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On Finite and Infinite Fork Algebras

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ON FINITE AND INFINITE FORK ALGEBRAS

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Abstract. A fork algebra is a relational algebra enriched with a new binary operation. They have been introduced because their equational calculus has applications in program construction, they also have some interesting connections with algebraic logic. In this paper, which stems from the crucial, albeit simple, observation of the interdefinability of fork and projections in fork algebras, we examine the finite and infinite fork algebras and contrast them with relational algebras. We show that the finite fork algebras are somewhat uninteresting, being essentially Boolean algebras, uniquely characterized by their relational reducts. This contrasts with the case of infinite fork algebras: we introduce a technique for the analysis of fork algebras and use some set-theoretical constructions to exhibit many relational algebras of each infinite cardinality with many expansions to fork algebras. We begin by providing some background about fork algebras and their reducts (relation and Boolean algebras): their abstract versions, some simple results concerning the algebraic structure of fork algebras, and their concrete, set-based, versions (fields of sets and proper relational and fork algebras). We then examine the Boolean fork algebras (where fork is Boolean meet), show that their relational reducts are Boolean relational algebras, and characterize them as subalgebras of direct powers of the two-element fork algebra. These results are then applied to finite fork algebras: they are described as finite direct powers of the two-element fork algebra, the only simple finite fork algebras being those with one or two elements. This shows that a finite fork algebra is completely determined by its relational reduct (which we call rigid), in contrast to the infinite fork algebras. We examine fork-expansions of relational algebras, with the purpose of comparing relational algebras and fork algebras. Then, we introduce a technique for the analysis of fork algebras and use some set-theoretical constructions to exhibit many (simple proper) relational algebras of each infinite cardinality with many expansions to fork algebras. Such (infinite) relational algebras demonstrate quite clearly the diversity of possible fork operations.

Key words: Fork algebras, relational algebras, Boolean algebras, expansions, proper set-based algebras, simple algebras, finite fork algebras, infinite fork algebras, fork expansions.

Resumo. Uma álgebra de fork é uma álgebra relacional enriquecida com uma nova operação binária. Tais álgebras foram introduzidas porque seu cálculo equacional tem aplicações em construção de programas, tendo também interessantes conexões com lógica algébrica. Neste trabalho, que se origina da observação crucial, se bem que simples, da interdefinibilidade de fork e projeções em álgebras de fork, examinamos as álgebras de fork finitas e infinitas, contrastando-as com as álgebras relacionais. Mostramos que as álgebras de fork finitas são de reduzido interesse, uma vez que são essencialmente álgebras de Boole, unicamente caracterizadas por seus redutos relacionais. Isto contrasta com o caso das álgebras de fork infinitas: introduzimos uma técnica para a análise de álgebras de fork e empregamos algumas construções com conjuntos para exibir diversas álgebras relacionais de cada cardinalidade infinita possuindo várias expansões a álgebras de fork. Começamos recordando conceitos e resultados relativos a álgebras de fork e seus redutos (álgebras relacionais e de Boole):

suas versões abstratas, alguns resultados simples acerca da estrutura algébrica das álgebras de fork, e suas versões concretas, baseadas em conjuntos (corpos de conjuntos e álgebras de relações e de fork próprias). Examinamos então as álgebras de fork Booleanas (em que fork é a conjunção Booleana), mostramos que seus redutos relacionais são álgebras relacionais Booleanas e as caracterizamos como subálgebras de potências diretas da álgebra de fork com dois elementos. Estes resultados são então aplicados às álgebras de fork finitas, as quais são descritas como potências diretas finitas da álgebra de fork com dois elementos, sendo as álgebra de fork com um ou dois elementos as únicas álgebras de fork finitas simples. Isto mostra que uma álgebra de fork finita fica completamente determinada por seu reduto relacional (que denominamos de rígido), em contraste com as álgebras de fork infinitas. Examinamos as expansões por fork de álgebras relacionais, a fim de comparar estas álgebras com álgebras de fork. Em seguida, introduzimos uma técnica para a análise de álgebras de fork e utilizamos algumas construções com conjuntos para exibir diversas álgebras relacionais de cada cardinalidade infinita possuindo várias expansões a álgebras de fork. Tais álgebras relacionais (infinitas) demonstram claramente a diversidade de possíveis operações fork.

Palavras chave: Álgebras de fork, álgebras relacionais, álgebras de Boole, expansões, álgebras próprias baseadas em conjuntos, álgebras simples, álgebras de fork finitas, álgebras de fork infinitas, expansões por fork.

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CONTENTS

1. INTRODUCTION

A fork algebra (FA, for short) is a relational algebra (RA, for short) enriched with a new binary operation, called fork. They have been introduced because their equational calculus has applications in program construction. They also have some interesting connections with algebraic logic.

In this paper, we examine the finite and infinite FA's and contrast them with RA's. We describe the finite FA's as the finite direct powers of the two-element FA, the only simple finite FA's being those with one or two elements. Thus, for each n≥0 there exists exactly one, up to isomorhism, FA of cardinality 2^n, which is essentially a Boolean algebra, rendering the finite FA's somewhat uninteresting. This contrasts with the case of infinite FA's: we introduce a technique for the analysis of FA's and use some set-theoretical constructions to exhibit many infinite (simple proper) RA's of each infinite cardinality with many expansions to FA's.

This paper stems from the crucial, though simple, observation of the interdefinability of fork and projections in FA's. The structure of the paper is as follows.

Section 2 provides some background about fork algebras and their reducts (relation and Boolean algebras), beginning with their abstract versions, examining some simple results concerning the algebraic structure of FA's, and proceeding to their concrete, set-based, versions: fields of sets and proper FA's and RA's. We then examine in section 3 the Boolean FA's (those where fork is Boolean meet), show that their reducts are Boolean RA's, and characterize them as subalgebras of direct powers of the two-element FA. In section 4, these results are applied to finite FA's: they are described as finite direct powers of the two-element FA, and the only simple finite FA's are those with one or two elements. This shows that a finite FA is completely determined by its RA-reduct (which we call rigid), in contrast with the case of infinite FA's. In section 5 we examine fork-expansions of RA's, with the purpose of comparing RA's and FA's. Then, in section 6, we introduce a technique for the analysis of FA's and use some set-theoretical constructions to exhibit many infinite (simple proper) RA's of each infinite cardinality with many expansions to FA's. Such (infinite) RA's demonstrate quite clearly the diversity of possible fork operations. Finally, section 7 presents some concluding remarks.

2. PRELIMINARIES: FORK ALGEBRAS AND THEIR REDUCTS

An abstract fork algebra (FA, for short) is a relational algebra enriched with a new binary operation, called fork. A relational algebra (RA, for short) is an expansion of a BA (short for Boolean algebra) with some Peircean operations and constant.

We shall use β for the signature <2,1,2> (with 2 constants, 1 unary operation and 2 binary operations) of the BA's, λ for the signature <3,2,2,>

of the RA's, and ϕ for the signature <3,2,3> of the FA's. Given algebraic signature σ, we use $\mathbf{Alg}(\sigma)$ to denote the class of all algebras with this signature.

2.1 Abstract Boolean, Relation and Fork Algebras

Let us briefly recall some concepts pertaining to (abstract) Boolean, relational and fork algebras.

A *Boolean algebra* (*BA*, for short) is an algebra $\mathbf{B}=<B,0,\infty,^-,+,\bullet>$ with signature β (so $0,\infty\in B$, $^-:B\to B$, and $+,\bullet:B\times B\to B$), satisfying well-known equations [Bell & Slomson '71; Burris & Sankappanavar '81; Halmos '63]. We shall use $\leq$ for the Boolean ordering (recall that $a\bullet b=a$ iff $a\leq b$ iff $a+b=b$).

A *relational algebra* (*RA*, for short) is an algebra $\mathbf{R}=<R,0,\infty,1,^-,^\dagger,+,\bullet,;>$ with signature λ, satisfying familiar equations, to the effect that

- its *BA-reduct* $\mathbf{R}_\beta=<R,0,\infty,^-,+,\bullet>$ is a BA with Boolean ordering $\leq$;
- the *Peircean reduct* $<R,1,^\dagger,;>$ is a semigroup with identity $1\in R$ and involution $^\dagger:R\to R$, so $1^\dagger=1$, $(r^\dagger)^\dagger=r$ and $(r;s)^\dagger=(s^\dagger);(r^\dagger)$;
- for all $r,s\in R$: $(r^\dagger);(r;s)^-\leq s^-$, i. e. $(r^\dagger);(r;s)^-+s^-=s^-$.

Recall that the simple RA's are those satisfying Tarski's rule: $\infty;r;\infty=\infty$ whenever $r\neq 0$ [Jónsson & Tarski '52, Theorem 4.10, p. 132, 133].

A pair of (*conjugated*) *quasiprojections* for RA $\mathbf{R}=<R,0,\infty,1,^-,^\dagger,+,\bullet,;>$ amounts to elements $\pi,\rho\in R$ such that: $\pi^\dagger;\pi\leq 1$, $\rho^\dagger;\rho\leq 1$, and $\pi^\dagger;\rho=\infty$. In fact, for a pair π and ρ of quasiprojections we have $\pi^\dagger;\pi=1=\rho^\dagger;\rho$ (since $1\leq\infty;\infty^\dagger=\pi^\dagger;\rho;\rho^\dagger;\pi\leq\pi^\dagger;1;\pi$, and similarly for $\rho^\dagger;\rho$). A *quasiprojective RA* (*QRA*, for short) is an RA that has a pair of quasiprojections [Tarski & Givant '87, p. 242].

Consider an algebra $\mathbf{R}=<R,0,\infty,1,^-,^\dagger,+,\bullet,;>$ of signature ρ. By adding a binary operation $\nabla:R\times R\to R$, we obtain an algebra $\mathbf{R}^\nabla\in\mathbf{Alg}(\phi)$, called its ∇-*expansion.* Note that in any such expansion, we have elements $\pi:=(1\nabla\infty)^\dagger$ and $\rho:=(\infty\nabla 1)^\dagger$.

A *fork algebra* (*FA*, for short) is an algebra $\mathbf{F}=<F,0,\infty,1,^-,^\dagger,+,\bullet,;,\nabla>$ with signature ϕ, such that

- its *RA-reduct* $\mathbf{F}_\lambda=<F,0,\infty,1,^-,^\dagger,+,\bullet,;>$ is an RA with Boolean ordering $\leq$;
- with $\pi:=(1\nabla\infty)^\dagger$ and $\rho:=(\infty\nabla 1)^\dagger$ as above

  for every $r,s,p,q\in F$: $(r\nabla s);(p\nabla q)^\dagger=(r;p^\dagger)\bullet(s;q^\dagger)$ (∇ vs. $\bullet$),

  for every $r,s\in F$: $r\nabla s=(r;\pi^\dagger)\bullet(s;\rho^\dagger)$ (∇-def),

  $\pi\nabla\rho\leq 1$ (i. e. $\pi\nabla\rho+1=1$) (∇ proj).

Since the class of RA's has an equational characterization [Chin & Tarski '50, Theorem 2.2, p. 350; Jónsson & Tarski '52; Veloso '74, p. 8], so does the class of FA's. We use **FA** for the variety of the FA's.

It is not difficult to see that, in any FA $\mathbf{F}$ the defined elements $\pi:=(1\nabla\infty)^\dagger$ and $\rho:=(\infty\nabla 1)^\dagger$ form a pair of quasiprojections (such that $(\pi;\pi^\dagger)\bullet(\rho;\rho^\dagger)\leq 1$) [Frias et al. '95, '96]. Thus, the RA-reducts of FA's are QRA's.

2.2 Algebraic Structure of Fork Algebras and some consequences

We now recall some simple results concerning the algebraic structure of FA's and their metamathematical consequences [Veloso '96]. They come from the crucial, though simple, observation that in an FA fork is definable by an RA-term from the elements $\pi^{\dagger}=1\nabla\infty$ and $\rho^{\dagger}=\infty\nabla 1$ in its carrier F. So, preservation of the RA operations as well as of π and ρ entails preservation of fork.

The first result characterizes the simple FA's as those with simple relational reducts.

Theorem *Simple FA's*

An FA **F** is simple iff its relational reduct $\mathbf{F}_{\lambda}$ is simple.

The next result characterizes the non-simple FA's as those with non-trivial direct decompositions, just like RA's.

Proposition *Non-simple FA's*

An FA **F** is non-simple iff $\mathbf{F}\cong\mathbf{G}\times\mathbf{H}$ for some non-trivial FA's **G** and **H**.

The next result provides subdirect decompositions for FA's paralleling the analogous result for RA's.

Theorem *Subdirect decomposition of FA's into simple components*

Every FA **F** is isomorphic to some subdirect product of simple homomorphic images of **F**.

We now consider a simple metamathematical consequence of the preceding results concerning the algebraic structure of the FA's.

Corollary *FA equations*

An FA equation ε holds in an FA algebra **F** iff equation ε holds in all (simple) homomorphic images of **F**.

2.3 Set-based Boolean, Relation and Fork Algebras

Now, let us briefly describe the concrete, or proper, versions of BA's, RA's and FA's. Much as BA's arise as abstractions from fields of sets, RA's (and FA's) are abstractions from their proper versions.

A *field of sets* (over set U) is an algebra $\mathbf{S}=\langle S,\varnothing,U,{\sim},\cup,\cap\rangle$ of signature β, with $S\subseteq\wp(U)$. Its Boolean ordering is set inclusion $\subseteq$. A field of sets over U is a Boolean algebra that can be embedded into the direct power $\mathbf{T}^U$ of the two-element BA **T** (of truth values).

By Stone Representation Theorem every (abstract) BA is isomorphic to some field of sets [Bell & Slomson '71; Burris & Sankappanavar '81; Halmos '63], and so can be embedded into a direct power of the two-element BA.

A *proper relation algebra* (over set U) is an algebra $\mathbf{P}=\langle P,\varnothing,V,I,{\sim},{}^{T},\cup,\cap,|\rangle$ of signature λ, where

- its BA-reduct $\mathbf{P}_{\beta}=\langle P,\varnothing,U,{\sim},\cup,\cap\rangle$ is a field of sets over $U^2=U\times U$;
- $I\in P$ is the identity (diagonal) relation over U: $I=id_U:=\{\langle u,v\rangle\in U^2/u=v\}$;
- operation ${}^{T}:P\to P$ is relation transposition, i. e. $p^T=\{\langle v,u\rangle\in U^2/\langle u,v\rangle\in p\}$;

- operation $|: P \times P \to P$ is relation composition, i. e.
  $r|s = \{<u,w> \in U^2 / \exists v \in U\ [<u,v> \in r \,\&\, <v,w> \in s]\}$.

The *full PRA over* set U is the proper RA $\mathbf{P}(U^2) := <\wp(U^2), \varnothing, U^2, I, \sim, ^T, \cup, \cap, |>$.

Recall that a PRA (short for proper relation algebra) $\mathbf{P} = <P, \varnothing, V, I, \sim, ^T, \cup, \cap, |>$ over U with $V = U^2$ is simple [Jónsson & Tarski '52, Theorem 4.28, p. 142; Veloso '74, p. 7, 12]. So, the full PRA $\mathbf{P}(U^2)$ and its subalgebras are simple.

In contrast with BA's, not every (abstract) RA can be represented as some proper RA (see e. g. [Maddux '91; Veloso '74].

Now, consider a function $*: V \to U$, where $V \subseteq U^2$. It induces a binary operation $\angle^*$ on relations over U (into $\wp(V)$), called *fork induced by* $*: V \to U$, defined by $r \angle^* s := \{<u,v> \in V / \exists v', v'' \in U\ [<v',v''> \in V \,\&\, v' * v'' = v \,\&\, <u,v'> \in r \,\&\, <u,v''> \in s]\}$ (so $r \angle^* s \subseteq V$).

A *proper fork algebra* (over set U) is an algebra $\mathbf{Q} = <Q, \varnothing, V, I, \sim, ^T, \cup, \cap, |, \angle>$ of signature ϕ, where

- its RA-reduct $\mathbf{Q}_\lambda = <Q, \varnothing, U, \sim, \cup, \cap>$ is a proper relation algebra over set U;
- there exists an underlying coding $*: U^2 \to U$ such that
  - $*$ is injective on $V \subseteq U^2$ (i. e. the restriction $*_{|V}: V \to U$ is one-to-one),
  - operation $\angle: Q \times Q \to Q$ is induced by the restriction $*_{|V}$ of $*$ to V, i. e.
    $r \angle s := r \angle^{*|V} s = \{<u,v> \in V / \exists <v',v''> \in V\ [v' * v'' = v \,\&\, <u,v'> \in r \,\&\, <u,v''> \in s]\}$.

It is not difficult to see that every PFA (short for proper fork algebra) is indeed a fork algebra [Frias et al. '96].

Notice that each injective function $*: U^2 \to U$ gives rise to a *full PFA over* set U $\mathbf{P}^*(U^2) := <\wp(U^2), \varnothing, U^2, I, \sim, ^T, \cup, \cap, |, \angle^*>$ as the $\angle^*$-expansion of the full PRA $\mathbf{P}(U^2)$ by the fork $\angle^*$ induced by coding $*: U^2 \to U$. Thus, the PFA's with $V = U^2$ are the subalgebras of the full PFA's, and they are all simple.

Also, for a (simple) PFA $\mathbf{Q} = <Q, \varnothing, V, I, \sim, ^T, \cup, \cap, |, \angle>$ with $V = U^2$, its underlying coding $*: U^2 \to U$ is bijective iff $V \angle^* V = V$. (For, $V \subseteq V \angle^* V$ iff $<u,v> \in V \angle^* V$ whenever $<u,v> \in V = U^2$ iff for each $v \in U$, we have $v', v'' \in U$ with $v' * v'' = v$.)

Moreover, like BA's and contrasting with RA's, every (abstract) FA can be represented as some proper FA [Frias et al. '95, '96], but we shall not make use of this result here.

3. Boolean Fork Algebras

Two very simple finite FA's are those with one and two elements. We shall presently show that these are the only finite simple FA's.

The trivial one-element FA $\mathbf{1}$ has single element $0 = 1 = \infty$. It is isomorphic to the full PFA $\mathbf{P}^*(\varnothing)$, with carrier $\wp(\varnothing) = \{\varnothing\}$, and underlying coding $* = \varnothing$.

The two-element FA **2** has two elements 0 and $1=\infty$ with $\infty\nabla\infty=\infty$. It is isomorphic to the full PFA $\mathbf{P}^*(\{<u,u>\})$ over singleton $\{<u,u>\}$, with carrier $\wp(\{<u,u>\})=\{\varnothing,\{<u,u>\}\}$, and underlying coding $*$ given by $u*u=u$.

These FA's **1** and **2** are simple. Also, the FA's **1** and **2** are the only FA's, up to isomorhism, with respectively 1 and 2 elements (since $0\nabla 0=0$ and $0\leq 1\leq\infty$). { If $0=\infty$ then $0=1=\infty$, and we have the one-element FA **1**. Otherwise $0\neq\infty$, and $1=\infty$ (since $1;\infty=\infty\neq 0$), also $\pi^\dagger\neq 0$ (since $\pi^\dagger;\rho=\infty\neq 0$), so $0\neq 1\nabla\infty\leq\infty\nabla\infty$ and $\infty\nabla\infty=\infty$, thus we have the two-element FA **2**. }

The RA-reducts of these simple FA's are QRA's, with $1=\infty$ as both quasiprojections. They are Boolean RA's as well.

Recall that a *Boolean* RA is one where ; is $\bullet$, † is the identity function, and $1=\infty$ [Jónsson & Tarski '52, p. 151]. Thus, a Boolean RA is a somewhat uninteresting expansion of a Boolean algebra to an RA. Clearly, every Boolean RA is a QRA, with $1=\infty$ as both quasiprojections.

By analogy with Boolean RA's, let us call an FA $\mathbf{F}=<F,0,\infty,1,^-,^\dagger,+,\bullet,;,\nabla>$ *Boolean* iff its fork ∇ is $\bullet$, i. e. **F** satisfies the equation $\forall x,y\ x\nabla y\approx x\bullet y$.

So, the FA's **1** and **2** are Boolean FA's and Boolean RA's. In fact, it is not difficult to see that the relational reduct of a Boolean FA is a Boolean RA.

Lemma *Boolean RA's and FA's*

The RA-reduct of a Boolean FA is a Boolean RA.

Proof

First, $\pi^\dagger=1\nabla\infty=1\bullet\infty=1$ and $\rho^\dagger=\infty\nabla 1=\infty\bullet 1=1$. So $\infty=\pi^\dagger;\rho=1^\dagger;1=1$.

Now $(r;p^\dagger)\bullet(s;q^\dagger)=(r\nabla s);(p\nabla q)^\dagger=(r\bullet s);(p\bullet q)^\dagger=(r\bullet s);(p^\dagger\bullet q^\dagger)$, so with $r=q=\infty=1$ we obtain $s;p^\dagger=(\infty\bullet s);(p^\dagger\bullet\infty^\dagger)=(1;p^\dagger)\bullet(s;1^\dagger)=p^\dagger\bullet s=s\bullet p^\dagger$. Thus $s;p=s\bullet p$.

From $r\leq\infty=1$ we get $r^\dagger=r$ by [Jónsson & Tarski '52, Theorem 4.6, p. 130].

QED

Thus, an FA **F** is Boolean FA iff $\mathbf{F}=<F,0,\infty,\infty,^-,id,+,\bullet,\bullet,\bullet>$. So, Boolean FA's, like Boolean RA's, are essentially Boolean algebras.

We can now characterize the Boolean FA's as the subalgebras of direct powers of the two-element FA **2**.

Proposition *Characterization of Boolean FA's*

A ϕ-algebra **F** is a Boolean FA iff **F** can be embedded into some direct power $\mathbf{2}^I$ of the two-element FA **2**.

Proof

($\Leftarrow$) Any subdirect power of (two-element) Boolean FA's is a Boolean FA.

($\Rightarrow$) Consider a Boolean FA $\mathbf{F}=<F,0,\infty,1,^-,^\dagger,+,\bullet,;,\nabla>$. Its BA-reduct $\mathbf{F}_\beta$ is a BA, and can be embedded into a field of sets over some set I; so we have a BA-embedding e of $\mathbf{F}_\beta=<R,0,\infty,^-,+,\bullet>$ into the direct power $(\mathbf{2}_\beta)^I$ of the two-element BA $\mathbf{2}_\beta$. By the lemma ; and ∇ are $\bullet$, † is the identity function, and

$1=\infty$. Thus, e is an FA-embedding of Boolean FA **F** into the direct power $\mathbf{2}^I$ of the two-element FA **2**.

QED

4. Finite Fork Algebras

We shall now describe the finite FA's as the finite direct powers of the simple ones, the latter being those with one and two elements.

By property of Boolean algebras, a finite FA must have cardinality 2^n, for some $n\geq 0$. The direct power $\mathbf{2}^n$ provide an (uninteresting) example of a (Boolean) FA cardinality 2^n for each $n\geq 0$. Thus, the finite spectrum of the FA's is the set $\{2^n/n\in\mathbf{N}\}$.

We now show that finite simple QRA's have at most two elements. (Notice that for a proper QRA **P** over a finite set U, we must have $|U|\leq 1$ [Tarski & Givant '87, p. 96].)

Lemma *Upper bound on finite simple QRA's*

There exists no finite simple QRA with more than two elements.

Proof

Consider a finite QRA $\mathbf{R}=\langle R,0,\infty,1,^{-},^{\dagger},+,\bullet,;\rangle$ with $n\geq 0$ elements.

Given any element $r\in R$, consider its iterates $r^k=r;\ldots;r$ (k times): $r^0:=1$ and $r^{k+1}:=r;r^k$. The iterates $r^0,r^1,\ldots,r^n$ cannot be all distinct.

In particular, for quasiprojection $\rho\in R$, for some $i\geq 0$ and $j>0$ $\rho^i=\rho^{i+j}$.

By applying $\rho^\dagger;\rho=1$ i times to $\rho^i;\rho^j=\rho^i$, we obtain $\rho^j=\rho^0=1$.

Now, by applying j times $\pi^\dagger;\rho=\infty$ to $\rho^j=1$, we have $(\pi^\dagger)^j=\infty$.

{ From $\rho;\rho^k=\rho^{k+1}\leq(\pi^\dagger)^l$ we get $\rho^k\leq\infty;\rho^k=\pi^\dagger;\rho;\rho^k\leq\pi^\dagger;(\pi^\dagger)^l=(\pi^\dagger)^{l+1}$;
so $\rho^j=1=(\pi^\dagger)^0$ yields $\rho=\rho^1\leq(\pi^\dagger)^{j-1}$, whence $\infty=\pi^\dagger;\rho\leq\pi^\dagger;(\pi^\dagger)^{j-1}=(\pi^\dagger)^j$. }

So, j applications of $\pi^\dagger;\pi=1$ yield $1=\infty$.

Finally, since **R** is simple, by Tarski's rule, for each $r\neq 0$, $\infty=\infty;r;\infty=1;r;1=r$.

Hence $R\subseteq\{0,\infty\}$, and thus $|R|\leq 2$.

QED

Since FA's have (quasi)projections, we can now see that, up to isomorphism, **1** and **2** are all the finite simple FA's.

Proposition *Description of finite simple FA's*

A ϕ-algebra **F** is a finite simple FA iff either $\mathbf{F}\cong\mathbf{1}$ or $\mathbf{F}\cong\mathbf{2}$.

We can now see that the finite FA's are not very interesting.

Lemma *Boolean finite FA's*

Every finite FA is Boolean.

Proof

By the subdirect decomposition theorem, FA **F** is isomorphic to a subdirect product of simple homomorphic images $\mathbf{F}_i$, $i\in I$. Now, each homomorphic image $\mathbf{F}_i$ is a simple finite FA, so with at most two elements. Thus, every $\mathbf{F}_i$ is Boolean, and so is a subdirect product of them.

QED

We now have a complete description of the finite FA's as the finite direct powers of the two-element simple FA **2**.

Theorem *Description of finite FA's*

A ϕ-algebra **F** is a finite FA iff **F** is isomorphic to some finite direct power $\mathbf{2}^n$ of the two-element FA **2**.

Proof

($\Leftarrow$) Clearly any finite direct power of (two-element) FA's is a finite FA.

($\Rightarrow$) Considering a finite FA **F** with carrier F, we proceed by induction on $|F|$.

a) For $|F|\leq 2$, we must have either $\mathbf{F}\cong\mathbf{1}\cong\mathbf{2}^0$ or $\mathbf{F}\cong\mathbf{2}\cong\mathbf{2}^1$.

b) Now, consider the case $|F|>2$.

Then, by the previous proposition, **F** cannot be simple.

By the proposition on non-simple FA's, we have a decomposition $\mathbf{F}\cong\mathbf{G}\times\mathbf{H}$ for some proper homomorphic images **G** and **H** of **F**. In particular $|G|<|F|$ and $|H|<|F|$. The inductive hypothesis yields $\mathbf{G}\cong\mathbf{2}^p$ and $\mathbf{H}\cong\mathbf{2}^q$, so $\mathbf{F}\cong\mathbf{G}\times\mathbf{H}\cong\mathbf{2}^p\times\mathbf{2}^q\cong\mathbf{2}^{p+q}$.

QED

Thus, the finite FA's are not very interesting FA's because they are the finite direct powers of the two-element simple FA **2**, so all are Boolean FA's, and essentially BA's. Thus, there exists exactly one, up to isomorphism, FA of each finite cardinality 2^n for $n\geq 0$. In particular, the RA-reduct of a finite RA has exactly one, up to isomorphism, expansion to an FA.

5. Fork-Expansions of Relational Algebras

In the sequel we will analyze infinite FA's, comparing and contrasting them with the finite ones. The finite FA's are characterized by their RA-reducts, which does not happen with the infinite FA's, as we will shortly see. So, this contrast between finite and infinite FA's is connected to a comparison between RA's and FA's.

The results concerning the algebraic structure of FA's and their metamathematical consequences are very similar to their analogs for RA's. This may give the impression of similarity in the behavior of RA's and FA's. But, representability, as mentioned previously, is already a clear difference. Further distinctions, indicating that they are quite different, will seen in the next section.

For the purpose of comparing RA's and FA's, we now examine some considerations and introduce some terminology of a somewhat ad-hoc nature.

What FA's have more than RA's is a fork operation. This difference vanishes in the Boolean FA's, including the finite ones, when fork is •. A not so extreme case is that where fork is • for the element ∞. Let us call *special* those FA's $\mathbf{F}$=<F,0,∞,1,$^{-}$,†,+,•,;,∇> where $\infty\nabla\infty=\infty$ (so $\infty\nabla\infty=\infty\bullet\infty$), but $1\nabla 1\neq 1$. Notice that special FA's are non-Boolean; we shall have occasion to construct many special FA's.

Now, consider an RA $\mathbf{R}$ with carrier R$\subseteq$F. We naturally call RA $\mathbf{R}$ *expandable by* binary operation ∇:F×F$\rightarrow$F iff R is closed under ∇: r∇s$\in$R whenever r,s$\in$R. The next result characterizes expandability of subalgebras of reducts.

Lemma *Expandability of subalgebras of reducts*

Consider an algebra $\mathbf{F}$ of FA-signature ϕ, with defined elements $\pi:=(1\nabla\infty)^{\dagger}$ and $\rho:=(\infty\nabla 1)^{\dagger}$, and a λ-subalgebra $\mathbf{R}$ of its reduct $\mathbf{F}_{\lambda}\in\mathbf{Alg}(\lambda)$.

a) If $\mathbf{F}$ satisfies axiom (∇-def), then subalgebra $\mathbf{R}$ of $\mathbf{F}_{\lambda}$ is expandable by ∇ :F×F$\rightarrow$F iff π and ρ are in R.

b) If π and ρ are in R and $\mathbf{F}$ is an FA, then so is the ∇-expansion $\mathbf{R}^{\nabla}\in\mathbf{Alg}(\phi)$.

Proof

a) Since $\mathbf{F}$ satisfies (∇-def), we have $r\nabla s=(r;\pi^{\dagger})\bullet(s;\rho^{\dagger})$ for every r,s$\in$R$\subseteq$F. Thus, R is closed under ∇ iff $\pi=(1\nabla\infty)^{\dagger}$ and $\rho=(\infty\nabla 1)^{\dagger}$ are in R.

b) Since R will be closed under ∇, the ∇-expansion $\mathbf{R}^{\nabla}\in\mathbf{Alg}(\phi)$ of subalgebra $\mathbf{R}$ of $\mathbf{F}_{\lambda}$ will be a ϕ-subalgebra of FA $\mathbf{F}\in\mathbf{FA}$, and thus $\mathbf{R}^{\nabla}\in\mathbf{FA}$.

QED

A tool for our comparison between RA's and FA's can use the (cardinal) number of non-isomorphic FA-expansions of a given RA $\mathbf{R}$, its expandability index $\varepsilon(\mathbf{R})$. Not surprisingly, some RA's (for instance, those that are not QRA's) have null expandability indices. Let us call an RA $\mathbf{R}$ *rigid* iff it has at most one, up to isomorphism, FA-expansion: $\varepsilon(\mathbf{R})\leq 1$. As we have seen, the RA-reducts of finite FA's are all rigid, with expandability index 1. At the other extreme, an RA may have many non-isomorphic FA-expansions.

Consider an RA $\mathbf{R}$, with cardinality $|R|=\kappa$. Since a possible fork is a binary operation ∇:R×R$\rightarrow$R, $\mathbf{R}$ may have at most $\kappa^{(\kappa.\kappa)}$ FA-expansions: $\varepsilon(\mathbf{R})\leq\kappa^{(\kappa.\kappa)}$. Thus, an infinite RA $\mathbf{R}$, with cardinality $|R|=\kappa\geq\aleph_0$ has $\varepsilon(\mathbf{R})\leq\kappa^{(\kappa.\kappa)}=2^{\kappa}$.

Among the non-rigid RA's we shall consider two kinds with high expandability indices. Consider an RA $\mathbf{R}$, with cardinality $|R|=\kappa$; we shall call RA $\mathbf{R}$ *flexible* (respectively *explosive*) iff it has expandability index $\varepsilon(\mathbf{R})\geq\kappa$ (respectively $\varepsilon(\mathbf{R})=2^{\kappa}$). Such RA's demonstrate quite clearly the diversity of possible fork operations. In the next section we will construct examples of such RA's with given infinite cardinality: a simple flexible PRA $\mathbf{P}$ with at least κ pairwise non-isomorphic special PFA-expansions, as well as κ pairwise non-isomorphic explosive RA's, each one of them with 2^{κ} non-isomorphic special FA-expansions.

The next result shows a simple case where homomorphisms for reducts can be guaranteed to preserve fork.

Lemma *Surjective factorization into homomorphism for reduct*

Consider algebras $\mathbf{F},\mathbf{G},\mathbf{H}\in\mathbf{Alg}(\phi)$ and a surjective ϕ-homomorphism $h:F\to H$ of $\mathbf{F}$ onto $\mathbf{H}$. Let $h:F\to H$ be a λ-homomorphism of $\mathbf{H}_\lambda$ onto $\mathbf{G}_\lambda$ such that the composite $g=f\circ h$ is a ϕ-homomorphism $g:F\to G$ of $\mathbf{F}$ into $\mathbf{G}$. Then, $f:H\to G$ is a ϕ-homomorphism of $\mathbf{H}$ into $\mathbf{G}$.

Proof

Since $h:F\to H$ is surjective, every $t\in H$ is $f(r)$ for some $r\in F$.

Now, for $r,s\in H$, we have $f[h(r)\nabla^{\mathbf{H}}h(s)]=f[h(r)]\nabla^{\mathbf{G}}f[h(s)]$.

{ Indeed, since h is a ϕ-homomorphism, $f[h(r)\nabla^{\mathbf{H}}h(s)]=f[h(r\nabla^{\mathbf{F}}s)]$; and, since $g=f\circ h$ is a ϕ-homomorphism $f[h(r\nabla^{\mathbf{F}}s)]=g(r\nabla^{\mathbf{F}}s)]=g(r)\nabla^{\mathbf{G}}g(s)=f\circ h(r)\nabla^{\mathbf{G}}f\circ h(s)$. }

Therefore, f preserves fork.

QED

An application of the preceding lemma to direct products is given in the next corollary. We shall use it later in connection with simple components.

Corollary *Homomorphism of reduct of direct-product component*

Consider a family of ϕ-algebras $\mathbf{F}_i\in\mathbf{Alg}(\phi)$, for $i\in I$, with direct product $\times_{i\in I}\mathbf{F}_i$, with projections p_i, and a ϕ-homomorphism $g:\times_{i\in I}F_i\to G$ of $\times_{i\in I}\mathbf{F}_i$ into ϕ-algebra $\mathbf{G}$. Then, every λ-homomorphism $f:F_i\to G$ of $(\mathbf{F}_i)_\lambda$ into $\mathbf{G}_\lambda$ such that $f\circ p_i=g$ is a ϕ-homomorphism of $\mathbf{F}_i$ into $\mathbf{G}$.

6. Infinite Fork Algebras

We shall now examine infinite FA's. We will see that there exist many (simple proper) FA's of each infinite cardinality (even with the same RA-reduct), in sharp contrast with the finite case.

For any infinite set U, U and $U^2=U\times U$ have the same cardinality. So, there exists an injective function $*:U^2\to U$. Now, each such injective function $*:U^2\to U$ gives rise to a full PFA $\mathbf{P}^*(U^2)=<\wp(U^2),\emptyset,U^2,I,\breve{},^T,\cup,\cap,|,\angle^*>$ over U. Notice that $\mathbf{P}^*(U^2)$ is a simple proper FA with cardinality 2^κ, where $\kappa=|U|$.

Thus, by Downward Löwenheim-Skolem Theorem, for each infinite cardinality $\kappa\geq\aleph_0$, there exists a simple FA with cardinality κ.

We shall exhibit some (simple proper) FA's of given infinite cardinality.

First, consider an infinite algebra $\mathbf{A}$. Given an infinite subset G with cardinality $|G|=\gamma$, let $\mathbf{A}[G]$ be the subalgebra of $\mathbf{A}$ generated by G. Note that $\mathbf{A}[G]$ has cardinality $|A[G]|=\gamma.\aleph_0=\gamma$ [Burris & Sankappanavar '81, p. 32].

In particular, for an infinite set U we have the infinite full PRA $\mathbf{P}(U^2)$ over set U. Given any infinite subset $G\subseteq\wp(U^2)$, the subalgebra $\mathbf{P}_G=\mathbf{P}(U^2)[G]$ of the full PFA $\mathbf{P}(U^2)$ generated by G is a simple PRA with cardinality $|G|\geq\aleph_0$.

Also, given an infinite set U with cardinality $\kappa\geq\aleph_0$, let $\wp_\omega(U^2)$ be the set of finite subsets of U^2, and notice that $|\wp_\omega(U^2)|=\kappa.\aleph_0=\kappa$. Given an injective function $*:U^2 \to U$, the subalgebra $\mathbf{P}^*_\omega(U^2) := \mathbf{P}^*(U^2)[\wp_\omega(U^2)]$ of the full PFA $\mathbf{P}^*(U^2)$ generated by $\wp_\omega(U^2)$ is a simple PFA with cardinality κ.

Proposition *Large simple proper FA's*

For each infinite cardinal $\kappa \geq \aleph_0$ there exists a simple PFA with cardinality κ.

We shall now construct some more such (simple proper) FA's.

First, we generalize the preceding construction. Each injective function $*:U^2 \to U$ induces a fork operation $\angle^*$ on relations over U, which gives rise to *induced projections* $p^* := (I \angle^* U^2)^T$ and $q^* := (U^2 \angle^* I)^T$.

Lemma *Large simple PRA's with PFA expansions*

Consider an infinite set U with cardinality $\kappa \geq \aleph_0$ and an injective function $*:U^2 \to U$ with induced projections p^* and q^*. Given an infinite set G with $\{p^*, q^*\} \subseteq G \subseteq \wp(U^2)$, let $\mathbf{P}_G = \mathbf{P}(U^2)[G]$ be the subalgebra of the full PRA $\mathbf{P}(U^2)$ generated by G. Then, $\mathbf{P}_G$ is a simple PRA with cardinality $|G|$ that is expandable by induced fork $\angle^*$ and the $\angle^*$-expansion of $\mathbf{P}_G$ is a simple PFA.

Proof

The RA-reduct of the full PFA $\mathbf{P}^*(U^2)$ is the full PRA $\mathbf{P}(U^2)$. So, $\mathbf{P}_G$ is a λ-subalgebra of $\mathbf{P}^*(U^2)_\lambda$ with $\{p^*, q^*\} \subseteq G \subseteq P^G$. Thus, the first assertion follows from the lemma on expandability of subalgebras of reducts. $\mathbf{P}_G$ and its $\angle^*$-expansion are simple PFA, for they are subalgebras of full proper algebras.

QED

We now introduce a tool for the analysis of FA's. Given an FA $\mathbf{F}$, we consider the set of its *stable sub-identities* $SsI(\mathbf{F}) := \{f \in F / f \leq 1 \ \& \ f \nabla f \leq f\}$.

Notice that any ϕ-isomorphism between algebras $\mathbf{F}$ and $\mathbf{G}$ in $\mathbf{Alg}(\phi)$ gives a bijection between $SsI(\mathbf{F})$ and $SsI(\mathbf{G})$, so $|SsI(\mathbf{F})| = |SsI(\mathbf{G})|$.

We are going to construct infinite simple PFA's whose sets of stable sub-identities have smaller cardinalities. We shall control the set of stable sub-identities by means of the set of fixpoints of the underlying coding.

Given a function $*:U^2 \to U$, we consider the its *set of fixpoints* $fxpt(*) := \{u \in U / u*u = u\}$. This set can be conveniently represented by its identity $Idfx(*) := \{\langle u,u \rangle \in U^2 / u*u = u\}$. Notice that $|Idfx(*)| = |fxpt(*)|$.

The next lemma shows a connection between the set of stable sub-identities of a (simple) PFA and the set of fixpoints of its underlying coding.

Lemma *Simple PFA connection*: SsI vs. Idfx

Consider a simple PFA $\mathbf{Q} = \langle Q, \varnothing, U^2, I, {}^-, {}^T, \cup, \cap, |, \angle^* \rangle$ with fork $\angle^*$ induced by coding $*:U^2 \to U$. Then $SsI(\mathbf{Q}) = \wp(Idfx(*)) \cap Q$.

Proof

For any $r \subseteq U^2$, with $r \subseteq I$, we have $r \angle^* r \subseteq r$ iff $r \subseteq Idfx(*)$.

{ If $r\angle^* r \subseteq r$ then, for any $<u,u> \in r \subseteq I$, $<u,u*u> \in r\angle^* r \subseteq r \subseteq I$, thus $u=u*u \in fxpt(*)$.
If $r \subseteq Idfx(*)$ then, for any $<u,v> \in r\angle^* r$, $v=v'*v''$ for some $<u,v'> \in r$, $<u,v''> \in r$; so $v'=u=v''$, and $v=u*u$ with $<u,u> \in r \subseteq Idfx(*)$, whence $u*u=u$ and $<u,v>=<u,u> \in r$. }
Hence, since $Idfx(*) \subseteq I$, $r \in SsI(\mathbf{Q})$ iff $r \in Q$ and $r \subseteq Idfx(*)$.

QED

To control the set of stable sub-identities of a (simple) PFA we control the set of fixpoints of its underlying coding. We do the latter by constructing special codings with a given set of fixpoints.

The next results presents a set-theoretical construction for a special coding on an infinite set U whose set of fixpoints is a given subset $S \subseteq U$ with smaller cardinality.

Proposition *Special coding with given set of fixpoints*

Consider an infinite set U with cardinality $\kappa \geq \aleph_0$. Given a subset $S \subseteq U$ with $|S| < \kappa$, there exists a bijective function $*_S : U^2 \to U$ such that $fxpt(*_S)=S$.

Proof

Let $I := \{<u,v> \in U^2 / u=v\}$ be the identity (diagonal) relation over U, and $I^\sim = \{<u,v> \in U^2 / u \neq v\}$. Then, $|U^2| = \kappa$, $|I| = \kappa$, and $|I^\sim| = \kappa$.

Given a subset $S \subseteq U$ with $|S| < \kappa$, consider its complement with respect to U: $S' := \{u \in U / u \notin s\}$. Since $|S| < \kappa$, we have $|S'| = \kappa$, and we can partition S' into disjoint subsets A and B of U, both with cardinality κ. Now, since $|I^\sim| = \kappa$, we have a bijection $f : I^\sim \to A$. Also, we have a bijection $g : S' \to B$ without fixpoints.
{ We can partition B into $\aleph_0$ subsets B_n, $n \in \mathbf{N}$, all with cardinality $\kappa = |B|$. So, we have bijections $g_A : A \to B_0$, and $g_n : B_n \to B_{n+1}$, $n \in \mathbf{N}$, with pairwise disjoint domains and images. Their disjoint union gives a bijective g from $S' = A \cup B$ onto $B = \cup_{n \in \mathbf{N}} B_n$ without fixpoints, as required. }

We now define $*_S : U^2 \to U$ as follows:

for $u \in S$ we set $u *_S u = u$ (notice that $u \notin A \cup B$);
for $u \in S'$ we set $u *_S u = g(u)$ (notice that $g(u) \in B$);
for $<v,w> \in I^\sim$ we set $v *_S w = f(v,w)$ (notice that $f(v,w) \in A$).

So, $*_S : U^2 \to U$ is a bijection, from $U^2 = id_S \cup id_{S'} \cup I^\sim$ onto $U = S \cup B \cup A$, because it is the disjoint union of bijections with pairwise disjoint domains and images. Also, $u *_S u = u$ iff $u \in S$, because for $u \notin S$ $u *_S u = g(u) \neq u$. Thus $fxpt(*_S) = S$.

QED

By a (special) *coding* on (infinite) set U *controlled* by subset $S \subseteq U$ (with $|S| < |U|$) we mean a bijective function $*_S : U^2 \to U$ with $fxpt(*_S) = S$ provided by the preceding proposition. It induces fork $\angle^S$ and projections p^S and q^S.

We now construct some simple infinite PFA's with sets of stable sub-identities of smaller cardinality, by putting together the preceding constructions.

Given an infinite set U with cardinality κ, consider the set of its relations with smaller cardinality: $\wp_\kappa(U^2):=\{r\subseteq U^2/|r|<\kappa\}$. Notice that $|\wp_\kappa(U^2)|\geq\kappa$.

Proposition *Large simple PRA's with controlled fork expansions*

Let U be an infinite set with cardinality $\kappa\geq\aleph_0$. For each subset $S\subseteq U$ with $|S|<\kappa$ consider a coding $*_S:U^2\to U$ on U controlled by S, inducing fork $\angle^S$ and projections p^S and q^S. Given a set K with $\{p^S,q^S\}\cup\wp_\kappa(U^2)\subseteq K\subseteq\mathbf{P}(U^2)$, consider the subalgebra $\mathbf{P}_K{}^S:=\mathbf{P}(U^2)[K]$ of the full PRA $\mathbf{P}(U^2)$ generated by K. Then, $\mathbf{P}_K{}^S$ is a simple non-Boolean PRA with cardinality $|K|$ that is expandable by fork $\angle^S$. Moreover, the $\angle^S$-expansion $\mathbf{Q}_K{}^S$ of $\mathbf{P}_K{}^S$ is a simple special PFA with set of stable sub-identities $SsI(\mathbf{Q}_K{}^S)=\wp(id_S)$, where $id_S=\{<u,u>\in U^2/u\in S\}$.

Proof

1. By the lemma on large simple PRA's with PFA expansions, we have the first assertion about $\mathbf{P}_K{}^S$ and the fact that $\angle^S$-expansion $\mathbf{Q}_K{}^S$ is a simple PFA.
2. By the remark on simple PFA's with bijective underlying coding, $V\angle^S V=V$.
3. Now $I=id_S\cup id_{S'}$, so $I\angle^S I=(id_S\angle^S id_S)\cup(id_S\angle^S id_{S'})\cup(id_{S'}\angle^S id_S)\cup(id_{S'}\angle^S id_{S'})$. Since $fxpt(*_S)=S$, $(id_{S'}\angle^S id_{S'})\cap I=\varnothing$. Thus $I\angle^S I\not\subseteq I$.
   { Indeed, if $<u,v>\in id_{S'}\angle^S id_{S'}$ we have $v=v'*v''$ with $<u,v'>,<u,v''>\in id_{S'}$, so $v'=u=v''\in S'$ and $v=u*u\neq u$, whence $<u,v>\notin I$. }
4. By the lemma on simple PFA connection between SsI and Idfx, we have $SsI(\mathbf{Q}_K{}^S)=\wp(Idfx(*_S))\cap Q_K{}^S$. But, for any $r\subseteq Idfx(*_S)$, $|r|\leq|Idfx(*_S)|=\gamma<\kappa$, so $r\in\wp_\kappa(U^2)\subseteq K\subseteq Q_K{}^S$. Thus $SsI(\mathbf{K}^S)=\wp(id_S)\cap Q_K{}^S=\wp(id_S)$.
5. By the lemma on Boolean RA's and FA's, RA $\mathbf{P}_K{}^S$ cannot be a Boolean RA.

QED

Let us call a PFA $\mathbf{Q}$ *size-controlled by* cardinal γ iff its set $SsI(\mathbf{Q})$ of stable sub-identities has cardinality $|SsI(\mathbf{Q})|=2^\gamma$.

For a denumerably infinite set U, we have $|\wp_\omega(U^2)|=\aleph_0=|U|$. For an infinite set U with cardinality $\kappa=2^\alpha$, we have $|\wp_\kappa(U^2)|=\kappa$ [Sigler '66, p. 62, 132].

We can now construct a simple PRA of given infinite cardinality with (special) PFA-expansions size-controlled by smaller cardinals.

Lemma *Large simple PRA's with smaller size-controlled PFA-expansions*

For each infinite cardinal $\kappa=\aleph_0$ or $\kappa=2^\alpha$ with $\alpha\geq\aleph_0$, there exists a simple non-Boolean PRA $\mathbf{P}$ of cardinality κ, such that, for each smaller cardinal $\gamma<\kappa$, $\mathbf{P}$ has a PFA-expansion $\mathbf{Q}_\gamma$ that is special and size-controlled by γ.

Proof

1. Given such a cardinal κ, the set of smaller cardinals $\gamma<\kappa$ has cardinality κ. For each smaller cardinal $\gamma<\kappa$, consider a subset $S_\gamma\subseteq U$ of cardinality $|S_\gamma|=\gamma$

and let $*_\gamma: U^2 \to U$ be a corresponding coding on U controlled by S_γ, with $fxpt(*_\gamma)=S_\gamma$, which induces fork $\angle^\gamma$ and projections p^γ and q^γ.

Set $H:=\wp_\kappa(U^2)\cup[\cup_{\gamma<\kappa}\{p^\gamma,q^\gamma\}]$ and note that it has cardinality $|H|=\kappa$.

2. By the proposition on large simple PRA's with controlled fork expansions, $\mathbf{P}:=\mathbf{P}(U^2)[H]$ is a simple PRA with cardinality $|Q|=|H|=\kappa$. Also, for each cardinal $\gamma<\kappa$, since $\{p^\gamma,q^\gamma\}\cup\wp_\kappa(U^2)\subseteq H$, PRA $\mathbf{P}$ is expandable by induced fork $\angle^\gamma$ to simple special PFA $\mathbf{Q}_\gamma$ with set $SsI(\mathbf{Q}_\gamma)$ of cardinality $|SsI(\mathbf{Q}_\gamma)|=2^\gamma$.

{ By the lemma on Boolean RA's and FA's, RA $\mathbf{P}$ cannot be a Boolean RA. }

QED

We can now show a simple PRA of given infinite cardinality that is flexible, expanding to many non-isomorphic special size-controlled PFA's.

Theorem *Large simple flexible PRA's: with many PFA-expansions*

For each infinite cardinal $\kappa=\aleph_0$ or $\kappa=2^\alpha$ with $\alpha\geq\aleph_0$, there exists a non-Boolean simple flexible PRA $\mathbf{P}$ of cardinality κ: PRA $\mathbf{P}$ has at least κ pairwise non-isomorphic PFA-expansions $\mathbf{Q}_\gamma$, (which are special).

Proof

Given such a cardinal κ, the set of smaller cardinals $\gamma<\kappa$ has cardinality κ

By the previous lemma, we have a simple PRA $\mathbf{P}$ of cardinality κ, which has a special PFA-expansion $\mathbf{Q}_\gamma$ size-controlled by each smaller cardinal $\gamma<\kappa$.

Now, consider distinct cardinalities $\eta,\delta<\kappa$.

Then PFA $\mathbf{Q}_\eta$ has $SsI(\mathbf{Q}_\eta)$ with cardinality 2^η and $\mathbf{Q}_\delta$ has $SsI(\mathbf{Q}_\delta)$ with cardinality 2^δ. Hence they cannot be isomorphic.

Therefore, there are κ non-isomorphic simple special PFA's $\mathbf{Q}_\gamma$ with $\gamma<\kappa$.

QED

We can now construct many explosive RA's of given infinite cardinality, each one of them expanding to really many non-isomorphic FA's.

Theorem *Many large explosive RA's: each one with many FA-expansions*

For each infinite cardinal $\kappa=\aleph_0$ or $\kappa=2^\alpha$ with $\alpha\geq\aleph_0$, there exist at least κ pairwise non-isomorphic non-Boolean explosive RA's of cardinality κ, each one of them having 2^κ pairwise non-isomorphic (special) FA-expansions.

Proof

1. Let $\mathbf{P}$ be the simple non-Boolean PRA of cardinality κ of the lemma on large simple PRA's with smaller size-controlled PFA-expansions Consider a subset with $0\in I\subseteq\kappa$, so $0<|I|\leq|\kappa|=\kappa$. Then, since each $\gamma\in I\subseteq\kappa$ is a cardinal $\gamma<\kappa$, PRA $\mathbf{P}$ has a simple special PFA expansion $\mathbf{Q}_\gamma$ whose set $SsI(\mathbf{Q}_\gamma)$ has cardinality 2^γ. The direct power $\mathbf{P}^I$ is a non-Boolean RA with cardinality $|P^I|=|I|.\kappa=\kappa$. Set $I':=I-\{0\}$.

2. Consider a subset $H\subseteq I'$ and set $H':=I-H$. This decomposes I into the union of disjoint subsets H and $H'=(I'-H)\cup\{0\}$. Now, for each $\gamma\in I$, let $\mathbf{F}_\gamma:=\mathbf{Q}_\gamma$ for $\gamma\in H$,

and $\mathbf{F}_\gamma := \mathbf{Q}_0$ for $\gamma \in H'$. The direct product $\mathbf{F}_H := \times_{\gamma\in I}\mathbf{F}_\gamma$ is a special FA, with carrier $F_H = \times_{\gamma\in I}F_\gamma = P^I$, and having $\{\mathbf{Q}_\eta / \eta \in H\} \cup \{\mathbf{Q}_0\}$ as its set of simple components. Moreover, since each $\mathbf{F}_\gamma$ is an FA-expansion of $\mathbf{P}$, the direct product $\mathbf{F}_M = \times_{\gamma\in I}\mathbf{F}_\gamma$ is an FA-expansion of the direct power $\mathbf{P}^I$.

3. Given two distinct subsets $M,N \subseteq I'$, the corresponding FA-expansions $\mathbf{F}_M$ and $\mathbf{F}_N$ of $\mathbf{P}^I$ are not isomorphic.

{ For definiteness, say $M \subset N \subseteq I'$ and consider an element $\nu \in (N-M) \neq \varnothing$.
Then, $\mathbf{F}_\nu = \mathbf{Q}_\nu$ is a simple component of $\mathbf{F}_N$ with $|SsI(\mathbf{F}_\nu)| = 2^\nu$.
The simple components $\mathbf{F}$ of $\mathbf{F}_M$ have $|SsI(\mathbf{F})| \in \{2^\mu / \mu \in M\} \cup \{2^0\}$.
Thus, since $\nu \notin M \cup \{0\}$, $\mathbf{F}_\nu$ cannot be a simple component of $\mathbf{F}_M$.
But, for the RA-reducts, $\mathbf{P} = (\mathbf{F}_\nu)_\lambda$ is a simple component of $\mathbf{P}^I = (\mathbf{F}_M)_\lambda$.
So, by the corollary on homomorphism of reduct of direct-product component, FA's $\mathbf{F}_M$ and $\mathbf{F}_N$ of $\mathbf{P}^I$ cannot be isomorphic. }

4. Hence, for $M,N \subseteq I'$, $\mathbf{F}_M$ and $\mathbf{F}_N$ are isomorphic iff $M=N$.
Thus, $\mathbf{P}^I$ has 2^κ non-isomorphic FA-expansions $\mathbf{F}_M$ with $M \in \wp(I')$.

5. We now examine the structure of the direct power RA $\mathbf{P}^I$. Let $|I| = \gamma$.
First, since $\mathbf{P}$ is a simple (infinite) PRA, it has no proper ideal elements.
Thus, the direct power RA $\mathbf{P}^I$ has exactly 2^γ ideal elements (see Appendix).

6. Now, consider two subsets $\Gamma,\Delta \subseteq I$ with $0 \in \Gamma \cap \Delta$ with distinct cardinalities.
Then, the direct powers $\mathbf{P}^\Gamma$ and $\mathbf{P}^\Delta$ cannot be isomorphic.
{ With $|\Gamma| = \gamma \neq \delta = |\Delta|$, $\mathbf{P}^\Gamma$ has 2^γ ideal elements, while $\mathbf{P}^\Delta$ has 2^γ ideal elements. }

7. Hence, for subsets $\Gamma,\Delta \subseteq I$ with $0 \in \Gamma \cap \Delta$, $\mathbf{P}^\Gamma$ and $\mathbf{P}^\Delta$ are isomorphic iff $|\Gamma| = |\Delta|$.
Thus, there are κ pairwise non-isomorphic RA's $\mathbf{P}^I$ with $|I| = |I'| \in (\kappa - \{0\})$.

QED

7. CONCLUSION

A fork algebra (FA, for short) is a relational algebra (RA, for short) enriched with a new binary operation, called fork. They have been introduced because their equational calculus has applications in program construction, in addition to some interesting connections with algebraic logic.

In this paper, which stems from the crucial, though simple, observation of the interdefinability of fork and projections in FA's, we examine the finite and infinite FA's and contrast them with RA's. We show that the finite FA's are somewhat uninteresting, being essentially Boolean algebras, by describing the finite FA's as the finite direct powers of the two-element FA. This contrasts with the case of infinite FA's: we introduce a technique for the analysis of FA's and use some set-theoretical constructions to exhibit many infinite (simple proper) RA's of each infinite cardinality with many expansions to FA's.

Some background about fork algebras and their reducts (relation and Boolean algebras) is provided in section 2: their abstract versions, some simple results concerning the algebraic structure of FA's, and their concrete, set-based, versions (fields of sets and proper FA's and RA's).

In section 3 we examine the Boolean FA's (those where fork is Boolean meet). We first show that their reducts are Boolean RA's, and thus essentially Boolean algebras. Then we characterize the Boolean FA's as the subalgebras of direct powers of the two-element FA.

These results are applied to finite FA's in section 4, . By showing that no finite simple QRA can have more than two elements, we can conclude that the only simple finite FA's are those with one or two elements. We then show that every finite FA is Boolean and describe the finite FA's as the finite direct powers of the two-element FA. This makes the finite FA's somewhat uninteresting: for each $n \geq 0$ there exists exactly one, up to isomorhism, FA of cardinality 2^n, which is essentially a BA, and each finite FA is completely determined by its RA-reduct. This contrasts sharply with the case of infinite FA's.

In section 5 we examine fork-expansions of RA's, aiming at comparing RA's and FA's. As a tool for this purpose we introduce the expandability index $\varepsilon(\mathbf{R})$ of an RA $\mathbf{R}$ as the (cardinal) number of its non-isomorphic FA-expansions and consider three kinds of RA's. At one extreme, we have the rigid ones, with $\varepsilon(\mathbf{R}) \leq 1$, which include the RA-reducts of finite FA's as well as those that are not QRA's. At the other extreme, we have RA's with many FA-expansions vis-à-vis their cardinalities: flexible and explosive RA's $\mathbf{R}$ with expandability index $\varepsilon(\mathbf{R}) \geq \kappa$ (respectively $\varepsilon(\mathbf{R}) = 2^{\kappa}$), where $\kappa = |R|$. Such (infinite) RA's demonstrate quite clearly the diversity of possible fork operations.

Then, in section 6, we examine infinite FA's and exhibit many infinite (simple proper) RA's of each infinite cardinality with many expansions to FA's. The existence of simple FA's of each given infinite cardinality is clear in view of the Downward Löwenheim-Skolem Theorem. To obtain simple proper RA's, and FA-expansions, we consider subalgebras of a full PFA generated by sets including $\wp_{\gamma}(U^2) = \{r \subseteq U^2 / |r| < \gamma\}$ with $\gamma \leq |U^2|$. The basic idea is quite straightforward: an infinite set U has (many) injective functions $*: U^2 \to U$ to be used as underlying codings, each such function inducing a fork operation $\angle^*$ on relations over U as well as projections $p^* := (I \angle^* U^2)^T$ and $q^* := (U^2 \angle^* I)^T$, by including the latter among the generators we guarantee fork-expandability. We then introduce a technique - set $\{f \in F / f \leq 1 \ \& \ f \nabla f \leq f\}$ of stable sub-identities - for the analysis of FA's and show that, in simple PFA's, it is connected to the set of fixpoints of the underlying coding. Some set-theoretical constructions, based on cardinality considerations, provide (bijective) codings with given sets of fixpoints. These considerations and constructions provide the tools for constructing many flexible and explosive non-Boolean RA's of given infinite cardinality κ: we exhibit a simple flexible PRA $\mathbf{P}$ of cardinality κ (with at least κ pairwise non-isomorphic PFA-expansions) as well as κ pairwise non-isomorphic explosive RA's of cardinality κ, each one of them having 2^{κ} pairwise non-isomorphic FA-expansions.

Two remarks should be made concerning our constructions of infinite RA's with many expansions to FA's. First, as a by-product of our constructions, the FA-expansions obtained turn out to be special (they have $\infty\nabla\infty=\infty$, but $1\nabla 1\neq 1$), so the RA's are non-Boolean. Second, as an alternative, and perhaps simpler, tool for the analysis of FA's one may consider the set of subrelations of $(1\nabla 1)\bullet 1=\pi^{\dagger}\bullet 1\bullet\rho^{\dagger}$. This suggests investigating descriptions of the FA-expansions of a given RA in terms of its possible projection pairs.

APPENDIX: IDEAL ELEMENTS OF DIRECT PRODUCTS OF (SIMPLE) RA'S

Our construction in the theorem on many large explosive RA's (each one with many FA-expansions) uses a result about ideal elements of direct powers of simple relational algebras. For completeness, we state and prove its direct-product version in this appendix.

Consider an RA $\mathbf{R}=\langle R,0,\infty,1,^{-},^{\dagger},+,\bullet,;\rangle$. Recall that an element $r\in R$ of $\mathbf{R}$ is called *ideal* iff $\infty;r;\infty=r$ [Jónsson & Tarski '52, p. 130, 131, Definition 4.5 (iv)]. We use $Idl(\mathbf{R})$ to denote the set $\{r\in R/\infty;r;\infty=\infty\}$ of ideal elements of RA $\mathbf{R}$.

Lemma *Ideal elements of direct products of RA's*

Given a set of RA's $\mathbf{R}_i$, $i\in I$, consider their direct product $\times_{i\in I}\mathbf{R}_i\in\mathbf{Alg}(\lambda)$. Then, $Idl(\times_{i\in I}\mathbf{R}_i)=\times_{i\in I}Idl(\mathbf{R}_i)$, i. e. an element $r\in\times_{i\in I}R_i$ is an ideal element of $\times_{i\in I}\mathbf{R}_i$ iff for each $i\in I$ projection $p_i(r)$ is an ideal element of $\mathbf{R}_i$.

Proof

Since the projection homomorphisms are jointly injective, we have $\infty;r;\infty=r$ iff for each $i\in I$ $p_i(\infty;r;\infty=r)=p_i(r)$ i. e. $\infty_i;p_i(r);\infty_i=p_i(r)$.

Thus, for $(r_i)_{i\in I}\in\times_{i\in I}R_i$, $(r_i)_{i\in I}\in Idl(\times_{i\in I}\mathbf{R}_i)$ iff for each $i\in I$ $r_i\in Idl(\mathbf{R}_i)$.

QED

Now, recall that a simple RA $\mathbf{R}$ has no proper ideal elements: $Idl(\mathbf{R})=\{0,\infty\}$ [Jónsson & Tarski '52, p. 132, 133 (Theorem 4.10)].

Corollary *Ideal elements of direct products of simple RA's*

Consider a set of simple non-trivial RA's $\mathbf{R}_i$ ($|R_i|\geq 2$), $i\in I$, with cardinality $|I|=\gamma$. Then, their direct product $\times_{i\in I}\mathbf{R}_i$ has 2^{γ} ideal elements: $|Idl(\times_{i\in I}\mathbf{R}_i)|=2^{\gamma}$.

Proof

For each simple non-trivial RA $\mathbf{R}_i$, $Idl(\mathbf{R}_i)=\{0_i,\infty_i\}$ with $0_i\neq\infty_i$, so $|Idl(\mathbf{R}_i)|=2$.

Thus, we have $|Idl(\times_{i\in I}\mathbf{R}_i)|=|\times_{i\in I}Idl(\mathbf{R}_i)|=|\times_{i\in I}\{0_i,\infty_i\}|=2^{\gamma}$.

QED

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