $f: I \rightarrow K$, yielding $g: J \rightarrow L$ and $h: K \rightarrow L$, as well as presentations $Q = \langle J, \Sigma \rangle$ and $R = \langle K, \Theta \rangle$. Given sentences σ of language J and τ of language K such that sentence $g(\sigma)\rightarrow h(\tau)$ of L is consequence of $g(\Sigma)\cup h(\Theta)$, there exists a po-interpolant sentence ρ of language I, such that $\Sigma\models\sigma\rightarrow e(\rho)$ and $\Theta\models f(\rho)\rightarrow\tau$.

Proof

The argument consists of two parts. We first construct an auxiliary presentation P by pre-image and use properties of faithful interpretations to show that it behaves as a po-interpolant presentation (steps 1 to 4). Then, compactness and the Deduction Theorem extract from P a po-interpolant sentence (step 5).

1. Let $P=\langle I,\Psi\rangle$ be the pre-image of $Q\cup\{\sigma\}=\langle J,\Sigma\cup\{\sigma\}\rangle$ under $e:I\rightarrow J$. By the lemma (Rf) on pre-image and faithfulness, e interprets P faithfully into $Q\cup\{\sigma\}$. Hence $e(\Psi)\subseteq Cn(\Sigma\cup\{\sigma\})$.
2. Clearly, f interprets $P=\langle I,\Psi\rangle$ into $\langle K,\Theta\cup f(\Psi)\rangle$.
   Thus, by Pushout Modularity PoM, h interprets $\langle K,\Theta\cup f(\Psi)\rangle$ faithfully into $\langle L,g(\Sigma\cup\{\sigma\})\cup h[\Theta\cup f(\Psi)]\rangle=\langle L,g(\Sigma\cup\{\sigma\})\cup h(\Theta)\cup h[f(\Psi)]\rangle$.
3. But, since Ψ consists of sentences of I, $h[f(\Psi)]=g[e(\Psi)]$.
   Thus, by 1, $h[f(\Psi)]\subseteq g[Cn(\Sigma\cup\{\sigma\})]$, and we have the equivalence $\langle L,g(\Sigma\cup\{\sigma\})\cup h(\Theta)\cup h[f(\Psi)]\rangle\equiv\langle L,g(\Sigma)\cup\{g(\sigma)\}\cup h(\Theta)\rangle$.
   Hence, h interprets $\langle K,\Theta\cup f(\Psi)\rangle$ faithfully into $\langle L,g(\Sigma)\cup\{g(\sigma)\}\cup h(\Theta)\rangle$.
4. Now, assume that $g(\Sigma)\cup h(\Theta)\models g(\sigma)\rightarrow h(\tau)$.
   Then, $g(\Sigma)\cup\{g(\sigma)\}\cup h(\Theta)\models h(\tau)$, whence, by 4, $\Theta\cup f(\Psi)\models\tau$.
5. Then, by compactness, there exist sentences $\rho_1,\dots,\rho_k\in\Psi$, such that $\Theta\cup\{f(\rho_1),\dots,f(\rho_k)\}\models\tau$. Letting ρ be the conjunction of $\rho_1,\dots,\rho_k$, we have $\Psi\models\rho$ and $\Theta\cup\{f(\rho)\}\models\tau$. The Deduction Theorem applied to the latter yields $\Theta\models f(\rho)\rightarrow\tau$. On the other hand, in view of 1, the former yields $\Sigma\cup\{\sigma\}\models e(\rho)$, whence $\Sigma\models\sigma\rightarrow e(\rho)$.

QED

4. The Auxiliary Properties and Constructions

We shall now examine more closely the auxiliary properties and constructions employed in establishing our generalisations. We first deal with the simple properties of pre-images and faithful interpretations and then give more details concerning the internalisation constructions.

4.1 Interpretations: pre-images and faithfulness

We now prove the simple properties of pre-images and of faithful interpretations.

First, recall the concept of pre-image. By the *pre-image* of presentation $R=\langle K,\Theta\rangle$ under translation $t:I\rightarrow K$ we mean the presentation ${}_tR:=\langle I,{}_t\Theta\rangle$, where ${}_t\Theta:=t^{-1}[Cn(\Theta)]$ is the set $\{\sigma\in Snt(I)/\Theta\models\sigma\}$.

Lemma *Pre-image and faithfulness* (Rf)

A translation $t{:}I \rightarrow K$ interprets the pre-image ${}_tR = \langle I, {}_t\Theta \rangle$ of $R = \langle K, \Theta \rangle$ under t faithfully into R.

Proof

By definition, for any sentence $\sigma \in Snt(I)$, we have $\sigma \in {}_t\Theta$ iff $t(\sigma) \in Cn(\Theta)$. Therefore, $\Theta \models t(\sigma)$ iff $\sigma \in {}_t\Theta$, whence t interprets ${}_tR$ faithfully into R.

QED

We now consider the properties of faithful interpretations. In the next lemma characterising faithfulness we add a model-theoretical characterisation, which is akin to the one for conservative extensions in terms of elementary substructures [Shoenfield, 1967; p. 95, exercise 9].

Lemma *Characterisation of faithfulness* (cF)

Consider a translation $t{:}I \rightarrow K$ as well as presentations $P = \langle I, \Gamma \rangle$ and $R = \langle K, \Theta \rangle$, such that t interprets P into R. Then, the following are equivalent.

a) Interpretation $t{:}P \rightarrow R$ is faithful.

b) For every $\Delta \subseteq Snt(I)$, t interprets $\langle I, \Gamma \cup \Delta \rangle$ faithfully into $\langle K, \Theta \cup t(\Delta) \rangle$.

c) Presentation $\langle K, \Theta \cup t(\Delta) \rangle$ is consistent whenever $\langle I, \Gamma \cup \Delta \rangle$ is so.

d) For any maximally consistent presentation $\langle I, \Xi \rangle$ extending $P = \langle I, \Gamma \rangle$, $\langle K, \Theta \cup t(\Xi) \rangle$ is consistent.

e) For every model $\mathcal{A} \in Mod[P]$, there exists $\mathcal{B} \in Mod[R]$ such that $\mathcal{B} \models t(Th[\mathcal{A}])$.

Proof

($a \Rightarrow b$) Consider a sentence σ of I such that $t(\tau) \in Cn[\Theta \cup t(\Delta)]$. Then, by compactness, there exist sentences $\delta_1, \ldots, \delta_k \in \Delta$, such that $\Theta \cup \{t(\delta_1), \ldots, t(\delta_k)\} \models t(\sigma)$. Letting δ be the conjunction of $\delta_1, \ldots, \delta_k$, we have $\Delta \models \delta$ and $\Theta \cup \{t(\delta)\} \models t(\sigma)$. The Deduction Theorem applied to the latter yields $\Theta \models [t(\delta) \rightarrow t(\sigma)]$. Thus, $(\delta \rightarrow \sigma)$ is a sentence of I such that $t(\delta \rightarrow \sigma) = [t(\delta) \rightarrow t(\sigma)] \in Cn(\Theta)$. So, by (a), $(\delta \rightarrow \sigma) \in Cn(\Gamma)$; whence $\sigma \in Cn(\Gamma \cup \Delta)$.

($b \Rightarrow c$) Assume $\langle K, \Theta \cup t(\Delta) \rangle$ inconsistent and consider a sentence σ of I. Then $t(\sigma \wedge \neg\sigma) = t(\sigma) \wedge \neg t(\sigma) \in Cn[\Theta \cup t(\Delta)]$. Thus, by (b), $\Gamma \cup \Delta \models (\sigma \wedge \neg\sigma)$.

($c \Rightarrow d$) Consider a maximally consistent extension $\langle I, \Xi \rangle$ of $\langle I, \Sigma \rangle$. Then $\langle I, \Sigma \cup \Xi \rangle \equiv \langle I, \Xi \rangle$ is consistent, and (c) yields the consistency of $\langle K, \Theta \cup t(\Xi) \rangle$.

($d \Rightarrow e$) Given $\mathcal{A} \in Mod[P]$, $\langle I, Th[\mathcal{A}] \rangle$ is a maximally consistent extension of Q. Thus, by (d), $\langle K, \Theta \cup t(Th[\mathcal{A}]) \rangle$ is consistent , whence by completeness it has some model $\mathcal{B}$. Then $\mathcal{B} \in Mod[R]$ and $\mathcal{B} \models t(Th[\mathcal{A}])$.

($e \Rightarrow a$) Given a sentence σ of I such that $\Gamma \not\models \sigma$, we will show $\Theta \not\models t(\sigma)$. Indeed, we have a model $\mathcal{A} \in Mod[P]$ such that $\mathcal{A} \not\models \sigma$, whence $\neg\sigma \in Th[\mathcal{A}]$.

By (e), we have some model $\mathcal{B}\in Mod[R]$ such that $\mathcal{B}\models t(Th[\mathcal{A}])$. Thus, $\mathcal{B}\models\Theta$ satisfies $\neg t(\sigma)=t(\neg\sigma)\in t(Th[\mathcal{A}])$, hence $\Theta\not\models t(\sigma)$.

QED

Corollary *Properties of faithful interpretations* (pF)

Consider a translation $t:I\rightarrow K$ as well as presentations $P=\langle I,\Gamma\rangle$ and $R=\langle K,\Theta\rangle$, such that t interprets P faithfully into R.

a) Presentation $P=\langle I,\Gamma\rangle$ is consistent iff $R=\langle K,\Theta\rangle$ is so.

b) If $R=\langle K,\Theta\rangle$ is complete then so is $P=\langle I,\Gamma\rangle$.

Proof

a) Since t interprets Q into R, we have the if-part. Since this interpretation is faithful, the converse follows from the implication $(a\Rightarrow c)$ in the preceding lemma with $\Delta=\varnothing$.

b) Clear: given $\sigma\in Snt(I)$ such $\Gamma\not\models\sigma$, by faithfulness, $\Theta\not\models t(\sigma)$, whence the completeness of R yields $t(\neg\sigma)=\neg t(\sigma)\in Cn(\Theta)$, and by faithfulness $\Gamma\models\neg\sigma$.

QED

4.2 Internalisations: diagram, kernel and coequaliser

We now give some more details about the internalisation techniques we used. We shall examine successively diagram, kernel and coequaliser internalisations.

We recall that a coproduct of alphabets is simply their disjoint union as sets [Arbib & Mannes '75; Goldblatt '79]. A coproduct of languages is formed by taking the coproduct of their alphabets and transferring the declarations of the symbols from their original languages [Ehrich '82; Ehrig & Mahr '85; Veloso '93].

A. *Diagram internalisation*

The diagram axiomatisation of translation $t:I\rightarrow K$ consists of the sentences $(i\leftrightarrow t(i))$ of language I+K asserting that each source symbol i is equivalent to its translation t(i) [Veloso '93, '95]. More precisely, the *diagram axiomatisation* $\Delta[t]$ of translation $t:I\rightarrow K$ consists of the sentences $\delta[i,k]$ of the coproduct language $L[t]=I+K$ for each pair of symbols i of I and k of K such that $t(i)=k$; where $\delta[i,k]$ is

$$\forall x_1,\ldots,x_m[i(x_1,\ldots,x_m)\leftrightarrow k(x_1,\ldots,x_m)],$$

if i is an m-ary predicate symbol;

$$\forall y\forall x_1,\ldots,x_n[y\approx i(x_1,\ldots,x_n)\leftrightarrow y\approx k(x_1,\ldots,x_n)],$$

if i is an n-ary function symbol.

Notice that for any formula φ of J, we have by induction $\Delta[t]\models(\varphi\leftrightarrow t(\varphi))$. Thus, for $\Gamma\subseteq Snt(I)$, we have $\langle I+K,\Delta[t]\cup\Gamma\rangle\equiv\langle I+K,\Delta[t]\cup t(\Gamma)\rangle$.

Proposition *Diagram Internalisation* (Di)

Consider a translation $t:I \rightarrow K$ with diagram axiomatisation $\Delta[t]$, and a presentation $R=\langle K,\Theta \rangle$ with diagram extension $\Delta[t] \cup R := \langle L[t],\Delta[t] \cup \Theta \rangle$.

a) $\Delta[t] \cup R$ is a conservative extension of R: $R \leq \Delta[t] \cup R$.

b) Consider a presentation $P=\langle I,\Gamma \rangle$. Then:

(i) t interprets P into R iff $\Delta[t] \cup R$ extends P, i. e. $P \sqsubseteq \Delta[t] \cup R$;

(ii) t interprets P faithfully into R iff $\Delta[t] \cup R$ extends P conservatively, i. e. $P \leq \Delta[t] \cup R$.

Proof

a) Every symbol i of I is introduced into $L[t]=I+K$ by means of a (single) axiom $\delta[i,t(i)] \in \Delta[t]$ of the form $i \leftrightarrow t(i)$, which defines i in terms of t(i). Thus, we have an extension by definitions, which is conservative.

b) For any $\sigma \in Snt(I)$, the above remark gives $\Delta[t] \cup \Theta \models \sigma$ iff $\Delta[t] \cup \Theta \models t(\sigma)$, and part (a) yields $\Delta[t] \cup \Theta \models t(\sigma)$ iff $\Theta \models t(\sigma)$; whence $\Delta[t] \cup \Theta \models \sigma$ iff $\Theta \models t(\sigma)$.

(i) Hence $t:P \rightarrow R$ iff $t[Cn(\Gamma)] \subseteq Cn(\Theta)$ iff $Cn(\Gamma) \subseteq Cn(\Delta[t] \cup \Theta)$ iff $P \sqsubseteq \Delta[t] \cup R$.

(ii) Also, since $t^{-1}[Cn(\Theta)]=Cn(\Delta[t] \cup \Theta) \cap Snt(I)$, t is faithful iff $P \leq \Delta[t] \cup R$.

QED

B. *Kernel internalisation*

Consider a translation $t:I \rightarrow K$ with kernel $ker(t)=\{<i,i'>:t(i)=t(i')\}$. The (internalised) kernel of this translation consists of the sentences $(i \leftrightarrow i')$ of source language I asserting the equivalence of source symbols with the same translation [Veloso '92; Maibaum & Veloso '95]. More precisely, the *(internalised) kernel* $\Lambda[t]:=\{\lambda[i,i']/<i,i'> \in ker(t)\}$ of translation $t:I \rightarrow K$ consists of the identifying sentences $\lambda[i,i']$ of language I for each pair of symbols i and i' of I such that t(i)=t(i'); where $\lambda[i,i']$ is

$$\forall x_1,\ldots,x_m[i(x_1,\ldots,x_m) \leftrightarrow i'(x_1,\ldots,x_m)],$$

if i is an m-ary predicate symbol;

$$\forall y \forall x_1,\ldots,x_n[y \approx i(x_1,\ldots,x_n) \leftrightarrow y \approx i'(x_1,\ldots,x_n)],$$

if i is an n-ary function symbol.

Notice that, by induction, $\Lambda[t] \models (\psi \leftrightarrow \theta)$ for all $\psi,\theta \in Frml(I)$ with $t(\psi)=t(\theta)$.

These two internalisation techniques are connected [Veloso '95]. Given a translation $t:I \rightarrow K$, its diagram axiomatisation $\Delta[t]$ is a conservative extension of its (internalised) kernel $\Lambda[t]$: $\langle I,\Lambda[t] \rangle \leq \langle I+K,\Delta[t] \rangle$. (For, any model $\mathcal{K} \models \Lambda[t]$ can be expanded to a structure $\mathcal{D}$ for I+K such that $\mathcal{D} \models \Delta[t]$.)

Proposition *(Internalised) Kernel* (iK)

Consider a translation $t:I \rightarrow K$ with (internalised) kernel $\Lambda[t]$. Given a presentation $P=\langle I,\Gamma \rangle$, interpretation t of $P=\langle I,\Gamma \rangle$ into the image presentation $t(P):=\langle K,t(\Gamma) \rangle$ is faithful iff the (internalised) kernel $\Lambda[t]$ consists of consequences of Γ, i. e. $\Lambda[t] \subseteq Cn(\Gamma)$.

Proof

($\Rightarrow$) Clear, since for any $\lambda\in\Lambda[t]$ $\varnothing\models t(\lambda)$.

($\Leftarrow$) By the above remark $\langle I,\Lambda[t]\rangle\leq\langle I+K,\Delta[t]\rangle$. So, by item (b) of characterisation (cF) of faithfulness, we have $\langle I,\Lambda[t]\cup\Gamma\rangle\leq\langle I+K,\Delta[t]\cup\Gamma\rangle$; whence $\langle I,\Lambda[t]\cup\Gamma\rangle\leq\langle I+K,\Delta[t]\cup t(\Gamma)\rangle$ since $\langle I+K,\Delta[t]\cup\Gamma\rangle\equiv\langle I+K,\Delta[t]\cup t(\Gamma)\rangle$.
If $\Lambda[t]\subseteq Cn(\Gamma)$, then $\langle I,\Gamma\rangle\equiv\langle I,\Lambda[t]\cup\Gamma\rangle\leq\langle I+K,\Delta[t]\cup t(\Gamma)\rangle$. Hence, item (b.ii) of proposition (Di) on diagram internalisation yields the faithfulness of interpretation $t:P\rightarrow t(P)$.

QED

C. *Coequaliser internalisation*

Given translations $e:I\rightarrow J$ and $f:I\rightarrow K$ with common source language I, we form the language L[e,f] as the coproduct J+K. The (internalised) coequaliser of these translations is the set $\Omega[e,f]$ of sentences $(e(i)\leftrightarrow f(i))$ of the coproduct language J+K expressing the equivalence of the translations of a source symbol. More precisely, the *(internalised)* coequaliser $\Omega[e,f]$ of translations $e:I\rightarrow J$ and $f:I\rightarrow K$ consists of the sentences $\omega[j,k]$ of the coproduct language L[e,f]=J+K for each pair of symbols j of J and k of K such that j=e(i) and k=f(i) for some symbol i of the common source language I; where $\omega[j,k]$ is

$\forall x_1,\ldots,x_m[j(x_1,\ldots,x_m)\leftrightarrow k(x_1,\ldots,x_m)]$,

if j and k are m-ary predicate symbols;

$\forall y\forall x_1,\ldots,x_n[y\approx j(x_1,\ldots,x_n)\leftrightarrow y\approx k(x_1,\ldots,x_n)]$,

if j and k are n-ary function symbols.

Notice that, by induction, $\Omega[e,f]\models(e(\varphi)\leftrightarrow f(\varphi))$ for every $\varphi\in Snt(I)$.

Lemma *(Internalised) Coequaliser and Diagram Axiomatisations* (CD)

Consider translations $e:I\rightarrow J$ and $f:I\rightarrow K$, with respective diagram axiomatisations $\Delta[e]=\{\delta[i,e(i)]/i\in I\}$ and $\Delta[f]=\{\delta[i,f(i)]/i\in I\}$, and (internalised) coequaliser $\Omega[e,f]$. Then the (internalised) coequaliser $\Omega[e,f]$ consists of consequences of $\Delta[e]\cup\Delta[f]$: $\Omega[e,f]\subseteq Cn(\Delta[e]\cup\Delta[f])$.

Proof

Clear. For a sentence $\omega[j,k]\in\Omega[e,f]=\{\omega[e(i),f(i)]/i\in I\}$, we have some symbol i of I such that j=e(i) and k=f(i) and $\omega[j,k]$ is $(e(i)\leftrightarrow f(i))$. We then have $(i\leftrightarrow e(i))\in\Delta[e]$ and $(i\leftrightarrow f(i))\in\Delta[f]$, and $\{(i\leftrightarrow e(i)),(i\leftrightarrow f(i))\}\models(e(i)\leftrightarrow f(i))$.

QED

Proposition *(Internalised) Coequaliser and Pushout Mediator* (CM)

Given a pushout rectangle of language translations $e:I\rightarrow J$ and $f:I\rightarrow K$, yielding $g:J\rightarrow L$ and $h:K\rightarrow L$, consider the (internalised) coequaliser $\Omega[e,f]=\{\omega[e(i),f(i)]/i\in I\}$ of $e:I\rightarrow J$ and $f:I\rightarrow K$ and the mediator $(g|h):J+K\rightarrow L$ of their pushouts $g:J\rightarrow L$ and $h:K\rightarrow L$.

a) The (internalised) kernel $\Lambda[(g|h)]$ consists of consequences of the (internalised) coequaliser $\Omega[e,f]$: $\Lambda[(g|h)] \subseteq Cn(\Omega[e,f])$.

b) The mediator (g|h) interprets $\langle J+K, \Omega[e,f] \rangle$ faithfully into $\langle L, \varnothing \rangle$.

Proof

By construction of the pushout, the alphabet of L is the quotient of the coproduct of the alphabets of J and K modulo the equivalence relation r^* generated by $r=\{<j,k> \in (J+K)\times(J+K) / \text{ for some } i \text{ in } I \ e(i)=j \ \& \ k=f(i)\}$. So, (g|h) maps each symbol c of J+K to its equivalence class [c] modulo r^*.

a) First, $ker[(g|h)] \subseteq r^*$. Because, for $<c,d> \in ker[(g|h)]$, $[c]=[d]$ so $<c,d> \in r^*$.
Now, consider relation $s:=\{<c,d> \in (J+K)\times(J+K) / \ \Omega[e,f] \models c \leftrightarrow d\}$.
Notice that relation s is reflexive, symmetric and transitive.
Now, $r \subseteq s$. Because, given $<j,k> \in r$, we have some i in I such that $e(i)=j$ and $k=f(i)$, whence $(j \leftrightarrow k) \in \Omega[e,f]$. Thus $r^* \subseteq s$.
Hence $ker[(g|h)] \subseteq s$. Now, for a sentence $(j \leftrightarrow j')$ of $\Lambda[(g|h)]$, we have $<j,j'>$ in $ker[(g|h)] \subseteq s$, so $\Omega[e,f] \models (j \leftrightarrow j')$.

b) Notice that (g|h) maps a sentence $(e(i) \leftrightarrow f(i)) \in \Omega[e,f]$, with e(i) in J and f(i) in K, to $(g|h)[e(i)] \leftrightarrow (g|h)[f(i)] = g[e(i)] \leftrightarrow h[f(i)]$, which is logically valid. Thus, (g|h) interprets $\langle J+K, \Omega[e,f] \rangle$ into $\langle L, \varnothing \rangle$, faithfulness following from (a) in view of proposition (iK) on (internalised) kernel.

QED

5. Conclusion

We have examined the generalisation of three known results concerning extensions to interpretations. More specifically, we have generalised two familiar logical theorems, namely Robinson's Joint Consistency (RJC) and Craig-Robinson Interpolation (CRI), as well as the Modularisation Theorem for logical specifications to interpretations and their pushouts.

The motivations for this investigation stem from two main sources in logic and in software development. From the logical side, we have known connections between modularity-like results such as Robinson's Joint Consistency and Craig Interpolation theorem [Barwise '77; Chang & Keisler, '73; Shoenfield '67]. From the standpoint of formal approach to program and specification development [Maibaum & Veloso '81; Maibaum et al. '84; Veloso et al. '85], modularity of interpretations is a crucial property in composing implementations and in instantiating parameterised specifications [Maibaum et al. '91; Maibaum & Veloso '95]. Also, its proofs [Veloso '92, '93; Maibaum & Veloso '95] involve (some version of) Craig Interpolation Lemma as well as internalisation techniques to reduce interpretations to extensions.

Now, a special case of the Modularisation Theorem, Extension Modularity (EM), is a slight generalisation of Robinson's Joint Consistency Theorem. Both deal with unions of presentations over a given presentation. But, while the latter guarantees preservation of a property - consistency - over a maximally consistent presentation, the former ensures preservation of the relationship of being conservative. Also, whereas both Craig Interpolation and Robinson's Joint Consistency concern extensions, the Modularisation Theorem involves both extensions and interpretations and deals with a pushout rectangle of extensions and interpretations. It is thus quite natural to consider the generalisation of these results to a more uniform situation where (conservative) extensions are replaced by (faithful) interpretations.

We have examined the generalisations of these three results to the situation involving theories over languages in a pushout rectangle of language translations. They are roughly as follows. Pushout Consistency asserts that the pushout of consistent interpretations over a maximally consistent theory is consistent. Pushout Modularity guarantees preservation of faithfulness under pushout constructions. Pushout Interpolation enables the decomposition derivations of certain sentences in the pushout theory into derivations in the given theories.

A crucial idea in deriving these generalisations from their known counterparts is internalisation These techniques code (part of) the information of language translations, thereby reducing - to a large extent - interpretations to extensions. We have resorted to two known internalisation constructions, namely kernel [Veloso '92; Maibaum & Veloso '95] and diagram [Veloso '93, '95], as well as to a novel one, coequaliser internalisation, which codes information about pairs of translations with common source. The simpler technique of kernel internalisation uses sentences of the source language, whereas the other two employ sentences of appropriate coproduct languages.

Let us now comment on some possible extensions of these techniques, including the many-sorted case, which is important for software specification and development.

Sometimes one considers interpretations mapping symbols to formulae, which define the translations in the target presentation [Enderton, '72; Turski & Maibaum '87]. This extended notion can be reduced to a language morphism into an extension by definitions. Another version considers interpretations with relativisation predicates, which can be handled as in the many-sorted case. In the many-sorted case, one sometimes considers a translation t mapping a sort s of source language I to a sequence $s_1,..,s_k$ of sorts, together with a relativisation predicate r over them in target presentation R [Turski & Maibaum '87]. The idea is that s is to be represented by the subsort, defined by the relativisation predicate r, of the product of sorts $s_1,..,s_k$. This can be reduced to a language morphism translating I into an extension of R by introduction of product sorts and subsorts. The latter is a definition-like extension obtained by adding axioms characterising product sorts and subsorts [Meré & Veloso '92, '94].

Ideas similar to the above ones can be used to adapt our internalisation techniques to the many-sorted translations. In this case one may wish to declare equivalent symbols over distinct sorts, say unary predicate symbols p, over sort s, and q, over sort t. To do this, we first extend conservatively the presentation by adding a new function symbol b from sort s to sort t together with axioms expressing that b is a bijection. (This amounts to regarding one as a subsort of the other with relativisation predicate defined by $x \approx x$.) We can then express that p and q are equivalent by means of this new function symbol b, namely $(\forall x{:}s)(\forall y{:}t)[b(x) \approx y \rightarrow (p(x) \leftrightarrow q(y))]$.

We have established our generalised results by linking them: PoC⇒PoM⇒PoI. It is not difficult to see that the converse implications also hold. We thus have PoC⇔PoM⇔PoI, showing that either property is a necessary and sufficient condition for the other two. Moreover, the connections PoC⇔PoM⇔PoI hinge on simple properties of pre-images and faithful interpretations, much as the connections RJC⇔EM⇔CRI rely on corresponding properties of restrictions and conservative extensions. This observation puts in perspective the role of our internalisation techniques: since they reduce - to a large extent - interpretations to extensions, they provide the key link - as in RJC⇒PoC - from the version concerning union of extensions to the generalised one involving pushout of translations.

Summing up, we have generalised three known results, namely the Modularisation Theorem for logical specifications, Robinson's Joint Consistency (RJC) and Craig-Robinson Interpolation, to pushouts of language translations. By relying on simple properties of pre-images and faithful interpretations, we have derived PoC from the familiar RJC by internalisation techniques, and then established the other two by showing that PoC⇒PoM⇒PoI.

These results suggest considering their extensions to more general frameworks: (Π-)institutions or categories (whose objects are sets of sentences, morphisms being (sets of) derivations, or appropriate equivalence classes). Preliminary efforts in the direction of Π-institutions are underway.

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