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ON THE ALGEBRAIC STRUCTURE OF FORK ALGEBRAS

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Abstract. A fork algebra is a relational algebra enriched with a new binary operation. They have been introduced because their equational calculus has applications in program construction. In this paper, we present some simple results concerning the algebraic structure of fork algebras and some of their metamathematical consequences. This paper stems from the crucial, though simple, observation of the interdefinability of fork and projections in fork algebras. We begin with some preliminaries about fork algebras and their reducts: relational and Boolean algebras. We introduce a basic expansion construction and proceed to examine its connections with homomorphisms and direct products. We establish the closure of the class of fork algebras under some of its cases. Then, these results are used to characterize the simple fork algebras as those with simple relational-reducts and to decompose fork algebras into subdirect products of their simple homomorphic images. We use these algebraic results to reduce equational (and Horn-clause) properties of fork algebras to their simple components. These simple results enable one to reduce some properties of fork algebras to corresponding ones about relational algebras.

Key words: Fork algebras, relational algebras, Boolean algebras, expansions, algebraic structure, simple algebras, subdirect decomposition, equations, Horn clauses.

Resumo. Uma álgebra de fork é uma álgebra relacional enriquecida com uma nova operação binária. Tais álgebras foram introduzidas porque seu cálculo equacional tem aplicações em construção de programas. Neste trabalho apresentamos alguns resultados simples sobre a estrutura algébrica das álgebras de fork bem como algumas de suas conseqüências metamatemáticas. Este trabalho deve sua origem á observação crucial, apesar de simples, da interdefinibilidade de fork e projeções em álgebras de fork. Começamos com alguns some preliminares sobre álgebras de fork e seus redutos: álgebras relacionais e de Boole. Introduzimos uma construção básica de expansão e passamos a examinar suas conexões com homomorfismos e produtos diretos, para então estabelecer o fechamento da classe das álgebras de fork sob alguns de seus caso. A seguir, estes resultados são empregados para caracterizar as álgebras de fork simples como aquelas com redutos relacionais simples bem como para decompor álgebras de fork em produtos subdiretos de suas imagens homomorfas simples. Esses resultados algébricos são aplicados para reduzir propriedades equacionais (e na forma de cláusulas de Horn) das álgebras de fork a suas componentes simples. Tais resultados permitem a redução de algumas propriedades de álgebras de fork a correspondentes sobre álgebras relacionais.

Palavras chave: Álgebras de fork, álgebras relacionais, álgebras de Boole, expansões, estrutura algébrica, álgebras simples, decomposição subdireta , equações, cláusulas de Horn.

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CONTENTS

1. INTRODUCTION

A fork algebra (FA, for short) is a relational algebra (RA, for short) enriched with a new binary operation, called fork. They have been introduced because their equational calculus has applications in program construction. In this paper, we present some simple results concerning the algebraic structure of FA's and some of their metamathematical consequences.

This paper stems from the crucial, though simple, observation of the interdefinability of fork and projections in FA's. The structure of the paper is as follows.

We begin in section 2 with some preliminaries about fork algebras and their reducts: relational and Boolean algebras. We then introduce in section 3 a basic expansion construction and proceed to examine its connections with homomorphisms and direct products. In section 4 we establish the closure of the class of FA's under some of its cases. Then, these results are used to characterize the simple FA's as those with simple RA-reducts and to decompose FA's into subdirect products of their simple homomorphic images, in section 5. We use these algebraic results in section 6 to reduce equational (and Horn-clause) properties of FA's to their simple components. Finally, section 7 presents some concluding remarks.

These results have quite simple proofs. For readability we present each result with an outline of its proof. More detailed proofs are presented in the appendix at the end of the paper.

An application of these results about the algebraic structure of FA's is the analysis of finite and infinite FA's, which provides some interesting comparison and contrasts between FA's and RA's. In a forthcoming paper [Veloso '96], we examine the finite and infinite FA's, contrasting them with the RA's. The finite FA's are described as finite direct powers of the two-element FA, the only simple finite FA's being those with one or two elements. This shows that for each $n \geq 0$ there exists exactly one, up to isomorhism, FA of cardinality 2^n. This contrasts with the case of infinite FA's: we exhibit many (simple proper) RA's of each infinite cardinality that have many expansions to (proper) FA's. Such RA's demonstrate quite clearly the diversity of possible fork operations.

2. PRELIMINARIES: FORK ALGEBRAS AND THEIR REDUCTS

An abstract fork algebra (FA, for short) is a relational algebra enriched with a new binary operation, called fork. A relational algebra (RA, for short) is an expansion of a BA (short for Boolean algebra) with some Peircean operations (conversion and composition) and constant (for identity).

We shall use β for the signature <2,1,2> (with 2 constants, 1 unary operation and 2 binary operations) of the BA's, λ for the signature <3,2,2,> of the RA's, and ϕ for the signature <3,2,3> of the FA's. Given algebraic signature σ, we use $Alg(\sigma)$ to denote the class of all algebras with this signature.

A *Boolean algebra* (*BA*, for short) is an algebra $\mathcal{B}=\langle B,0,\infty,^{-},+,\bullet\rangle$ with signature β (so $0,\infty\in B$, $^{-}:B\to B$, and $+,\bullet:B\times B\to B$), satisfying well-known equations [Bell & Slomson '71; Burris & Sankappanavar '81; Halmos '63]. We shall use $\leq$ for the Boolean ordering (recall that $a\bullet b=a$ iff $a\leq b$ iff $a+b=b$).

A *relational algebra* (*RA*, for short) is an algebra $\mathcal{R}=\langle R,0,\infty,1,^{-},^{\dagger},+,\bullet,;\rangle$ with signature λ, satisfying familiar equations, to the effect that

- its *BA-reduct* $\mathcal{R}_{\beta}=\langle R,0,\infty,^{-},+,\bullet\rangle$ is a BA with Boolean ordering $\leq$;
- the *Peircean reduct* $\langle R,1,^{\dagger},;\rangle$ is a semigroup with identity $1\in R$ and involution $^{\dagger}:R\to R$, so $1^{\dagger}=1$, $(r^{\dagger})^{\dagger}=r$ and $(r;s)^{\dagger}=(s^{\dagger});(r^{\dagger})$;
- for all $r,s\in R$: $(r^{\dagger});(r;s)^{-}\leq s^{-}$, i. e. $(r^{\dagger});(r;s)^{-}+s^{-}=s^{-}$.

Consider an algebra $\mathcal{R}=\langle R,0,\infty,1,^{-},^{\dagger},+,\bullet,;\rangle$ of signature ρ. By adding a binary operation $\nabla:R\times R\to R$, we obtain an algebra $\mathcal{R}^{\nabla}\in Alg(\phi)$, called its ∇-*expansion*. Note that in any such expansion, we have elements $\pi:=(1\nabla\infty)^{\dagger}$ and $\rho:=(\infty\nabla 1)^{\dagger}$.

A *fork algebra* (*FA*, for short) is an algebra $\mathcal{F}=\langle F,0,\infty,1,^{-},^{\dagger},+,\bullet,;,\nabla\rangle$ with signature ϕ, such that

- its *RA-reduct* $\mathcal{F}_{\lambda}=\langle F,0,\infty,1,^{-},^{\dagger},+,\bullet,;\rangle$ is an RA with Boolean ordering $\leq$;
- with $\pi:=(1\nabla\infty)^{\dagger}$ and $\rho:=(\infty\nabla 1)^{\dagger}$ as above
  - for every $r,s,p,q\in F$: $(r\nabla s);(p\nabla q)^{\dagger}=(r;p^{\dagger})\bullet(s;q^{\dagger})$ (∇ vs. $\bullet$),
  - for every $r,s\in F$: $r\nabla s=(r;\pi^{\dagger})\bullet(s;\rho^{\dagger})$ (∇-def),
  - $\pi\nabla\rho\leq 1$ (i. e. $\pi\nabla\rho+1=1$) (∇ proj).

Since the class of RA's has a finite equational characterization [Chin & Tarski '50, Theorem 2.2, p. 350; Jónsson & Tarski '52; Veloso '74, p. 8], so does the class of FA's. We use FA for the variety of the FA's.

3. Basic Expansion Construction

The crucial, though simple, observation is this: in an FA fork is definable by an RA-term from the elements $\pi^{\dagger}=1\nabla\infty$ and $\rho^{\dagger}=\infty\nabla 1$ in its carrier F. So, preservation of the RA operations as well as of π and ρ entails preservation of fork.

This observation is the motivation for the following basic construction. Consider an algebra $\mathcal{F}$ of the FA-signature ϕ, with relational reduct $\mathcal{F}_{\lambda}\in Alg(\lambda)$. We have the elements $\pi^{\dagger}=1\nabla\infty$ and $\rho^{\dagger}=\infty\nabla 1$ in its carrier F. Given an algebra $\mathcal{R}$ of the RA-signature λ and a function $h:F\to R$, we have $h(\pi^{\dagger}),h(\rho^{\dagger})\in R$, and we define binary operation $\nabla^{h}:R\times R\to R$ by setting $r\nabla^{h}s:=[r;h(\pi^{\dagger})]\bullet[s;h(\rho^{\dagger})]$ (notice that R is closed under ∇^{h}). This provides a ∇-expansion, which we call the *expansion of* $\mathcal{R}$ *by* h and denote by $\mathcal{R}^{h}$.

Notice that this construction of $\mathcal{R}^{h}$ from $\mathcal{R}$ and $h:F\to R$ relies only on the existence of elements $\pi^{\dagger}$ and $\rho^{\dagger}$ in F and the fact that R is closed under ; and $\bullet$. The next two result relies on the preservation of ; and $\bullet$ by h.

Lemma *Homomorphism for expansion*

Consider an algebra $\mathcal{F}$ of FA-signature ϕ with reduct $\mathcal{F}_{\lambda}\in Alg(\lambda)$. If $\mathcal{F}$ satisfies axiom (∇-def), then any λ-homomorphism h of $\mathcal{F}_{\lambda}$ into an algebra $\mathcal{R}\in Alg(\lambda)$ is a homomorphism of $\mathcal{F}$ into the expansion $\mathcal{R}^{h}$ of $\mathcal{R}$ by h.

Proof outline

Since h preserves ; and $\bullet$, preservation of ∇ by h follows from (∇-def) and the definition of ∇^h.

QED

When we have an RA-homomorphism into a direct product, there are two natural ways to obtain fork expansions: expanding the direct product or expanding its components. It is not difficult to see that the expansion construction commutes with direct products, which is the content of the next lemma.

Lemma *Expansions and direct product*

Consider a family of λ-algebras $\mathcal{R}_i \in \mathrm{Alg}(\lambda)$, for $i \in I$, yielding the direct product $\times_{i\in I}\mathcal{R}_i$, with projections p_i. Given an algebra $\mathcal{F}$ of FA-signature ϕ with reduct $\mathcal{F}_\lambda \in \mathrm{Alg}(\lambda)$, and a function $h: F \to \times_{i\in I} R_i$, consider the expansion $\mathcal{R}_i^{h_i}$ of $\mathcal{R}_i$ by the composite $h_i = p_i \circ h$, for $i \in I$. Then, the direct product of the expansions and the expansion of the direct product by h coincide: $\times_{i\in I}(\mathcal{R}_i^{h_i}) = (\times_{i\in I}\mathcal{R}_i)^h$.

Proof outline

To check that the direct-product fork $\nabla^\times$ and the induced fork ∇^h coincide, we rely on the jointly injectivity of the direct-product projections, and show that $p_i(r\nabla^\times s) = p_i(r\nabla^h s)$, for $r,s \in \times_{i\in I} R_i$ and $i \in I$. The latter follows from the definitions ∇^h and $\nabla^\times$, and the facts that each p_i is a homomorphism.

QED

Note that the construction of $\mathcal{R}^h$ from $\mathcal{R}$ and $h: F \to R$, as well as the above properties, hinge on the new operation(s) being definable by old term(s) with parameters in F. So, they carry over to this more general case.

A possible dual construction could expand an algebra $\mathcal{R} \in \mathrm{Alg}(\lambda)$ to a ϕ-algebra on the basis of λ-homomorphism h of $\mathcal{R}$ into the reduct $\mathcal{F}_\lambda$ of an algebra $\mathcal{F} \in \mathrm{Alg}(\phi)$. M. Frias proposed a version of this dual construction in connection with representability [Frias et al. '96]. Our basic construction was suggested by an abstraction of this special dual construction.

4. Closure under Expansion Construction

The following results use the equational character of the FA's. We obtain some simple closure properties of the variety $FA \subseteq \mathrm{Alg}(\phi)$ of the FA's.

The next result establishes the closure of FA under expansions by surjective RA-homomorphisms. It follows immediately from the basic lemma on homomorphism for expansion.

Proposition *FA expansion of homomorphic image*

Consider an FA $\mathcal{F}$ with relational reduct $\mathcal{F}_\lambda$. Given any surjective RA-homomorphism h of $\mathcal{F}_\lambda$ onto RA $\mathcal{R}$, the expansion $\mathcal{R}^h$ of $\mathcal{R}$ by h is an FA.

Proof outline

By the lemma on homomorphism for expansion, h is a surjective homomorphism of $\mathcal{F}\in$ FA onto $\mathcal{R}^h$.

QED

The next result follows easily from this proposition and the lemma on expansions and direct product. It shows that the product mediator of surjective RA-homomorphisms is an FA-homomorphism into the direct product of their expansions.

Proposition *Direct product of homomorphic images*

Consider an FA $\mathcal{F}$ with relational reduct $\mathcal{F}_\lambda$ and a family of surjective RA-homomorphisms $h_i:F\to R_i$ of $\mathcal{F}_\lambda$ onto RA's $\mathcal{R}_i$, for $i\in I$. Then the direct product $\times_{i\in I}(\mathcal{R}_i{}^{h_i})$ of the expansions is an FA and the mediator $h:F\to\times_{i\in I}R_i$ is an FA-homomorphism of $\mathcal{F}$ into it.

Proof outline

By the preceding proposition $\times_{i\in I}(\mathcal{R}_i{}^{h_i})$ is an FA. By the lemma on expansions and direct product $\times_{i\in I}(\mathcal{R}_i{}^{h_i})=(\times_{i\in I}\mathcal{R}_i)^h$. So, the assertion follows from the lemma on homomorphism for expansion.

QED

5. Simple Fork Algebras and Product Decompositions

We now examine some results concerning the algebraic structure of the FA's. We shall characterize the simple FA's and establish subdirect decomposition of FA's into them.

As usual, an algebra $\mathcal{A}$ is called *simple* iff it has no proper homomorphic image, i. e. whenever function $h:A\to B$ is a surjective homomorphism h of $\mathcal{A}$ onto algebra $\mathcal{B}$, h is a bijection or $B\in$ Triv (the carrier B is a singleton).

The next result characterizes the simple FA's as those with simple relational reducts.

Theorem *Simple FA's*

An FA $\mathcal{F}$ is simple iff its relational reduct $\mathcal{F}_\lambda$ is simple.

Proof outline

($\Leftarrow$) Any FA-homomorphism is an RA-homomorphism of their RA-reducts.

($\Rightarrow$) By the proposition on FA expansion of homomorphic image, any RA-homomorphism h of $\mathcal{F}_\lambda$ onto RA $\mathcal{R}$, will be an FA homomorphism of $\mathcal{F}$ onto the FA-expansion $\mathcal{R}^h$ of $\mathcal{R}$. So, either $h:F\to R$ is a bijection or $R\in$ Triv.

QED

The next result characterizes the non-simple FA's as those with non-trivial direct decompositions, just like RA's.

Proposition *Non-simple FA's*

An FA $\mathcal{F}$ is non-simple iff $\mathcal{F}\cong\mathcal{G}\times\mathcal{H}$ for some non-trivial FA's $\mathcal{G}$ and $\mathcal{H}$.

Proof outline

($\Leftarrow$) Clearly, such an algebra $\mathcal{G}\times\mathcal{H}$ cannot be simple.

(⇒) If $\mathcal{F}$ is not simple then its non-simple relational reduct $\mathcal{F}_\lambda$ has a decomposition $\mathcal{F}_\lambda \cong \mathcal{P} \times \mathcal{Q}$ for some proper homomorphic images of $\mathcal{F}_\lambda$ [Jónsson & Tarski '52, Theorems 4.14 , p. 135]. Now, by the proposition on direct product of homomorphic images, $\mathcal{P}$ and $\mathcal{Q}$ have ∇-expansions $\mathcal{P}^\nabla$ and $\mathcal{Q}^\nabla$ in FA such that $\mathcal{F} \cong \mathcal{P}^\nabla \times \mathcal{Q}^\nabla$ with $\mathcal{P}^\nabla$ and $\mathcal{Q}^\nabla$ non-trivial FA's.

QED

The next result provides subdirect decompositions for FA's paralleling the analogous result for RA's.

Theorem *Subdirect decomposition of FA's into simple components*

Every FA $\mathcal{F}$ is isomorphic to some subdirect product of simple homomorphic images of $\mathcal{F}$: there exist a set of simple homomorphic images $\mathcal{F}_i$, $i \in I$, of $\mathcal{F}$ and a ϕ-embedding of $\mathcal{F}$ into the direct product $\times_{i \in I} \mathcal{F}_i$.

Proof outline

In view of the subdirect decomposition of RA's [Jónsson & Tarski '52, Theorem 4.14, p. 135], we have we have surjective RA-homomorphisms h_i of the RA-reduct $\mathcal{F}_\lambda$ of $\mathcal{F}$ onto simple RA $\mathcal{R}_i$, $i \in I$, and an RA-embedding e of $\mathcal{F}_\lambda$ into the direct product $\times_{i \in I} \mathcal{R}_i$. By the propositions on FA expansion of homomorphic image and on simple FA's, each h_i is an FA-homomorphism of $\mathcal{F}$ onto the simple expansion $\mathcal{R}_i^{h_i} \in$ FA. Thus, by the proposition on direct product of homomorphic images, injective e is an FA-homomorphism of $\mathcal{F}$ into FA $\times_{i \in I}(\mathcal{R}_i^{h_i})$.

QED

6. Some Metamathematical Consequences

We now consider some simple metamathematical consequences of the preceding results concerning the algebraic structure of the FA's.

Corollary *FA equations*

An FA equation ε holds in an FA algebra $\mathcal{F}$ iff equation ε holds in all (simple) homomorphic images of $\mathcal{F}$.

Proof outline

(⇒) An equation holding in an algebra must hold in its (simple) homomorphic images

(⇐) By the preceding theorem, $\mathcal{F}$ can be embedded into a direct product $\times_{i \in I} \mathcal{F}_i$ of its simple homomorphic images If equation ε to hold in all (simple) homomorphic images $\mathcal{F}_i$ of $\mathcal{F}$ then $\mathcal{F}_i \vDash \varepsilon$, for $i \in I$, whence $\mathcal{F} \vDash \varepsilon$.

QED

We shall use SFA to denote the class of simple FA's. Recall that the simple RA's are those satisfying Tarski's rule τ: $\forall x(\neg x \approx 0 \rightarrow \infty;x;\infty \approx \infty)$ [Jónsson & Tarski '52, Theorem 4.10, p. 132, 133]. So, for $\mathcal{F} \in$ FA, $\mathcal{F} \in$ SFA iff $\mathcal{F} \vDash \tau$.

By a *fork calculus* FC we mean an equational axiomatization for the class of FA's, consisting of

- a (finite) set of equations characterizing BA's;

- equations characterizing RA's, say:

  $1^\dagger\approx 1$, $\forall x\ (x^\dagger)^\dagger\approx x$, $\forall x,y\ (x;y)^\dagger\approx(y^\dagger);(x^\dagger)$ and $\forall x,y\ (x^\dagger);(x;y)^-+y^-\approx y^-$;
- fork equations corresponding to (∇ vs. $\bullet$), (∇-def) and (∇ proj):

  $\forall x,y,u,v\ (x\nabla y);(u\nabla v)^\dagger\approx(x;u^\dagger)\bullet(y;v^\dagger)$ ($\nabla;\nabla^\dagger$),

  $\forall x,y\ x\nabla y\approx[x;(1\nabla\infty)]\bullet[y;(\infty\nabla 1)]$ ($\nabla\approx$),

  $(1\nabla\infty)^\dagger\nabla(\infty\nabla 1)^\dagger+1\approx 1$ ($\pi\nabla\rho$).

By a *Horn clause* (in the language of FA's) we mean a sentence in prenex form whose prefix consists only of universal quantifiers $\forall x_1\ldots\forall x_m$ and whose matrix is a conditional equation $\varepsilon_1\wedge\ldots\wedge\varepsilon_n\rightarrow\varepsilon_0$ where $\varepsilon_0,\varepsilon_1\ldots,\varepsilon_n$ are equations between terms and $n\geq 0$. Such Horn clauses are special universal Horn sentences, which are known to be preserved under subdirect products [Shoenfield '67, p. 94, 95; Burris & Sankappanavar '81, p. 204].

Proposition *FA Horn clauses*

For a Horn clause σ (in the language of FA's) the following are equivalent.

a) Horn clause σ holds in all simple FA's: SFA$\models\sigma$.

b) Horn clause σ holds in all FA's: FA$\models\sigma$.

c) Horn clause σ is derivable from an FA calculus FC: FC$\vdash\sigma$.

d) Horn clause σ is derivable from an FA calculus FC together with Tarski's rule τ: FC$\vdash\tau\rightarrow\sigma$.

Proof outline

($a\Rightarrow b$) Follows from the subdirect decomposition theorem and the preceding remark, since σ holds in all simple FA's $\mathcal{F}_i$'s, we have $\mathcal{F}\models\sigma$.

($b\Rightarrow c$) & ($c\Rightarrow d$) Clear.

($d\Rightarrow a$) Clear by the preceding characterization of simple RA's.

QED

Now, equations are special cases of Horn clauses. Thus, by completeness of equational reasoning, we have the following property of FA equations.

Theorem *FA equations*

For an equation ε the following are equivalent.

a) Equation ε holds in all simple FA's: SFA$\models\varepsilon$.

b) Equation ε holds in all FA's: FA$\models\varepsilon$.

c) Equation ε is derivable from an FA equational calculus FC by equational reasoning.

7. CONCLUSION

This paper originates from the crucial, albeit simple, observation of the interdefinability of fork and projections in FA's. We present some simple results concerning the algebraic structure of FA's and some of their metamathematical consequences. These results have quite simple proofs, whose outlines are presented in the body of the paper, leaving the details for the appendix at the end of the paper.

We recall in section 2 some preliminaries about fork algebras and their reducts: relational and Boolean algebras. We then introduce in section 3 a basic expansion construction and examine its connections with homomorphisms and direct products. In section 4 we establish the closure of the variety of FA's under some of its cases. Next, these results are used to characterize the simple FA's as those with simple RA-reducts and to show that FA's, like RA's, can be decomposed as subdirect products of their simple homomorphic images, in section 5. We use these algebraic results in section 6 to reduce equational (and Horn-clause) properties of FA's to their simple components. Finally, section 7 presents some concluding remarks.

The results concerning the basic expansion construction and their properties enable one to reduce some properties of FA's to corresponding ones about RA's. The characterization of the simple FA's as those with simple RA-reducts provides an example of such reduction. It shows that a product decomposition of the RA-reduct of an FA into homomorphic images yields a corresponding decomposition for the original FA.

The metamathematical consequences examined, though simple, have some importance in connection with FA calculi. By reducing equational properties of an FA to their simple components, it allows one to check such an equational property in simple FA's, which is much easier. Also, the reduction of Horn-clause properties of FA's to the simple FA's has the advantage of simplifying both proofs, where one can use Tarski's rule, and the search for possible counterexamples. Similar remarks apply to the special case of equations, which is of importance in connection with equational calculi for reasoning about programs, one of the original motivations for the introduction of FA's.

These results about the algebraic structure of FA's and their metamathematical consequences are very similar to their analogs for RA's. This may give the impression of similarity in the behavior of RA's and FA's. That they are not so similar can easily be seen, for instance, by considering representability [Frias et al. '95, '96].

Another application of these results about the algebraic structure of FA's is the analysis of finite and infinite FA's, which provides some interesting comparison and contrasts between FA's and RA's. In a forthcoming paper [Veloso '96], we examine the finite and infinite FA's, contrasting them with the RA's. We analyze the Boolean FA's (those where fork is Boolean meet), show that their reducts are Boolean RA's, and characterize them as subalgebras of direct powers of the two-element FA. Then, these results are applied to finite FA's: they are described as finite direct powers of the two-element FA, and the only simple finite FA's are those with one or two elements. This shows that for each $n \geq 0$ there exists exactly one, up to isomorhism, FA of cardinality 2^n. This contrasts with the case of infinite FA's: we introduce a technique for the analysis of FA's and use some set-theoretical constructions to exhibit many (simple proper) RA's of each infinite cardinality that have many expansions to (proper) FA's. Such RA's demonstrate quite clearly the diversity of possible fork operations.

APPENDIX: DETAILED PROOFS OF THE RESULTS

We present in this appendix detailed proofs of the results.

Lemma *Homomorphism for expansion*

Consider an algebra $\mathcal{F}$ of FA-signature ϕ with reduct $\mathcal{F}_\lambda \in Alg(\lambda)$. If $\mathcal{F}$ satisfies axiom (∇-def), then any λ-homomorphism h of $\mathcal{F}_\lambda$ into an algebra $\mathcal{R} \in Alg(\lambda)$ is a homomorphism of $\mathcal{F}$ into the expansion $\mathcal{R}^h$ of $\mathcal{R}$ by h.

Proof

We must show that h preserves ∇, i. e. $h(f \nabla g)=h(f)\nabla^h h(g)$.
First, since $\mathcal{F}$ satisfies (∇-def), we have $f\nabla g=(f;\pi^\dagger)\bullet (g;\rho^\dagger)$.
Now, since h preserves ; and $\bullet$, $h(f\nabla g)=h[(f;\pi^\dagger)\bullet(g;\rho^\dagger)]$ equals $[h(f);h(\pi^\dagger)]\bullet[h(g);h(\rho^\dagger)]$, which is $h(f)\nabla^h h(g)$, by definition of ∇^h.

QED

Lemma *Expansions and direct product*

Consider a family of λ-algebras $\mathcal{R}_i \in Alg(\lambda)$, for $i\in I$, yielding the direct product $\times_{i\in I}\mathcal{R}_i$, with projections p_i. Given an algebra $\mathcal{F}$ of FA-signature ϕ with reduct $\mathcal{F}_\lambda \in Alg(\lambda)$, and a function $h:F\to\times_{i\in I}R_i$, consider the expansion $\mathcal{R}_i{}^{h_i}$ of $\mathcal{R}_i$ by the composite $h_i=p_i\circ h$, for $i\in I$. Then, the direct product of the expansions and the expansion of the direct product by h coincide: $\times_{i\in I}(\mathcal{R}_i{}^{h_i})=(\times_{i\in I}\mathcal{R}_i)^h$.

Proof

Letting $\nabla^\times$ denote the direct-product fork, we must check $\nabla^\times=\nabla^h$.
Since the direct-product projections are jointly injective, it suffices to show that $p_i(r\nabla^\times s)=p_i(r\nabla^h s)$, for $r,s\in\times_{i\in I}R_i$ and $i\in I$.
By definition of expansion, we have $r\nabla^h s=[r;h(\pi^\dagger)]\bullet[s;h(\rho^\dagger)]$ in $(\times_{i\in I}\mathcal{R}_i)^h$ and $r_i\nabla^{h_i}s_i=[r_i;h_i(\pi^\dagger)]\bullet[s_i;h_i(\rho^\dagger)]$ in $\mathcal{R}_i{}^{h_i}$.
By definition of direct product, $\nabla^\times$ is defined componentwise and each p_i is a homomorphism of $\times_{i\in I}(\mathcal{R}_i{}^{h_i})$ onto $\mathcal{R}_i{}^{h_i}$. So, $p_i(r\nabla^\times s)=p_i(r)\nabla^{h_i}p_i(s)$.
Since p_i preserves ; and $\bullet$, we have the equality
$p_i([r;h(\pi^\dagger)]\bullet[s;h(\rho^\dagger)])=[p_i(r);p_i(h(\pi^\dagger))]\bullet[p_i(r);p_i(h(\rho^\dagger))])$.
Now, since $p_i\circ h=h_i$, $p_i(h(\pi^\dagger))=h_i(\pi^\dagger)$ and $p_i(h(\rho^\dagger))=h_i(\rho^\dagger)$.
Thus, we have: $p_i(r\nabla^\times s)=p_i(r)\nabla^{h_i}p_i(s)=[p_i(r);h_i(\pi^\dagger)]\bullet[p_i(s);h_i(\rho^\dagger)] =$
$= [p_i(r);p_i(h(\pi^\dagger))]\bullet[p_i(r);p_i(h(\rho^\dagger))])=p_i([r;h(\pi^\dagger)]\bullet[s;h(\rho^\dagger)])=p_i(r\nabla^h s)$.
Therefore, $\nabla^\times=\nabla^h$ and $(\times_{i\in I}\mathcal{R}_i)^h=\times_{i\in I}(\mathcal{R}_i{}^{h_i})$.

QED

Proposition *FA expansion of homomorphic image*

Consider an FA $\mathcal{F}$ with relational reduct $\mathcal{F}_\lambda$. Given any surjective RA-homomorphism h of $\mathcal{F}_\lambda$ onto RA $\mathcal{R}$, the expansion $\mathcal{R}^h$ of $\mathcal{R}$ by h is an FA.

Proof

By the lemma on homomorphism for expansion, $h:F\to R$ will be a surjective homomorphism of $\mathcal{F}$ onto $\mathcal{R}^h$. Since $\mathcal{F}$ is in the variety FA, so is its homomorphic image $\mathcal{R}^h$.

QED

Proposition *Direct product of homomorphic images*

Consider an FA $\mathcal{F}$ with relational reduct $\mathcal{F}_\lambda$ and a family of surjective RA-homomorphisms $h_i:F\to R_i$ of $\mathcal{F}_\lambda$ onto RA's $\mathcal{R}_i$, for $i\in I$. Then the direct product $\times_{i\in I}(\mathcal{R}_i{}^{h_i})$ of the expansions is an FA and the mediator $h:F\to\times_{i\in I}R_i$ is an FA-homomorphism of $\mathcal{F}$ into it.

Proof

By the preceding proposition, each surjective $h_i:F\to R_i$ will be an FA-homomorphism of $\mathcal{F}$ onto $\mathcal{R}_i{}^{h_i}$. Thus $\times_{i\in I}(\mathcal{R}_i{}^{h_i})$ is an FA.

Letting $p_i:P\to R_i$ be the direct-product projections, we have $p_i\circ h=h_i$. So, by the lemma on expansions and direct product, $\times_{i\in I}(\mathcal{R}_i{}^{h_i})=(\times_{i\in I}\mathcal{R}_i)^h$.

Finally, since $\mathcal{F}$ satisfies (∇-def), the lemma on homomorphism for expansion yields that h is a homomorphism of $\mathcal{F}$ into $(\times_{i\in I}\mathcal{R}_i)^h$.

QED

Theorem *Simple FA's*

An FA $\mathcal{F}$ is simple iff its relational reduct $\mathcal{F}_\lambda$ is simple.

Proof

($\Leftarrow$) Assume that the RA-reduct $\mathcal{F}_\lambda$ is simple.

Given any FA-homomorphism h of $\mathcal{F}$ onto FA $\mathcal{G}$, function $h:F\to G$ that is an RA-homomorphism of $\mathcal{F}_\lambda$ onto the RA-reduct $\mathcal{G}_\lambda$.

So, either $h:F\to G$ is a bijection or $G\in$Triv.

($\Rightarrow$) Assume that FA $\mathcal{F}$ simple.

Given any RA-homomorphism h of $\mathcal{F}_\lambda$ onto RA $\mathcal{R}$, by the proposition on FA expansion of homomorphic image, $h:F\to R$ will be an FA homomorphism of $\mathcal{F}$ onto the FA-expansion $\mathcal{R}^h$ of $\mathcal{R}$. So, either $h:F\to R$ is a bijection or $R\in$Triv.

QED

Proposition *Non-simple FA's*

An FA $\mathcal{F}$ is non-simple iff $\mathcal{F}\cong\mathcal{G}\times\mathcal{H}$ for some non-trivial FA's $\mathcal{G}$ and $\mathcal{H}$.

Proof

($\Leftarrow$) Clearly, such an algebra $\mathcal{F}\cong\mathcal{G}\times\mathcal{H}$ has proper homomorphic images $\mathcal{G}$ and $\mathcal{H}$, so it cannot be simple.

($\Rightarrow$) Assume $\mathcal{F}$ non-simple. Then so is its relational reduct $\mathcal{F}_\lambda$.

By properties of RA's [Jónsson & Tarski '52, Theorems 4.14 , p. 135], we have a decomposition $\mathcal{F}_\lambda\cong\mathcal{P}\times\mathcal{Q}$ for some proper homomorphic images of $\mathcal{F}_\lambda$.

{ Non-simple RA $\mathcal{F}_\lambda$ has an ideal element $f\in F$ distinct from 0 and ∞ [Jónsson & Tarski '52, Theorems 4.10 and 4.13, p. 132-134], which gives rise to proper homomorphic images $\mathcal{P}:=\mathcal{F}_\lambda(f)$ and $\mathcal{Q}:=\mathcal{F}_\lambda(f^-)$ with $\mathcal{F}_\lambda\cong\mathcal{P}\times\mathcal{Q}$. }

By the proposition on direct product of homomorphic images, $\mathcal{P}$ and $\mathcal{Q}$ have ∇-expansions $\mathcal{P}^\nabla$ and $\mathcal{Q}^\nabla$ in FA such that $\mathcal{F}\cong\mathcal{Q}^\nabla\times\mathcal{Q}^\nabla$ with $\mathcal{P}^\nabla$ and $\mathcal{Q}^\nabla$ non-trivial.

QED

Theorem *Subdirect decomposition of FA's into simple components*
Every FA $\mathcal{F}$ is isomorphic to some subdirect product of simple homomorphic images of $\mathcal{F}$: there exist a set of simple homomorphic images $\mathcal{F}_i$, $i \in I$, of $\mathcal{F}$ and a ϕ-embedding of $\mathcal{F}$ into the direct product $\times_{i\in I}\mathcal{F}_i$.

Proof
Consider the RA-reduct $\mathcal{F}_\lambda$ of FA $\mathcal{F}$.
In view of the subdirect decomposition of RA's [Jónsson & Tarski '52, Theorem 4.14, p. 135], there exist a set of simple homomorphic images $\mathcal{R}_i$, $i \in I$, of $\mathcal{F}_\lambda$ and a subdirect product $\mathcal{S}$ of the RA's $\mathcal{R}_i$ such that $\mathcal{F}_\lambda$ is isomorphic to $\mathcal{S}$. So, we have surjective RA-homomorphisms h_i:F→R_i of $\mathcal{F}_\lambda$ onto simple RA $\mathcal{R}_i$, as well as an injective RA-homomorphism e of $\mathcal{F}_\lambda$ into the direct product $\mathcal{P}:=\times_{i\in I}\mathcal{R}_i$ with projections p_i:P→R_i, such that, for each $i \in I$, h_i is the composite $p_i \circ e$.
By the proposition on FA expansion of homomorphic image, each h_i is an FA-homomorphism of $\mathcal{F}$ onto the expansion $\mathcal{R}_i^{h_i}$, the latter being simple by the preceding proposition on simple FA's.
Thus, by the proposition on direct product of homomorphic images, $\times_{i\in I}(\mathcal{R}_i^{h_i})$ is an FA and injective $e:F\to\times_{i\in I}R_i$ is a homomorphism of $\mathcal{F}$ into it.
QED

Corollary *FA equations*
An FA equation ε holds in an FA algebra $\mathcal{F}$ iff equation ε holds in all (simple) homomorphic images of $\mathcal{F}$.

Proof
(⇒) If equation ε holds in algebra $\mathcal{F}$ then clearly it must hold in every (simple) homomorphic image of $\mathcal{F}$.
(⇐) Assume equation ε to hold in all (simple) homomorphic images of $\mathcal{F}$.
By the preceding theorem, we can decompose $\mathcal{F}$ into simple homomorphic images $\mathcal{F}_i$, $i \in I$, with $\mathcal{F}$ embedded into direct product $\mathcal{G}:=\times_{i\in I}\mathcal{F}_i$. Since $\mathcal{F}_i \models \varepsilon$, for $i \in I$, we have $\mathcal{G} \models \varepsilon$, whence $\mathcal{F} \models \varepsilon$.
QED

Proposition *FA Horn clauses*
For a Horn clause σ (in the language of FA's) the following are equivalent.
a) Horn clause σ holds in all simple FA's: $SFA \models \sigma$.
b) Horn clause σ holds in all FA's: $FA \models \sigma$.
c) Horn clause σ is derivable from an FA calculus FC: $FC \vdash \sigma$.
d) Horn clause σ is derivable from an FA calculus FC together with Tarski's rule τ: $FC \vdash \tau \to \sigma$.

Proof
(a⇒b) Assume that σ holds in all simple FA's.

By the subdirect decomposition theorem, we have simple FA's $\mathcal{F}_i$, $i \in I$, such that $\mathcal{F}$ is isomorphic to a subdirect product of the $\mathcal{F}_i$'s. By the preceding remark, since σ holds in all simple FA's $\mathcal{F}_i$'s, we have $\mathcal{F} \models \sigma$.

($b \Rightarrow c$) Clear by the completeness of first-order logic.

($c \Rightarrow d$) Clear.

($d \Rightarrow a$) Clear by the preceding characterization of simple RA's.

QED

Theorem *FA equations*

For an equation ε the following are equivalent.

a) Equation ε holds in all simple FA's: $SFA \models \varepsilon$.

b) Equation ε holds in all FA's: $FA \models \varepsilon$.

c) Equation ε is derivable from an FA equational calculus FC by equational reasoning.

Proof

($a \Leftrightarrow b$) By the preceding proposition.

($b \Leftrightarrow c$) Clear by soundness and completeness of equational reasoning.

QED

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