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ON INFINITE FORK ALGEBRAS AND THEIR RELATIONAL REDUCTS

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Abstract. A fork algebra is a relational algebra enriched with a new binary operation. They have been introduced because their equational calculus has applications in program construction, they also have some interesting connections with algebraic logic. In this paper, we concentrate on the infinite fork algebras, with the purpose of contrasting them with relational algebras and with the finite fork algebras. For this purpose, we introduce some concepts for the analysis of fork algebras and construct some special infinite relational algebras and fork algebras. We try to correct a somewhat embarrassing mistake (and simplify some constructions) that occurred in a previous report. The finite fork algebras are somewhat uninteresting, being essentially Boolean algebras, uniquely characterized by their relational reducts. This contrasts with the case of infinite fork algebras: we introduce concepts and methods for the analysis of fork algebras and use some set-theoretical constructions to exhibit many relational algebras of each infinite cardinality with many expansions to fork algebras. We begin by providing some background about fork algebras and their reducts (relation and Boolean algebras): their abstract versions, some simple results concerning the algebraic structure of fork algebras, and their concrete, set-based, versions (fields of sets and proper relational and fork algebras). We then recall some results concerning the Boolean fork algebras (where fork is Boolean meet) and the finite fork algebras (they are completely determined by their relational reducts, which we call rigid, in contrast to the infinite fork algebras). We examine fork-expansions of relational algebras, with the purpose of comparing relational algebras and fork algebras. Then, we introduce methods for the analysis and construction of fork algebras and, finally, use some set-theoretical constructions to exhibit many (simple, proper) relational algebras of each infinite cardinality with many expansions to fork algebras. Such (infinite) relational algebras demonstrate quite clearly the diversity of possible fork operations.

Key words: Fork algebras, relational algebras, Boolean algebras, expansions, simple algebras, infinite fork algebras, fork expansions, proper relational algebras, proper fork algebras.

Resumo. Uma álgebra de fork é uma álgebra relacional enriquecida com uma nova operação binária. Tais álgebras foram introduzidas porque seu cálculo equacional tem aplicações em construção de programas, tendo também interessantes conexões com lógica algébrica. Este trabalho se concentra em álgebras de fork infinitas, contrastando-as com as álgebras relacionais e com as álgebras de fork finitas. Com este objetivo, são introduzidos alguns conceitos para a análise de álgebras de fork bem como construções de algumas álgebras de fork infinitas especiais. Também tentamos corrigir um embaraçoso engano (e simplificar algumas construções) em um trablaho anterior. As álgebras de fork finitas são de reduzido interesse, uma vez que são essencialmente álgebras de Boole, unicamente caracterizadas por seus redutos relacionais. Isto contrasta com o caso das álgebras de fork infinitas: introduzimos conceitos e métodos para a análise de álgebras de fork e empregamos algumas construções com conjuntos para exibir diversas álgebras relacionais de cada cardinalidade infinita possuindo várias expansões a álgebras de fork. Começamos recordando conceitos e resultados relativos a álgebras de fork e seus redutos (álgebras relacionais e de Boole): suas versões abstratas, alguns resultados simples acerca da estrutura algébrica das álgebras de fork, e suas versões concretas, baseadas em conjuntos (corpos de conjuntos e álgebras de relações e de fork próprias). A seguir, são recordadso alguns resultados referentes às álgebras de fork Booleanas (em que fork é a conjunção Booleana) e às álgebras de fork finitas (estas ficam completamente determinadas por seus reduto relacionais, que denominamos de rígido, em contraste com as álgebras de fork infinitas). Examinamos as expansões por fork de álgebras relacionais, a fim de comparar estas álgebras com álgebras de fork. Em seguida, introduzimos métodos para a análise de álgebras de fork e utilizamos algumas construções com conjuntos para exibir diversas álgebras relacionais (simples e próprias) de cada cardinalidade infinita possuindo várias expansões a álgebras de fork. Tais álgebras relacionais (infinitas) demonstram claramente a diversidade de possíveis operações fork.

Palavras chave: Álgebras de fork, álgebras relacionais, álgebras de Boole, álgebras simples, álgebras de fork infinitas, expansões por fork, álgebras relacionais próprias, álgebras de fork próprias.

CONTENTS

1. INTRODUCTION 1

2. PRELIMINARIES: FORK ALGEBRAS AND THEIR REDUCTS 2

2.1. Abstract Boolean, Relational and Fork Algebras 2

2.2. Algebraic Structure of Fork Algebras and some consequences 3

2.3 Set-based Boolean, Relation and Fork Algebras 4

3. BOOLEAN AND FINITE FORK ALGEBRAS 5

3.1 Boolean Relational and Fork Algebras 5

3.2 Finite Fork Algebras 6

4. INFINITE FORK ALGEBRAS 7

5. RELATIONAL ALGEBRAS AND FORK EXPANSIONS 8

5.1 Fork Expansions of Relational Algebras 8

5.2 Direct-product Factors of Fork Algebras 10

5.3 Analysis of (Proper) Fork Algebras 12

5.4 Construction of (Infinite, Simple) Proper Fork Algebras 13

6. FORK EXPANSIONS OF INFINITE RELATIONAL ALGEBRAS 14

6.1 Special Codings and Sets of Fixpoints 14

6.2 Large Simple Proper Relational and Fork Algebras 15

6.3 Large non-Boolean Relation and Fork Algebras 16

7. CONCLUSION 19

APPENDIX: DETAILED PROOFS OF THE RESULTS 21

REFERENCES 33

1. INTRODUCTION

A fork algebra (FA, for short) is a relational algebra (RA, for short) enriched with a new binary operation, called fork. Such algebras have been introduced because their equational calculus has applications in program construction. They also have some interesting connections with algebraic logic.

In this report, we concentrate on the infinite FA's, aiming at contrasting them with RA's and with the finite FA's. For this purpose, we introduce some concepts for the analysis of fork algebras and construct some special infinite RA's and FA's. We also try to correct a somewhat embarrassing mistake (and simplify some constructions) that appeared in a previous report [Veloso '96b].

The finite FA's, being essentially Boolean algebras, are somewhat uninteresting. Also, a finite FA is completely characterised by its RA reduct. This contrasts with the richness of the infinite FA's: we introduce some methods for the analysis of FA's and use some set-theoretical constructions to exhibit many infinite (simple proper) RA's of each infinite cardinality with many expansions to FA's.

We employ the following main ideas. We control the size of RA's by controlling the sizes of their sets of generators. We guarantee that an RA has an FA expansion by putting the defined projections of the latter in its carrier. We force simple proper (set-based) FA's to be non-isomorphic by controlling the sizes of the set of fixpoints of their underlying codings. Finally, we construct non-isomorphic (non-simple) RA's by controlling the sizes of their sets of ideal elements or by controlling their prime (simple and non-trivial) direct-product factors.

For readability we present each result with an outline of its proof. More detailed proofs are presented in the appendix at the end of the paper. The structure of this report is as follows (Sections 2 and 3 are mainly background and review of known results [Veloso '96a,b]. The main body consists of sections 4, 5 and 6, which cover mainly infinite FA's and RA's.)

Section 2 provides some preliminary background about fork algebras and their reducts (relation and Boolean algebras), beginning with their abstract versions, examining some simple results concerning the algebraic structure of FA's, and proceeding to their more concrete, set-based, versions: fields of sets and proper FA's and RA's. We then review in section 3 some results concerning the Boolean FA's (those where fork is Boolean meet) and the finite FA's: the Boolean FA's (whose reducts turn out to be Boolean RA's) are characterised as subalgebras of direct powers of the two-element FA; the finite FA's are described as finite direct powers of the two-element FA, and the only simple finite FA's are those with one or two elements. This shows that a finite FA is completely determined by its RA reduct (which we call rigid), in contrast with the case of infinite FA's. In section 4 we establish the existence of simple non-Boolean (proper) FA's of each infinite cardinality. Then, in section 5, we suggest some concepts and methods for the analysis of FA's and their RA reducts, with the purpose of comparing RA's and FA's. We examine fork expansions of RA's, introduce some indices for the fork-expandability of RA's, and examine the direct-product factors of FA's. Then we consider some concepts for the analysis of (proper) FA's, examine how they can be controlled by means of the underlying coding, and consider also some methods for constructing (infinite, simple) proper FA's. In section 6, we apply these ideas to fork expansions of infinite RA's: we use some set-theoretical constructions to exhibit many infinite (simple, proper) non-Boolean RA's with many expansions to FA's. Such (infinite) RA's demonstrate quite clearly the diversity of possible fork operations. Finally, section 7 presents some concluding remarks.

2. PRELIMINARIES: FORK ALGEBRAS AND THEIR REDUCTS

An abstract fork algebra (FA, for short) is a relational algebra enriched with a new binary operation, called fork. A relational algebra (RA, for short) is an expansion of a BA (short for Boolean algebra) with some Peircean operations and constant.

We shall use β for the signature $<2,1,2>$ (with 2 constants, 1 unary operation and 2 binary operations) of the BA's, λ for the signature $<3,2,2>$ of the RA's, and ϕ for the signature $<3,2,3>$ of the FA's.

We shall call an algebra *trivial* when its carrier has more than one element. As usual, an algebra is *simple* iff it has no proper homomorphic images. We shall call an algebra *prime* iff it is simple and non-trivial (its carrier has more than one element).

Within the context of a fixed class of algebras, we shall use $\nu(\kappa)$ for the (cardinal) number of isomorphism classes of algebras with cardinality κ, and similarly, $\sigma(\kappa)$ for the (cardinal) number of isomorphism classes of simple algebras with cardinality κ. { Clearly $\sigma(\kappa)\leq\nu(\kappa)$. }

2.1 Abstract Boolean, Relational and Fork Algebras

We now briefly recall some concepts pertaining to (abstract) Boolean, relational and fork algebras.

A *Boolean algebra* (*BA*, for short) is an algebra $\mathcal{B}=<B,0,\infty,^-,+,\bullet>$ with signature $\beta=<2,1,2>$ (so $0,\infty\in B$, $^-:B\rightarrow B$, and $+,\bullet:B\times B\rightarrow B$), satisfying well-known equations [Bell & Slomson '71; Burris & Sankappanavar '81; Halmos '63]. We shall use $\leq$ for the Boolean ordering ($a\bullet b=a$ iff $a\leq b$ iff $a+b=b$).

A *relational algebra* (*RA*, for short) is an algebra $\mathcal{R}=<R,0,\infty,1,^-,^\dagger,+,\bullet,;>$ with signature $\lambda=<3,2,2>$, satisfying familiar equations, to the effect that

- its *BA reduct* $\mathcal{R}_\beta:=<R,0,\infty,^-,+,\bullet>$ is a BA with Boolean ordering $\leq$;
- the *Peircean reduct* $<R,1,^\dagger,;>$ is a semigroup with identity $1\in R$ and involution $^\dagger:R\rightarrow R$, so $1^\dagger=1$, $(r^\dagger)^\dagger=r$ and $(r;s)^\dagger=(s^\dagger);(r^\dagger)$,
- for all $r,s\in R$: $(r^\dagger);(r;s)^-\leq s^-$, i. e. $(r^\dagger);(r;s)^-+s^-=s^-$.

Recall that the simple RA's are those satisfying Tarski's rule: $\infty;r;\infty=\infty$ whenever $r\neq 0$ [Jónsson & Tarski '52, Theorem 4.10, p. 132, 133].

A pair of (*conjugated*) *quasiprojections* for RA $\mathcal{R}=<R,0,\infty,1,^{-},^{\dagger},+,\bullet,;>$ amounts to elements $\pi,\rho\in R$ such that: $\pi^{\dagger};\pi\leq 1$, $\rho^{\dagger};\rho\leq 1$, and $\pi^{\dagger};\rho=\infty$. {In fact, for such a pair π and ρ of quasiprojections we have $\pi^{\dagger};\pi=1=\rho^{\dagger};\rho$}. A *quasiprojective RA* (*QRA*, for short) is an RA that has a pair of quasiprojections [Tarski & Givant '87, p. 242].

Consider an algebra $\mathcal{R}=<R,0,\infty,1,^{-},^{\dagger},+,\bullet,;>$ of signature $\lambda=<3,2,2>$. By adding a binary operation $\nabla:R\times R\rightarrow R$, we obtain an algebra $\mathcal{R}^{\nabla}$ of signature $\phi=<3,2,3>$, called its ∇-*expansion*. Note that in any such expansion, we have elements $\pi:=(1\nabla\infty)^{\dagger}$ and $\rho:=(\infty\nabla 1)^{\dagger}$.

A *fork algebra* (*FA*, for short) is an algebra $\mathcal{F}=<F,0,\infty,1,^{-},^{\dagger},+,\bullet,;,\nabla>$ with signature $\phi=<3,2,3>$, such that

- its *RA reduct* $\mathcal{F}_{\lambda}:=<F,0,\infty,1,^{-},^{\dagger},+,\bullet,;>$ is an RA with Boolean ordering $\leq$;
- with $\pi:=(1\nabla\infty)^{\dagger}$ and $\rho:=(\infty\nabla 1)^{\dagger}$ as above
  - for every $r,s,p,q\in F$: $(r\nabla s);(p\nabla q)^{\dagger}=(r;p^{\dagger})\bullet(s;q^{\dagger})$ (∇ vs. $\bullet$),
  - for every $r,s\in F$: $r\nabla s=(r;\pi^{\dagger})\bullet(s;\rho^{\dagger})$ (∇-def),
  - $\pi\nabla\rho\leq 1$ (i. e. $\pi\nabla\rho+1=1$) (∇ proj).

Since the class of RA's has an equational characterisation [Chin & Tarski '50, Theorem 2.2, p. 350; Jónsson & Tarski '52; Veloso '74, p. 8], so does the class of FA's. We use **FA** for the variety of the FA's.

It is not difficult to see that, in any FA $\mathcal{F}$ the defined elements $\pi:=(1\nabla\infty)^{\dagger}$ and $\rho:=(\infty\nabla 1)^{\dagger}$ form a pair of quasiprojections (such that $(\pi;\pi^{\dagger})\bullet(\rho;\rho^{\dagger})\leq 1$) [Frias et al. '95, '96]. Thus, the RA reducts of FA's are QRA's.

2.2 Algebraic Structure of Fork Algebras

We now recall some simple results concerning the algebraic structure of FA's [Veloso '96a]. They come from the crucial, though simple, observation that in an FA fork is definable by an RA-term from the elements $\pi^{\dagger}=1\nabla\infty$ and $\rho^{\dagger}=\infty\nabla 1$ in its carrier F. So, preservation of the RA operations as well as of π and ρ entails preservation of fork.

The first result characterises the simple FA's as those with simple relational reducts.

Theorem *Simple FA's*

An FA $\mathcal{F}$ is simple iff its relational reduct $\mathcal{F}_{\lambda}$ is simple.

The next result characterises the non-simple FA's as those with non-trivial direct decompositions, just like RA's.

Proposition *Non-simple FA's*

An FA $\mathcal{F}$ is non-simple iff $\mathcal{F}\cong\mathcal{G}\times\mathcal{H}$ for some non-trivial FA's $\mathcal{G}$ and $\mathcal{H}$.

The next result provides subdirect decompositions for FA's into simple components (their simple homomorphic images), paralleling the analogous result for RA's.

Theorem *Subdirect decomposition of FA's into simple components*
Every FA $\mathcal{F}$ is isomorphic to a subdirect product of simple homomorphic images of $\mathcal{F}$.

We also mention a simple metamathematical consequence of the preceding results concerning the algebraic structure of the FA's.

Corollary *FA equations*
An FA equation ε holds in an FA algebra $\mathcal{F}$ iff equation ε holds in all (simple) homomorphic images of $\mathcal{F}$.

2.3 Set-based Boolean, Relation and Fork Algebras

Now, let us briefly describe the proper, set-based, versions of BA's, RA's and FA's. Much as BA's arise as abstractions from fields of sets, RA's (and FA's) are abstractions from their set-based versions.

A *field of sets* (over set U) is an algebra $\mathcal{S}=<S,\emptyset,U,\sim,\cup,\cap>$ of signature $\beta=<2,1,2>$, with $S\subseteq \wp(U)$. Its Boolean ordering is set inclusion $\subseteq$. A field of sets over U is a Boolean algebra that can be embedded into the direct power $\mathcal{T}^U$ of the two-element BA $\mathcal{T}$ (of truth values).

By Stone Representation Theorem every (abstract) BA is isomorphic to some field of sets [Bell & Slomson '71; Burris & Sankappanavar '81; Halmos '63], and so can be embedded into a direct power of the two-element BA.

A *proper relation algebra* (over set U) is an algebra $\mathcal{P}=<P,\emptyset,V,1_U,\sim,^T,\cup,\cap,|>$ of signature $\lambda=<3,2,2>$, where

- its BA reduct $\mathcal{P}_\beta=<P,\emptyset,V,\sim,\cup,\cap>$ is a field of sets over $U^2=U\times U$;
- $1_U\in P$ is the identity (diagonal) relation over U: $1_U:=\{<u,v>\in U^2/u=v\}$;
- operation $^T:P\rightarrow P$ is relation transposition, i. e. $p^T=\{<v,u>\in U^2/<u,v>\in p\}$;
- operation $|:P\times P\rightarrow P$ is relation composition, i. e.
  $r|s=\{<u,w>\in U^2/\exists v\in U\ [<u,v>\in r\&<v,w>\in s]\}$.

The *full PRA over* set U is the proper RA $\mathcal{P}(U^2):=<\wp(U^2),\emptyset,U^2,1_U,\sim,^T,\cup,\cap,|>$.

Recall that a PRA (short for proper relation algebra) $\mathcal{P}=<P,\emptyset,V,1_U,\sim,^T,\cup,\cap,|>$ over U with $V=U^2$ is simple [Jónsson & Tarski '52, Theorem 4.28, p. 142; Veloso '74, p. 7, 12]. So, the full PRA $\mathcal{P}(U^2)$ and its subalgebras are simple.

In contrast with BA's, not every (abstract) RA can be represented as some proper RA (see e. g. [Maddux '91; Veloso '74]).

Now, a function $*:V\rightarrow U$, where $V\subseteq U^2$, induces a binary operation $\angle^*$ on relations over U (into $\wp(V)$), called *fork induced by* $*:V\rightarrow U$, defined by
$r\angle^* s:=\{<u,v>\in V/\exists v',v''\in U\ [<v',v''>\in V\&v'*v''=v\&<u,v'>\in r\&<u,v''>\in s]\}$.

A *proper fork algebra* (over set U) is an algebra $\mathcal{Q}=<Q,\emptyset,V,1_U,\sim,^T,\cup,\cap,|,\angle>$ of signature $\phi=<3,2,3>$, where

- its RA reduct $\mathcal{Q}_\lambda=<Q,\emptyset,V,1_U,\sim,^T,\cup,\cap,|>$ is a proper relation algebra over U;
- there exists an underlying coding $*:U^2\rightarrow U$ such that
  $*$ is injective on $V\subseteq U^2$ (i. e. the restriction $*_{|V}:V\rightarrow U$ is one-to-one),

operation $\angle$:Q×Q→Q is induced by the restriction $*_{|V}$ of $*$ to V, i. e.

$r\angle s := r\angle^{*|V} s = \{<u,v>\in V / \exists <v',v''>\in V\ [v'*v''=v \& <u,v'>\in r \& <u,v''>\in s]\}$.

It is not difficult to see that every PFA (short for proper fork algebra) is indeed a fork algebra [Frias et al. '96].

Notice that each injective function $*:U^2\to U$ gives rise to a *full PFA over* set U $\mathcal{P}^*(U^2):=<\wp(U^2),\varnothing,U^2,1_U,\smile,^T,\cup,\cap,|,\angle^*>$ as the $\angle^*$-expansion of the full PRA $\mathcal{P}(U^2)$ by the fork $\angle^*$ induced by coding $*:U^2\to U$. Thus, the PFA's with $V=U^2$ are the subalgebras of the full PFA's, and they are all simple.

Also, for a simple PFA $\mathcal{Q}=<Q,\varnothing,V,1_U,\smile,^T,\cup,\cap,|,\angle>$ with $V=U^2$, its underlying injective coding $*:U^2\to U$ is surjective iff $V\angle^*V=V$.

{For, $V\angle^*V\subseteq V$ and $V\subseteq V\angle^*V$ iff $<u,v>\in V\angle^*V$ whenever $<u,v>\in V=U^2$ iff for each $v\in U$, we have $v',v''\in U$ with $v'*v''=v$.}

Moreover, like BA's and contrasting with RA's, every (abstract) FA can be represented as some proper FA [Frias et al. '95, '96], but we shall not make use of this result here.

3. Boolean and Finite Fork Algebras

We now briefly recall some simple results concerning the Boolean FA's (where fork is Boolean meet) and the finite FA's [Veloso '96b].

Two very simple finite FA's are those with one and two elements. It is not difficult to show that these are the only finite simple FA's (see 3.2).

The trivial one-element FA *1* has single element $0=1=\infty$. It is isomorphic to the full PFA $\mathcal{P}^*(\varnothing)$, with carrier $\wp(\varnothing)=\{\varnothing\}$, and underlying coding $*=\varnothing$.

The two-element FA *2* has two elements 0 and $1=\infty$ with $\infty\nabla\infty=\infty$. It is isomorphic to the full PFA $\mathcal{P}^*(\{<u,u>\})$ over singleton $\{<u,u>\}$, with carrier $\wp(\{<u,u>\})=\{\varnothing,\{<u,u>\}\}$, and underlying coding $*$ given by $u*u=u$.

These FA's *1* and *2* are simple. Also, FA's *1* and *2* are the only FA's, up to isomorhism, with respectively 1 and 2 elements (since $0\nabla 0=0$ and $0\leq 1\leq\infty$).

3.1 Boolean Relational and Fork Algebras

The RA reducts of the simple FA's *1* and *2* are QRA's, with $1=\infty$ as both quasiprojections. They are Boolean RA's as well.

Recall that a *Boolean* RA is one where ; is $\bullet$, † is the identity function, and $1=\infty$ [Jónsson & Tarski '52, p. 151], i. e. one satisfying the equations $1=\infty$, $\forall x\ x^\dagger=x$ and $\forall x,y\ x;y=x\bullet y$, . Thus, a Boolean RA is a somewhat uninteresting expansion of a Boolean algebra to an RA.

Clearly, every Boolean RA is a QRA, with $1=\infty$ as both quasiprojections. Also, a direct product $\times_{i\in I}\mathcal{R}_i$ of RA's is Boolean iff every $\mathcal{R}_i$, $i\in I$, is Boolean.

By analogy with Boolean RA's, let us call an FA $\mathcal{F}=<F,0,\infty,1,^-,^\dagger,+,\bullet,;,\nabla>$ *Boolean* iff its fork ∇ is $\bullet$, i. e. $\mathcal{F}$ satisfies the equation $\forall x,y\ x\nabla y=x\bullet y$.

By the remark in 2.3, a simple PFA whose underlying coding $*:U^2\to U$ is not surjective cannot be Boolean (since $V\angle^*V\subset V$).

The FA's *1* and *2* are Boolean FA's and Boolean RA's. In fact, it is not difficult to see that the relational reduct of a Boolean FA is a Boolean RA.

Lemma *Boolean FA's and RA reducts*

The RA reduct of a Boolean FA is a Boolean RA.

Thus, an FA $\mathcal{F}$ is a Boolean FA iff $\mathcal{F}$=<F,0,∞,∞,$^{-}$,1_F,+,•,•,•>.

So, Boolean FA's, like Boolean RA's, are essentially Boolean algebras.

We can characterise the Boolean FA's as the subalgebras of direct powers of the two-element FA *2*.

Proposition *Characterisation of Boolean FA's*

A ϕ-algebra $\mathcal{F}$ is a Boolean FA iff $\mathcal{F}$ can be embedded into some direct power 2^I of the two-element FA *2*.

3.2 Finite Fork Algebras

We now describe the finite FA's as the finite direct powers of the simple ones, the latter being those with one and two elements.

By a property of Boolean algebras, a finite FA must have cardinality 2^n, for some n≥0. The direct power 2^n provide an (uninteresting) example of a (Boolean) FA cardinality 2^n for each n≥0. Thus, the finite spectrum of the FA's is the set $\{2^n/n\in \mathbf{N}\}$.

We now recall that finite simple QRA's have at most two elements. (Notice that for a proper QRA $\mathcal{P}$ over a finite set U, we must have |U|≤1 [Tarski & Givant '87, p. 96].)

Lemma *Upper bound on finite simple QRA's*

There exists no finite simple QRA with more than two elements.

Since FA's have (quasi)projections, we can see that, up to isomorphism, *1* and *2* are all the finite simple FA's.

Proposition *Description of finite simple FA's*

A ϕ-algebra $\mathcal{F}$ is a finite simple FA iff either $\mathcal{F}\cong 1$ or $\mathcal{F}\cong 2$.

We can now see that the finite FA's are not very interesting.

Lemma *Boolean finite FA's*

Every finite FA is Boolean.

We now have a complete description of the finite FA's as the finite direct powers of the two-element simple FA *2*.

Theorem *Description of finite FA's*

A ϕ-algebra $\mathcal{F}$ is a finite FA iff $\mathcal{F}$ is isomorphic to some finite direct power 2^n of the two-element FA *2*.

Thus, the finite FA's are not very interesting FA's because they are the finite direct powers of the two-element simple FA *2*, so all are Boolean FA's, and essentially BA's. Thus, there exists exactly one, up to isomorphism, FA of each finite cardinality 2^n for n≥0. In particular, the RA reduct of a finite RA has exactly one, up to isomorphism, expansion to a fork algebra.

This description of the finite FA's is summarised in the following table:

n=0	*1* {trivial}	$\sigma(2^0)=1$	$\nu(2^0)=1$
n=1	*2* {prime}	$\sigma(2^1)=1$	$\nu(2^1)=1$
n>1	2^n {non-simple}	$\sigma(2^n)=0$	$\nu(2^n)=1$

4. Infinite Fork Algebras

We now examine infinite FA's. We shall later see that there exist many (simple proper) FA's of each infinite cardinality (even with the same RA reduct), in sharp contrast with the finite case.

Each BA can be expanded to a (Boolean) FA. Thus, there exists an FA of each infinite cardinality. We shall now establish the existence of simple non-Boolean (proper) FA's of each infinite cardinality.

For any infinite set U, U and $U^2=U\times U$ have the same cardinality. So, there exists an injective function $*:U^2\rightarrow U$. Now, each such injection $*:U^2\rightarrow U$ gives rise to a full PFA $\mathcal{P}^*(U^2)=<\wp(U^2),\varnothing,U^2,1_U,\bar{\ },^T,\cup,\cap,|,\angle^*>$ over U. Notice that $\mathcal{P}^*(U^2)$ is a simple PFA with cardinality 2^κ, where $\kappa=|U|$. Also, $\mathcal{P}^*(U^2)$ will be non-Boolean if $*:U^2\rightarrow U$ is not surjective.

Since removal of an element from an infinite set does not alter its cardinality, it is easy to obtain injective, non-surjective codings.

Lemma *Non-surjective codings for infinite sets*

For each infinite set U, there exists an injective, non-surjective coding function $*:U^2\rightarrow U$.

Thus, an application of the Downward Löwenheim-Skolem Theorem yields the existence of a simple non-Boolean FA of each infinite cardinality.

Proposition *Large simple non-Boolean FA's*

For each infinite cardinal $\kappa\geq\aleph_0$ there exists a simple non-Boolean FA with cardinality κ.

Proof outline

Consider an injective, non-surjective coding function $*:U^2\rightarrow U$, where $|U|=\kappa$. The full PFA $\mathcal{P}^*(U^2)$ is a model, with cardinality $2^\kappa>\kappa$, of set Σ, consisting of the FA equations together with Tarski's rule $\forall x(\neg x\approx 0\rightarrow\infty;x;\infty\approx\infty)$ and the sentence $\exists x,y\ \neg\ x\nabla y=x\bullet y$. Thus, set Σ is consistent; and the Downward Löwenheim-Skolem Theorem yields the existence of a simple non-Boolean FA $\mathcal{F}$ of infinite cardinality $\kappa<2^\kappa$.

QED

We shall now exhibit some simple proper FA's of given infinite cardinality. Given an algebra $\mathcal{A}$ and a subset G of its carrier, let $\mathcal{A}[G]$ be the subalgebra of $\mathcal{A}$ generated by G. Note that, if G is infinite with cardinality $|G|=\gamma$, then $\mathcal{A}[G]$ has cardinality $|A[G]|=\gamma.\aleph_0=\gamma$ [Burris & Sankappanavar '81, p. 32].

In particular, for an infinite set U we have the infinite full PRA $\mathcal{P}(U^2)$ over set U. Given any infinite subset $G\subseteq\wp(U^2)$, the subalgebra $\mathcal{P}_G:=\mathcal{P}(U^2)[G]$ of the full PFA $\mathcal{P}(U^2)$ generated by G is a simple PRA with cardinality $|G|\geq\aleph_0$.

Also, given an infinite set U with cardinality $\kappa\geq\aleph_0$, let $\wp_\omega(U^2)$ be the set of finite subsets of U^2, and notice that $|\wp_\omega(U^2)|=\kappa.\aleph_0=\kappa$. Given an injective function $*:U^2\to U$, the subalgebra $\mathcal{P}^*_\omega(U^2):=\mathcal{P}^*(U^2)[\wp_\omega(U^2)]$ of the full PFA $\mathcal{P}^*(U^2)$ generated by $\wp_\omega(U^2)$ is a simple PFA with cardinality κ. Again, $\mathcal{P}^*_\omega(U^2)$ will be non-Boolean if $*:U^2\to U$ is not surjective.

Proposition *Large simple non-Boolean PFA's*

For each infinite cardinal $\kappa\geq\aleph_0$ there exists a simple non-Boolean PFA $\mathcal{P}^*_\omega(U^2)$ with cardinality κ.

Proof outline

Consider an injective, non-surjective coding function $*:U^2\to U$, where $|U|=\kappa$, and the full PFA $\mathcal{P}^*(U^2)$. Then the subalgebra $\mathcal{P}^*_\omega(U^2)$ of the full PFA $\mathcal{P}^*(U^2)$ generated by $\wp_\omega(U^2)$ is a simple non-Boolean PFA with cardinality κ.

QED

The information, so far, on the infinite (proper) FA's is summarised below:

$\kappa\geq\aleph_0$	$\mathcal{P}^*_\omega(U^2)$ {non-Boolean}	$1\leq\sigma(\kappa)\leq\nu(\kappa)$

5. Relational Algebras and Fork Expansions

As mentioned, we wish to analyse the infinite FA's, comparing and contrasting them with the finite ones. The finite FA's are characterised by their RA reducts, which does not happen with the infinite FA's, as we will shortly see. So, this contrast between finite and infinite FA's is connected to a comparison between RA's and FA's.

The results concerning the algebraic structure of FA's and their metamathematical consequences are very similar to their analogues for RA's. This may give the impression of similarity in the behaviour of RA's and FA's. But, representability, as mentioned previously, is already a clear difference. Further distinctions, indicating that they are quite different, will seen in the next section.

For the purpose of comparing (mainly infinite) RA's and FA's, we now examine some considerations and introduce some terminology of a somewhat ad-hoc nature.

5.1 Fork Expansions of Relational Algebras

What FA's have more than RA's is a fork operation. This difference vanishes in the Boolean FA's, when fork is $\bullet$. A not so extreme case is that where fork is $\bullet$ for some elements, say ∞. Let us call *special* those FA's $\mathcal{F}=\langle F,0,\infty,1,^-,^\dagger,+,\bullet,;,\nabla\rangle$ where $\infty\nabla\infty=\infty$ (so $\infty\nabla\infty=\infty\bullet\infty$), but $1\nabla1\neq1$.

Notice that special FA's are non-Boolean (since $1\nabla1\neq1\bullet1$). We shall have occasion to examine and construct a few special FA's.

A non-trivial direct product $\times_{i\in I}\mathcal{F}_i$ of special FA's $\mathcal{R}_i$, $i\in I$, is a special FA.

Now, consider an RA $\mathcal{R}$ with carrier $R\subseteq F$. We naturally call RA $\mathcal{R}$ *expandable by* binary operation $\nabla:F\times F\to F$ iff R is closed under ∇: $r\nabla s\in R$

whenever $r,s\in R$. In such case we use $\mathcal{R}^{\nabla}:=(\mathcal{R},\nabla)$ to denote its ∇-*expansion*. The next result characterises expandability of subalgebras of reducts.

Lemma *Expandability of subalgebras of reducts*

Consider an algebra $\mathcal{F}$ of FA-signature ϕ, with defined elements $\pi:=(1\nabla\infty)^{\dagger}$ and $\rho:=(\infty\nabla 1)^{\dagger}$, and a λ-subalgebra $\mathcal{R}$ of its λ-reduct $\mathcal{F}_{\lambda}$.

a) If $\mathcal{F}$ satisfies axiom (∇-def), then λ-subalgebra $\mathcal{R}$ of $\mathcal{F}_{\lambda}$ is expandable by $\nabla:F\times F\to F$ iff π and ρ are in R.

b) If π and ρ are in R and $\mathcal{F}$ is an FA, then so is the ∇-expansion $\mathcal{R}^{\nabla}$.

Proof outline

a) Since $\mathcal{F}$ satisfies (∇-def), R is closed under ∇ iff both π and ρ are in R.

b) The ∇-expansion $\mathcal{R}^{\nabla}=(\mathcal{R},\nabla)$ is a ϕ-subalgebra of FA $\mathcal{F}$, and thus $\mathcal{R}^{\nabla}\in$ **FA**.

QED

As a tool for comparing RA's and FA's, we can consider using the (cardinal) number of non-isomorphic FA expansions of an RA: its fork index.

More precisely, given an RA $\mathcal{R}$, consider the set $\mathbf{FRK}(\mathcal{R}):=\{\mathcal{F}\in\mathbf{FA}/\mathcal{F}_{\lambda}=\mathcal{R}\}$ of FA expansions of RA $\mathcal{R}$. We have the equivalence relation $\cong$ of being ϕ-isomorphic between ϕ-algebras, which gives rise to the quotient $\mathbf{FRK}(\mathcal{R})/\cong$. The fork index of RA $\mathcal{R}$ is the cardinality $\varphi(\mathcal{R}):=|\mathbf{FRK}(\mathcal{R})/\cong|$

Not surprisingly, some RA's (for instance, those that are not QRA's) have null fork indices. Let us call an RA $\mathcal{R}$ *rigid* iff it has at most one, up to isomorphism, FA expansion: $\varphi(\mathcal{R})\leq 1$. As we have seen, the finite RA's are all rigid (the RA reducts of finite FA's having fork index 1). At the other extreme, an RA may have many non-isomorphic FA expansions.

Consider an RA $\mathcal{R}$, with cardinality $|R|=\kappa$. Since a possible fork is a binary operation $\nabla:R\times R\to R$, RA $\mathcal{R}$ may have at most $\kappa^{(\kappa.\kappa)}$ FA expansions $(\mathcal{R},\nabla)$: $|\mathbf{FRK}(\mathcal{R})|\leq\kappa^{(\kappa.\kappa)}$. Thus, an infinite RA $\mathcal{R}$, with cardinality $|R|=\kappa\geq\aleph_0$ has $\varphi(\mathcal{R})\leq\kappa^{(\kappa.\kappa)}=2^{(\kappa.\kappa)}=2^{\kappa}$.

A sharper upper bound can be obtained by means of the preceding lemma.

Proposition *Upper bound on fork indices of RA's*

An RA $\mathcal{R}$ with cardinality $|R|=\kappa$ has fork index $\varphi(\mathcal{R})\leq\kappa^2$.

Proof outline

We have a function $P:\mathbf{FRK}(\mathcal{R})\to R^2$ by assigning to FA expansion $\mathcal{F}$ of RA $\mathcal{R}$ its defined projections $\pi_{\mathcal{F}}:=(1\nabla_{\mathcal{F}}\infty)^{\dagger}$ and $\rho_{\mathcal{F}}:=(\infty\nabla_{\mathcal{F}}1)^{\dagger}$ (in R, by the lemma).

This function $P:\mathbf{FRK}(\mathcal{R})\to R^2$ is injective, in view of axiom (∇-def).

Thus, $|\mathbf{FRK}(\mathcal{R})|\leq|R^2|=|R|^2=\kappa^2$, and $\varphi(\mathcal{R})\leq|\mathbf{FRK}(\mathcal{R})|$; so $\varphi(\mathcal{R})\leq\kappa^2$.

QED

Corollary *Upper bound on fork indices of infinite RA's*

An infinite RA $\mathcal{R}$, with cardinality $|R|=\kappa\geq\aleph_0$ has fork index $\varphi(\mathcal{R})\leq\kappa$.

Among the non-rigid RA's we shall consider those with high fork indices.

Since our constructions will produce special FA's, we may as well relativise the preceding considerations to this class of FA's, and consider the (cardinal) number of non-isomorphic special FA expansions of an RA $\mathcal{R}$, its special index $\theta(\mathcal{R})$. Given an RA $\mathcal{R}$, we consider the set **SPC**$(\mathcal{R})$ of special FA expansions of RA $\mathcal{R}$, and use as the special index of RA $\mathcal{R}$ the cardinality $\theta(\mathcal{R}):=|\mathbf{SPC}(\mathcal{R})/\cong|$. { Note that $\theta(\mathcal{R})\leq\varphi(\mathcal{R})$. }

We shall call RA $\mathcal{R}$ *elastic* iff it has special index $\theta(\mathcal{R})=|R|^2$. Such RA's (with $\theta(\mathcal{R})=\varphi(\mathcal{R})=|R|^2$) exhibit quite clearly the diversity of possible fork operations; notice that they cannot be Boolean.

The expandability of the RA reducts of the finite FA's is as follows:

$n\geq 0$ $\qquad (\mathcal{2}^n)_\lambda$ {Boolean} $\qquad \theta[(\mathcal{2}^n)_\lambda]=0 \qquad \varphi[(\mathcal{2}^n)_\lambda]=1$

5.2 Direct-product Factors of Fork Algebras

In the next section we will construct examples of such RA's with given infinite cardinality: (simple, proper) elastic RA's $\mathcal{P}$ with $|P|$, pairwise non-isomorphic, special PFA-expansions. In these constructions we shall have occasion to use some lower bounds on the special indices of some direct powers and products of RA's.

For this purpose, we consider prime algebras and recall the construction of the relativisation of an RA to one of its ideal elements, extending it to FA's.

We first recall concept of ideal element of an RA and the construction of the relativisation of an RA to one of its ideal elements.

Consider an RA $\mathcal{R}=\langle R,0,\infty,1,^-,^\dagger,+,\bullet,;\rangle$. An element $j\in R$ of $\mathcal{R}$ is called *ideal* iff $\infty;j;\infty=r$ [Jónsson & Tarski '52, p. 129, 130, Definition 4.5 (iv)]. We use $J(\mathcal{R})$ to denote the set $\{j\in R/\infty;j;\infty=j\}$ of ideal elements of relational algebra $\mathcal{R}$. Recall that $\mathcal{J}[\mathcal{R}]:=\langle J(\mathcal{R}),0,\infty,^-,+,\bullet\rangle$ is a Boolean algebra, such that $i;j=i\bullet j$ and $i^\dagger=i$ for $i,j\in J(\mathcal{R})$ [Jónsson & Tarski '52, p. 130, 131, Theorem 4.6 (viii, ix)].

We now extend the construction of the relativisation of an RA to an ideal element to FA's. For an FA $\mathcal{F}=(\mathcal{F}_\lambda,\nabla)$, we set $\mathcal{J}[\mathcal{F}]:=\mathcal{J}[\mathcal{F}_\lambda]$. By the theorem on simple FA's in 2.2 and [Jónsson & Tarski '52, p. 132, 133, Theorem 4.10], FA $\mathcal{F}$ is simple iff $J(\mathcal{F})\subseteq\{0,\infty\}$. So, $\mathcal{F}$ is prime iff $J(\mathcal{F})\subseteq\{0,\infty\}$ and $0\neq\infty$.

Recall that the relativisation of RA $\mathcal{R}$ to ideal element $i\in J(\mathcal{R})$ is the λ-algebra $\mathcal{R}^{(i)}:=\langle R^{(i)},0,i,1\bullet i,^{-(i)},^\dagger,+,\bullet,;\rangle$, where $R^{(i)}:=\{r\in R/r\leq i\}$ and $^{-(i)}(r):=r^-\bullet i$ [Jónsson & Tarski '52, p. 132, Definition 4.8]. It is an RA, a homomorphic image of RA $\mathcal{R}$ under relativisation $\_^{(i)}:R\rightarrow R^{(i)}$, defined by $\_^{(i)}(r):=r\bullet i$ [Jónsson & Tarski '52, p. 132, Theorem 4.9].

Lemma *Relativisation of FA to ideal elements*

Consider an FA $\mathcal{F}=(\mathcal{R},\nabla)$ with carrier R. Given an ideal element $i\in J(\mathcal{F})=J(\mathcal{R})$, consider the set of elements below i: $F^{(i)}:=\{f\in F/f\leq i\}$..

a) Set $F^{(i)}=\{f\in F/f\leq i\}$ is closed under fork $\nabla:F\times F\rightarrow F$.

b) Relativisation $\_^{(i)}:F\rightarrow F^{(i)}$, defined by $\_^{(i)}(f):=f\bullet i$, is a ϕ-homomorphism from FA $\mathcal{F}=(\mathcal{R},\nabla)$ onto the ∇-expansion $\mathcal{F}^{(i)}:=(\mathcal{R}^{(i)},\nabla)$, with kernel $\ker(\_^{(i)})=F^{(j)}$ where $j=i^-$. Thus $\mathcal{F}^{(i)}\cong\mathcal{F}/F^{(j)}$ with $j=i^-\in J(\mathcal{F})$.

c) $\mathcal{F}^{(i)}:=(\mathcal{R}^{(i)},\nabla)$ is an FA with $J(\mathcal{F}^{(i)})=J(\mathcal{F})\cap F^{(i)}=\{j\bullet i/j\in J(\mathcal{F})\}$.

d) FA $\mathcal{F}^{(i)}$ is prime iff i is an atom of BA $\mathcal{J}[\mathcal{F}]=\mathcal{J}[\mathcal{R}]$.

Proof outline

First, notice that, for $j,k\in J(\mathcal{F})=J(\mathcal{R})$, $j\nabla k=(j;\pi^\dagger)\bullet(k;\rho^\dagger)\leq(j;\infty)\bullet(k;\infty)\leq j\bullet k$.

a) Thus, for $r,s\in F^{(i)}$, $r\nabla s\leq i\bullet i=i$; and so $r\nabla s\in F^{(i)}$.

b) By part (a), $\mathcal{R}^{(i)}$ is expandable by fork ∇ to ϕ-algebra $\mathcal{F}=(\mathcal{R},\nabla)$.
To see that $\_^{(i)}:F\to F^{(i)}$ preserves fork, consider elements $r=r\bullet i+r\bullet i^-$ and $s=s\bullet i+s\bullet i^-$ of F, and notice that $r\nabla s=(r\bullet i)\nabla(s\bullet i)+0+0+(r\bullet i^-)\nabla(s\bullet i^-)$;
whence $(r\nabla s)\bullet i=(r\bullet i)\nabla(s\bullet i)$.

c) We clearly have $J(\mathcal{F}^{(i)})\subseteq J(\mathcal{F})\cap F^{(i)}\subseteq\{j\bullet i/j\in J(\mathcal{F})\}$.
{ If $k\in J(\mathcal{F}^{(i)})\subseteq F^{(i)}$ then $k\leq i$ and $i;k;i=k$ so $k=k\bullet i$ and $\infty;k;\infty=\infty;i;k;i;\infty=k$. }
We also have $\{j\bullet i/j\in J(\mathcal{F})\}\subseteq J(\mathcal{F}^{(i)})$.
{ If $k=i\bullet j$ with $j\in J(\mathcal{F})$ then $k=i\bullet j\leq i$ and $k=i\bullet j\in J(\mathcal{F})$, so $i;k;i=i\bullet k\bullet i=k$. }

d) FA $\mathcal{F}^{(i)}$ is simple iff $J(\mathcal{F}^{(i)})\subseteq\{0,i\}$ iff for every $j\in J(\mathcal{F})$: $j\bullet i\in\{0,i\}$.
Thus, FA $\mathcal{F}^{(i)}$ is prime iff $i\neq 0$ and for all $j\in J(\mathcal{F})$, $j\bullet i\in\{0,i\}$.

QED

The next result characterises the direct product factorisations and the factors of an FA in terms of its ideal elements. We call FA $\mathcal{G}$ a *factor* of FA $\mathcal{F}$ iff $\mathcal{F}\cong\mathcal{G}\times\mathcal{H}$ for some non-trivial FA $\mathcal{H}$.

Proposition *Factors of FA and ideal elements*

Consider an FA $\mathcal{F}$ with set of ideal elements $J(\mathcal{F})$.

a) $\mathcal{F}\cong\mathcal{G}\times\mathcal{H}$ iff $\mathcal{G}\cong\mathcal{F}^{(g)}$ and $\mathcal{H}\cong\mathcal{F}^{(h)}$ for some ideal elements $g=h^-\in J(\mathcal{F})$.

b) FA $\mathcal{G}$ is a factor of FA $\mathcal{F}$ iff $\mathcal{G}\cong\mathcal{F}^{(g)}$ for some element $g\in(J(\mathcal{F})-\{\infty\})$.

Proof outline

Similar to [Jónsson & Tarski '52, p. 134, 135, Theorem 4.13].

a) We use the characterisation of relativisation in the preceding lemma.
($\Leftarrow$) With $g=h^-\in J(\mathcal{F})$, we have FA's $\mathcal{F}^{(g)}\cong\mathcal{F}/F^{(h)}$ and $\mathcal{F}^{(h)}\cong\mathcal{F}/F^{(g)}$ such that $\mathcal{F}^{(g)}\times\mathcal{F}^{(h)}\cong\mathcal{F}/[F^{(h)}\cap F^{(g)}]\cong\mathcal{F}$.
($\Rightarrow$) Given FA's $\mathcal{G}$ and $\mathcal{H}$, call $\mathcal{P}:=\mathcal{G}\times\mathcal{H}$, set $g:=<\infty_{\mathcal{G}},0_{\mathcal{H}}>$ and $h:=<0_{\mathcal{G}},\infty_{\mathcal{H}}>$.
Notice that $g=h^-\in J(\mathcal{P})$.
Now, consider the product projections $p:G\times H\to G$ and $q:G\times H\to H$.
Notice that their kernels are $ker(p)=P^{(h)}$ and $ker(q)=P^{(g)}$;
thus $\mathcal{G}\cong\mathcal{P}/P^{(h)}\cong\mathcal{P}^{(g)}$ and $\mathcal{H}\cong\mathcal{P}/P^{(g)}\cong\mathcal{P}^{(h)}$.
Hence, $\mathcal{P}=\mathcal{G}\times\mathcal{H}\cong\mathcal{P}^{(g)}\times\mathcal{P}^{(h)}$ with $g=h^-\in J(\mathcal{P})$.
Now, given an isomorphism m between $\mathcal{P}=\mathcal{G}\times\mathcal{H}$ and $\mathcal{F}$, we have $\mathcal{F}\cong\mathcal{F}^{(m(g))}\times\mathcal{F}^{(m(h))}$ with $m(g)=m(h)^-\in J(\mathcal{F})$.

b) With $h=g^-\in J(\mathcal{F})$, relativisation $\mathcal{F}^{(h)}$ is non-trivial iff $h\neq 0$ iff $g=h^-\neq\infty$.

QED

The next result describes the prime factors of a direct product of prime FA's.

Corollary *Prime factors of direct product of prime FA's*

Consider a direct product $\mathcal{F}:=\times_{i\in I}\mathcal{F}_i$ of prime FA's $\mathcal{F}_i$, $i\in I$. Any prime factor $\mathcal{G}$ of FA $\mathcal{F}$ is isomorphic to some factor $\mathcal{F}$: $\mathcal{G}\cong\mathcal{F}_i$, for some $i\in I$.

Proof outline

Consider the (ideal) elements $\underline{i}:=<\ldots 0_j,\ldots,\infty_i,\ldots,0_k,\ldots>$, $i\in I$, of FA $\mathcal{F}$.

They are the atoms of the BA $\mathcal{J}[\mathcal{F}]$, and $\mathcal{F}_i\cong\mathcal{F}^{(\underline{i})}$, for each $i\in I$.

By the proposition and lemma, $\mathcal{G}\cong\mathcal{F}^{(g)}$ with g an atom of BA $\mathcal{J}[\mathcal{F}]$.

Thus, for some $i\in I$, $g=\underline{i}$: whence $\mathcal{G}\cong\mathcal{F}^{(g)}=\mathcal{F}^{(\underline{i})}\cong\mathcal{F}_i$.

QED

The situation is entirely similar to the case of RA's: the prime factors of a direct product $\times_{i\in I}\mathcal{R}_i$ of prime RA's are the components $\mathcal{R}_i$, $i\in I$.

The next lemma gives a lower bound on the special index of a direct power of prime RA's.

Lemma *Lower bound on the special index of direct power of prime RA's*

For each prime RA $\mathcal{R}$, if $I\neq\varnothing$ then $\theta(\mathcal{R}^I)\geq\theta(\mathcal{R})$.

Proof outline

Since RA $\mathcal{R}$ is prime, for each special expansion $\mathcal{F}$ of $\mathcal{R}$, the direct power $\mathcal{F}^I$ is a special FA expansion of $\mathcal{R}^I$ with single prime factor $\mathcal{F}$.

Now, prime $\mathcal{G},\mathcal{H}\in$ **SPC**$(\mathcal{R})$ are isomorphic FA's iff the direct powers $\mathcal{G}^I,\mathcal{H}^I\in$ **SPC**$(\mathcal{R}^I)$ are isomorphic FA's.

Hence, $\theta(\mathcal{R}^I)\geq\theta(\mathcal{R})$.

QED

The next lemma gives a lower bound on the special index of some direct products of prime RA's.

Lemma *Special index of direct product of prime, special RA's*

Let $\mathcal{R}$ be a prime RA of cardinality $|\mathcal{R}|=\kappa$. Given a set of special prime FA's $\mathcal{Q}_i$ of cardinality $|\mathcal{Q}_i|=\zeta_i$ with RA reducts $\mathcal{P}_i$, $i\in I$, consider the direct product $\mathcal{R}^*:=(\times_{i\in I}\mathcal{P}_i)\times\mathcal{R}$. If $\kappa\notin\{\zeta_i/i\in I\}$ then $\theta(\mathcal{R}^*)\geq\theta(\mathcal{R})$.

Proof outline

Since RA $\mathcal{R}$ is prime, for each special expansion $\mathcal{F}$ of $\mathcal{R}$, the direct product $\mathcal{F}^*:=(\times_{i\in I}\mathcal{Q}_i)\times\mathcal{F}$ is a special FA expansion of RA $\mathcal{R}^*$ with set of prime factors $\{\mathcal{Q}_i/i\in I\}\cup\{\mathcal{F}\}$ (with set of cardinalities $\{\zeta_i/i\in I\}\cup\{\kappa\}$).

Now prime $\mathcal{G},\mathcal{H}\in$ **SPC**$(\mathcal{R})$ are isomorphic FA's iff the direct products $\mathcal{G}^*=(\times_{i\in I}\mathcal{Q}_i)\times\mathcal{G}$ and $\mathcal{H}^*=(\times_{i\in I}\mathcal{Q}_i)\times\mathcal{H}$ are isomorphic FA's (since $|\mathcal{R}|=\kappa\notin\{\gamma_i/i\in I\}$).

Hence, $\theta(\mathcal{R}^*)\geq\theta(\mathcal{R})$.

QED

5.3 Analysis of (Proper) Fork Algebras

We now introduce a tool for the analysis of FA's. Given a ϕ-algebra $\mathcal{F}$, let $2:=1\nabla 1$, and consider the set of its *sub-identities of* 2: $SI2(\mathcal{F}):=\{f\in F/f\leq 2\bullet 1\}$.

Notice that any ϕ-isomorphism between ϕ-algebras $\mathcal{F}$ and $\mathcal{G}$ provides a bijection between SI2($\mathcal{F}$) and SI2($\mathcal{G}$), so $|SI2(\mathcal{F})|=|SI2(\mathcal{G})|$.

We shall call a ϕ-algebra $\mathcal{F}$ *size-controlled by* cardinal γ iff its set SI2($\mathcal{F}$) of sub-identities of 2 has cardinality $|SI2(\mathcal{F})|=2^{\gamma}$.

In view of the preceding remark, ϕ-algebras $\mathcal{G}$ and $\mathcal{H}$ that are size-controlled by distinct cardinals cannot be isomorphic.

In a PFA $\mathcal{Q}=<Q,\varnothing,V,1_U,{}^{\sim},{}^{T},\cup,\cap,|,\angle^*>$ with fork $\angle^*$ induced by coding $*:U^2\to U$, its set of sub-identities of 2 is $SI2(\mathcal{Q})=\{q\in Q/q\subseteq 2\cap 1_U\}=\wp(2\cap 1_U)$. In case PFA $\mathcal{Q}$ is simple ($V=U^2$), the element $2\cap 1_U$ is connected to the set of fixpoints of its underlying coding.

Given a function $*:U^2\to U$, consider its *set of fixpoints* $fxpt(*):=\{u\in U/u*u=u\}$. This set of fixpoints can also be conveniently represented by its identity $1_{fxpt(*)}:=\{<u,u>\in U^2/u*u=u\}$. Notice that $|1_{fxpt(*)}|=|fxpt(*)|$.

The next result shows a connection between the set of sub-identities of 2 of a simple PFA and the set of fixpoints of its underlying coding.

Proposition *Simple PFA connection*: SI2 vs. fxpt(*)

Consider a simple PFA $\mathcal{Q}=<Q,\varnothing,U^2,1_U,{}^{\sim},{}^{T},\cup,\cap,|,\angle^*>$ with fork $\angle^*$ induced by coding $*:U^2\to U$. Then $2\cap 1_U=1_{fxpt(*)}$; so $SI2(\mathcal{Q})=\wp[1_{fxpt(*)}]\cap Q$.

Proof outline

First, we see that $2\cap 1_U=1_{fxpt(*)}$, since $<u,v>\in 2\cap 1_U$ iff $v=u\in fxpt(*)$.

Hence, since $Q\subseteq\wp(U^2)$, $r\in SI2(\mathcal{Q})$ iff $r\in Q\cap\wp[1_{fxpt(*)}]$.

QED

In a simple PFA, the set of fixpoints of its underlying coding provides a tool for checking whether it is special, as shown in the next result.

Lemma *Set of fixpoints of coding and special simple PFA's*

Consider a simple PFA $\mathcal{Q}=<Q,\varnothing,U^2,1_U,{}^{\sim},{}^{T},\cup,\cap,|,\angle^*>$ with fork $\angle^*$ induced by underlying coding $*:U^2\to U$ with set of fixpoints $fxpt(*)=\{u\in U/u*u=u\}$.

Then, simple PFA $\mathcal{Q}$ is special iff $*:U^2\to U$ is surjective and $fxpt(*)\neq U$.

Proof outline

We have $2=1_U\angle^*1_U$, and, by the preceding proposition, $2\cap 1_U=1_{fxpt(*)}$.

Thus, $1_U\subseteq 1_U\angle^*1_U$ iff $U=fxpt(*)$. Also, $1_U\angle^*1_U\subseteq 1_U$ iff $fxpt(*)=U$.

So, $1_U\angle^*1_U=1_U$ iff $fxpt(*)=U$. Also $U^2\angle^*U^2=U^2$ iff $*:U^2\to U$ is surjective.

QED

5.4 Construction of (Infinite, Simple) Proper Fork Algebras

In the next section, we shall construct some infinite (simple, proper) non-Boolean RA's that have many (special) fork expansions. We shall control the cardinalities of the RA's by means of the sizes of their sets of generators. We shall control their fork expansions by means of the sizes of the sets of sub-identities of 2.

First, controlling the cardinality of an infinite simple PRA by means of the sizes of its sets of generators is quite easy, in view of the remark, in section 4, on subalgebras generated by infinite subsets.

Lemma *Infinite simple PRA's and sets of generators*

Given an infinite cardinal $\kappa \geq \aleph$, algebra $\mathcal{P}$ is a simple PRA with cardinality $|P|=\kappa$ over set U iff $\mathcal{P}$ is the subalgebra $\mathcal{P}(U^2)[G]$ of the full PRA $\mathcal{P}(U^2)$ generated by some subset $G \subseteq \wp(U^2)$ with cardinality $|G|=\kappa$.

We now analyse and generalise the construction involved in the proposition on large simple non-Boolean PFA's in section 4. Each injective function $*: U^2 \rightarrow U$ induces a fork operation $\angle^*$ on relations over U, which gives rise to *induced projections* $p^* := (1_U \angle^* U^2)^T$ and $q^* := (U^2 \angle^* 1_U)^T$.

Proposition *Simple PRA's and PFA expansions*

Let $\mathcal{P}$ be a simple PRA over set U. Consider any injective function $*: U^2 \rightarrow U$ inducing fork operation $\angle^*$ and projections p^* and q^*.

a) PRA $\mathcal{P}$ is expandable by the induced fork $\angle^*$ iff induced projections p^* and q^* are in its carrier P.

b) If $\{p^*,q^*\} \subseteq P$ and $fxpt(*) \in 1^P(U)$ {where $1^P(U) := \{S \subseteq U / \wp(1_S) \subseteq P\}$, then PRA $\mathcal{P}$ has fork expansion $\mathcal{P}^* := (\mathcal{P}, \angle^*)$ size-controlled by cardinal $\gamma := |fxpt(*)|$: $\mathcal{P}^*$ has set of sub-identities of 2 $SI2(\mathcal{P}^*) = \wp[1_{fxpt(*)}]$.

Proof outline

a) By the lemma on expandability of subalgebras of reducts in 5.1.

b) By the proposition on simple PFA connection between SI2 and fixpoint in 5.3, since $\wp[1_{fxpt(*)}] \subseteq P$ and $\gamma = |fxpt(*)| = |1_{fxpt(*)}|$.

QED

6. Fork Expansions of Infinite Relational Algebras

We will now construct some infinite (simple, proper) non-Boolean RA's that have many (special) fork expansions.

We first construct some infinite simple non-Boolean PRA's that have special fork expansions controlled by the sizes of their possible sets of sub-identities of 2.

6.1 Special Codings and Sets of Fixpoints

To control the set of fixpoints of its underlying coding, and so the set of sub-identities of 2 of a simple PFA, we construct special codings with a given set of fixpoints.

The next result presents a set-theoretical construction for a special coding on an infinite set U with controlled set of fixpoints.

Proposition *Special coding with controlled set of fixpoints*

Consider an infinite set U of cardinality $\kappa \geq \aleph_0$. For each subset $T \subseteq U$ with $|T|=\kappa$, there exists a bijection $*^S: U^2 \rightarrow U$ with $fxpt(*^S)=S$, where $S:=U-T$.

Proof outline

First, since set U is infinite, the following sets have the same infinite cardinality $|U|=\kappa \geq \aleph_0$: its square $U^2 = U \times U$, its identity (diagonal) relation

$1_U:=\{<u,v>\in U^2/u=v\}$ and its complement $1_U{}^{\sim}=\{<u,v>\in U^2/u\neq v\}$.

Thus, we can partition T into disjoint subsets A and B of U, both with cardinality κ. So we have a bijection $f:1_U{}^{\sim}\to A$.

We also have a bijection $g:T\to B$ without fixpoints.

{ We have bijections $g_0:A\to B_0$, and $g_{n+1}:B_n\to B_{n+1}$, $n\in \mathbf{N}$, with pairwise disjoint domains and images. Their disjoint union gives a bijection g from $T=A\cup B$ onto $B=\cup_{n\in N}B_n$ without fixpoints, as required. }

We now define $*^S:U^2\to U$ as follows:

for $u\in S$ we set $u*^Su:=u$ (notice that $u\notin A\cup B$);

for $u\in T$ we set $u*^Su:=g(u)$ (notice that $g(u)\in B$);

for $<v,w>\in 1_U{}^{\sim}$ we set $v*^Sw:=f(v,w)$ (notice that $f(v,w)\in A$).

So, $*^S:U^2\to U$ is a bijection, from $U^2=1_S\cup 1_T\cup 1_U{}^{\sim}$ onto $U=S\cup B\cup A$, since it is the disjoint union of bijections with pairwise disjoint domains and images. Also, $u*^Su=u$ iff $u\in S$, because for $u\notin S$ $u*^Su=g(u)\neq u$. Thus $fxpt(*^S)=S$.

QED

By a *special coding* on set U *fixing* a subset $S\subseteq U$ we mean a bijective function $*^S:U^2\to U$ with $fxpt(*^S)=S$. It induces fork $\angle^S$ over $\wp(U^2)$ and projections p^S and q^S in $\wp(U^2)$.

Corollary *Simple PFA and special coding with smaller set of fixpoints*

Consider a simple PFA $\mathcal{Q}=<Q,\emptyset,U^2,1_U,{}^{\sim},{}^{T},\cup,\cap,|,\angle^*>$ over set U where $\angle^*$ is the fork induced by some special coding $*:U^2\to U$ with $|fxpt(*)|<|U|$. Then $\mathcal{Q}$ is a special FA, and hence non-Boolean.

Proof outline

By the lemma on set of fixpoints of coding and special simple PFA's in 5.3.

QED

Corollary *Many special codings with smaller sets of fixpoints*

Consider an infinite set U. For each subset $S\subseteq U$ with cardinality $|S|<|U|$, there exists a special coding $*^S:U^2\to U$ with $fxpt(*^S)=S$.

Proof outline

We apply the previous proposition on special coding with controlled set of fixpoints to the complement $T:=U-S=\{u\in U/u\notin S\}$ (with $|T|=|U|$, since $|S|<|U|$).

QED

6.2 Large Simple Proper Relational and Fork Algebras

We will now construct many infinite special (prime, proper) fork algebras, by putting together the preceding considerations.

Proposition *Many large prime, special PFA's*

For each infinite cardinal $\kappa\geq\aleph_0$, there exist at least κ, pairwise non-isomorphic, prime special PFA's.

Proof outline

We select a set U with cardinality $|U|=\kappa\geq\aleph_0$, and for each subset $S\subseteq U$ with cardinality $|S|=\gamma<\kappa$, we obtain a simple special PFA Q_S size-controlled by γ. The corollary on many special codings with smaller sets of fixpoints yields a special coding $*^S:U^2\to U$ with fxpt$(*^S)=S$, inducing fork $\angle^S$ and projections p^S and q^S. Set $G_S:=\wp(1_S)\cup\{p^S,q^S\}\cup\wp_\omega(U^2)$ (note that $\kappa\le|G_S|\le 2^\gamma+2+\kappa=\kappa$) and let $\mathcal{P}_S:=\mathcal{P}(U^2)[G_S]$ be the subalgebra of the full PRA $\mathcal{P}(U^2)$ generated by G_S. Then, $\mathcal{P}_S$ is a simple PRA of cardinality $|\mathcal{P}_S|=\kappa\ge\aleph_0$ (so prime), which has a special fork expansion $Q_S=(\mathcal{P}_S,\angle^S)$ with $|SI2(Q_S)|=2^\gamma$ (in view of section 5). Therefore, there are at least κ, pairwise non-isomorphic, prime special PFA's Q_S with cardinality $|Q_S|=\kappa$, for $|S|<\kappa$.

QED

This information about the infinite (prime) PFA's is summarised below:

$\kappa\ge\aleph_0$ $\quad$ $\sigma(\kappa)\ge\kappa$ $\quad$ $\nu(\kappa)\ge\kappa$

We now wish to strengthen the preceding result to the effect that we have many large prime, special PFA's with the same RA reduct. For this purpose, we now construct a prime, non-Boolean, elastic PRA of each infinite cardinality (with special PFA-expansions size-controlled by smaller cardinals).

Theorem *Large prime, non-Boolean, elastic PRA's*

For each infinite cardinal $\kappa\ge\aleph_0$, there exists a prime, non-Boolean, elastic PRA $\mathcal{P}_\kappa$ of cardinality $|\mathcal{P}_\kappa|=\kappa$: PRA $\mathcal{P}_\kappa$ has κ, pairwise non-isomorphic, special FA expansions Q_γ for each smaller cardinal $\gamma<\kappa$.

Proof outline

The set $U:=\kappa$ is such that each cardinal $\xi<\kappa$, ξ is a subset $\xi\subseteq U$ with $|\xi|=\xi<\kappa$. So, by the corollary on many special codings with smaller set of fixpoints, for each cardinal $\gamma<\kappa$, we have a special coding $*^\gamma:U^2\to U$, with fxpt$(*^\gamma)=\gamma$, inducing fork $\angle^\gamma$ and projections p^γ and q^γ.

Set $H:=\wp_\omega(U^2)\cup\bigcup_{\gamma<\kappa}[\{p^\gamma,q^\gamma\}\cup\wp(1_\gamma)]$ (notice that $\kappa\le|H|\le\kappa+\kappa.(2+\kappa)=\kappa$) and consider the subalgebra $\mathcal{P}_\kappa:=\mathcal{P}(U^2)[H]$ of the full PRA $\mathcal{P}(U^2)$ generated by H.

Then, $\mathcal{P}_\kappa$ is a simple PRA of cardinality $|\mathcal{P}_\kappa|=\kappa\ge\aleph_0$ (so prime), which has a special fork expansion $Q_\gamma=(\mathcal{P}_\kappa,\angle^\gamma)$ with $|SI2(Q_\gamma)|=2^\gamma$, for each $\gamma<\kappa$.

Therefore, $\mathcal{P}_\kappa$ is a prime, non-Boolean, elastic PRA of cardinality $|\mathcal{P}_\kappa|=\kappa$.

QED

This information about the infinite (prime) PRA's is summarised below:

$\kappa\ge\aleph_0$ $\quad$ $\mathcal{P}_\kappa$ {prime} $\quad$ $\theta(\mathcal{P}_\kappa)=\kappa$ $\quad$ $\varphi(\mathcal{P}_\kappa)=\kappa$

6.3 Large non-Boolean Relation and Fork Algebras

We now wish to show that there are many infinite elastic RA's and non-Boolean FA's of each given infinite cardinality. These (non-simple) algebras will be constructed from the prime ones obtained in 6.2.

We shall now exhibit many non-Boolean elastic RA's of each infinite cardinality, each one of them with many non-isomorphic special FA expansions. We examine two constructions, namely direct powers and products; in each case we obtain non-simple FA's, which we guarantee to be non-isomorphic by controlling the sizes of their sets of ideal elements.

We first use a direct-power construction to exhibit an infinite collection of pairwise non-isomorphic non-Boolean elastic RA's of each given infinite cardinality.

Theorem *Infinitely many large non-Boolean elastic RA's*

For each infinite cardinal $\kappa \geq \aleph_0$, there exist infinitely many pairwise non-isomorphic non-Boolean elastic RA's of cardinality κ.

Proof outline

By the preceding theorem on large prime, non-Boolean, elastic PRA's, we have a prime, non-Boolean, elastic PRA $\mathcal{P}$ of cardinality $|\mathcal{P}|=\kappa$.

For each $n \in \mathbf{N}$, consider the direct power $\mathcal{P}^{n+1}$.

Then, $\mathcal{P}^{n+1}$ is an RA, of cardinality $|\mathcal{P}^{n+1}|=\kappa^n.\kappa=\kappa$ and with special index $\theta(\mathcal{P}^{n+1}) \geq \theta(\mathcal{P})=\kappa$, which has exactly 2^{n+1} ideal elements.

Thus, each RA $\mathcal{P}^{n+1}$ is elastic, so non-Boolean, with cardinality $|\mathcal{P}^{n+1}|=\kappa$.

Hence, there are at least $\aleph_0$ pairwise non-isomorphic elastic RA's $\mathcal{P}^{n+1}$ of cardinality κ, for $n \in \mathbf{N}$.

QED

Thus, for the elastic (non-Boolean) RA's, we have:

$\kappa \geq \aleph_0$	$\mathcal{P}^{n+1}$ {elastic RA}	$\theta(\mathcal{P}^{n+1})=\kappa$	$\varphi(\mathcal{P}^{n+1})=\kappa$
$\kappa \geq \aleph_0$	Elastic RA's	$\sigma(\kappa) \geq 1$	$\nu(\kappa) \geq \aleph_0$

We now use a direct-product construction to exhibit large collections of pairwise non-isomorphic non-Boolean elastic RA's of each given infinite cardinality.

Theorem *Many large non-Boolean elastic RA's*

For each infinite cardinal $\kappa \geq \aleph_0$, there exist at least κ pairwise non-isomorphic non-Boolean elastic RA's $\mathcal{R}[\gamma]$ of cardinality κ (each one of them with special index $\theta(\mathcal{R}[\gamma])=\kappa$).

Proof outline

By the theorem on large prime, non-Boolean, elastic PRA's, we have

a prime, non-Boolean, elastic PRA Q of cardinality $|Q|=\aleph_0$;

a prime, non-Boolean, elastic PRA $\mathcal{P}$ of cardinality $|\mathcal{P}|=\kappa$.

For each cardinal $\gamma<\kappa$, form the direct product $\mathcal{R}[\gamma]:=Q^{\gamma} \times \mathcal{P}$.

Then, $\mathcal{R}[\gamma]$ is an RA, of cardinality $\kappa=|\mathcal{P}| \leq |\mathcal{R}[\gamma]| \leq \kappa.\kappa=\kappa$ and with special index $\theta(\mathcal{R}[\gamma]) \geq \theta(\mathcal{P})=\kappa$, which has exactly $2^{\gamma+1}$ ideal elements.

Thus, each RA $\mathcal{R}[\gamma]$ is elastic, so non-Boolean, with cardinality $|\mathcal{R}[\gamma]|=\kappa$.

Thus, there are at least κ pairwise non-isomorphic elastic RA's $\mathcal{R}[\gamma]=Q^{\gamma} \times \mathcal{P}$ of cardinality κ, for $\gamma<\kappa$.

QED

Thus, for the elastic (non-Boolean) RA's, we can actually state:

$\kappa\geq\aleph_0$	$\mathcal{R}[\gamma]$ {elastic RA}	$\theta(\mathcal{R}[\gamma])=\kappa$	$\varphi(\mathcal{R}[\gamma])=\kappa$
$\kappa\geq\aleph_0$	Elastic RA's	$\sigma(\kappa)\geq 1$	$\nu(\kappa)\geq\kappa$

We can consider refining the direct-product construction of the preceding theorem to exhibit even larger collections of pairwise non-isomorphic non-Boolean elastic RA's of each given infinite cardinality. In this refined construction we guarantee the resulting RA's to be non-isomorphic by controlling their sets prime factors.

We first consider some notations for cardinals. Given a set I of cardinals, let $\Pi_{\gamma\in I}\gamma:=|\times_{\gamma\in I}\gamma|$ be the cardinality of the direct product $\times_{\gamma\in I}\gamma$. Given a cardinal κ, let $\kappa?:=\Pi_{\gamma\in\kappa}2^{\gamma}$.

Proposition *Collections of large non-Boolean elastic RA's*

For each infinite cardinal $\kappa>\aleph_0$ such that $\kappa?\leq\kappa$, there exist at least 2^{κ} pairwise non-isomorphic non-Boolean elastic RA's $\mathcal{R}[I]$ with cardinality κ.

Proof outline

By the theorem on large prime, non-Boolean, elastic PRA's,

for each infinite cardinal $\aleph_0\leq\xi\leq\kappa$, we have:

a prime, non-Boolean, elastic PRA $\mathcal{P}_\xi$ of cardinality $|\mathcal{P}_\xi|=\xi$.

For each set I of infinite cardinals strictly below κ ($I\subseteq\kappa-\aleph_0$ and $|\kappa-\aleph_0|=\kappa$), form the direct product $\mathcal{R}[I]:=(\times_{\gamma\in I}\mathcal{P}_\gamma)\times\mathcal{P}_\kappa$.

Then, $\mathcal{R}[I]$ is an RA, with cardinality $\kappa\leq|\mathcal{R}[I]|\leq\kappa?\leq\kappa$ and with special index $\theta(\mathcal{R}[I])\geq\theta(\mathcal{P}_\kappa)=\kappa$, which has set of prime factors $\{\mathcal{P}_\gamma/\gamma\in I\}\cup\{\mathcal{P}_\kappa\}$.

Thus, each RA $\mathcal{R}[I]$ is elastic, so non-Boolean, with cardinality $|\mathcal{R}[I]|=\kappa$.

Therefore, there are at least 2^{κ} pairwise non-isomorphic elastic RA's $\mathcal{R}[I]=(\times_{\gamma\in I}\mathcal{P}_\gamma)\times\mathcal{P}_\kappa$ of cardinality κ, for $I\in\wp(\kappa-\aleph_0)$.

QED

For an infinite successor cardinal $\kappa=2^{\alpha}$ with $\alpha\geq\aleph_0$, we have $\kappa?=\Pi_{\gamma\in\kappa}2^{\gamma}\leq\kappa$.

Theorem *Very large collections of large non-Boolean elastic RA's*

For each infinite successor cardinal $\kappa=2^{\alpha}$ with $\alpha\geq\aleph_0$, there exist at least 2^{κ} pairwise non-isomorphic non-Boolean elastic RA's $\mathcal{R}[I]$ with cardinality κ.

Thus, for the elastic (non-Boolean) RA's, we can actually state:

$\kappa=2^{\alpha}$ with $\alpha\geq\aleph_0$	$\mathcal{R}[I]$ {elastic RA}	$\theta(\mathcal{R}[I])=\kappa$	$\varphi(\mathcal{R}[I]])=\kappa$
$\kappa=2^{\alpha}$ with $\alpha\geq\aleph_0$	Elastic RA's	$\sigma(\kappa)\geq 1$	$\nu(\kappa)\geq 2^{\kappa}$

In view of the preceding theorem and the proposition on many large prime, special PFA's in 6.2, we can now see that there are large collections of pairwise non-isomorphic special FA's, for each infinite cardinality.

Corollary *Large collections of large special FA's*

Consider an infinite cardinal $\kappa\geq\aleph_0$.

a) There exist at least κ pairwise non-isomorphic special (simple, proper) FA's with cardinality κ.

b) If κ is a successor cardinal $\kappa=2^{\alpha}$ with $\alpha\geq\aleph_0$, then there exist at least 2^{κ}

pairwise non-isomorphic special FA's with cardinality κ.

The situation for the infinite special FA's is summarised below:

$\kappa \geq \aleph_0$	$\sigma(\kappa) \geq \kappa$	$\nu(\kappa) \geq \kappa$
$\kappa = 2^\alpha$ with $\alpha \geq \aleph_0$	$\sigma(\kappa) \geq \kappa$	$\nu(\kappa) \geq 2^\kappa$

7. CONCLUSION

A fork algebra (FA, for short) is a relational algebra (RA, for short) enriched with a new binary operation, called fork. They have been introduced because their equational calculus has applications in program construction, as well as some interesting connections with algebraic logic.

In this report, we have concentrated on the infinite FA's, contrasting them with RA's and with the finite FA's. For this purpose, we have introduced some concepts for the analysis of fork algebras and for the construction of some special infinite RA's and FA's. We also have simplified and extended some results appearing in previous reports [Veloso '96a,b].

Section 2 has provided some background [Veloso '96a] about fork algebras and their reducts (relational and Boolean algebras), the algebraic structure of FA's, and their concrete, set-based, versions (fields of sets and proper FA's and RA's) Section 3 has reviewed some results about the Boolean FA's (those where fork is Boolean meet) and the finite FA's [Veloso '96b].

The main body of the report consists of sections 4, 5 and 6, which cover mainly infinite FA's and RA's. In section 4 we have established the existence of simple non-Boolean (proper) FA's of each infinite cardinality. In section 5, we have suggested some concepts and methods for the analysis of FA's and their RA reducts, with the purpose of comparing RA's and FA's. We have examined fork expansions of RA's, introducing in 5.1 some indices for the fork-expandability of RA's, and examining the direct-product factors of FA's in 5.2. Then we have considered in 5.3 some concepts for the analysis of (proper) FA's, examining how they can be controlled by means of the underlying coding, as well as some methods for constructing (infinite, simple) proper FA's in 5.4. In section 6, we have applied these ideas to fork expansions of infinite RA's: we use a set-theoretical construction, examined in 6.1, to exhibit many infinite (simple, proper) non-Boolean RA's with many expansions to FA's in 6.2 and 6.3.

The infinite elastic RA's constructed in section 6 demonstrate quite clearly the diversity of possible fork operations. They also show the contrast between finite and infinite FA's.

The results recalled in section 3 show that, up to isomorphism, the finite FA's are the finite direct powers 2^n of the two-element (Boolean) FA and that the only simple finite FA's are *1* and *2*. Thus, a finite FA, in addition to being Boolean, is completely determined by its RA reduct (which we call rigid). This contrasts with the case of infinite FA's.

Let us compare finite and infinite FA's as well as infinite RA's.

We consider first a comparison in terms of the cardinalities of the isomorphism classes of (simple) FA's with cardinality κ.

The situation for finite FA's is as follows:

$n=0$	*1* {trivial}	$\sigma(2^0)=1$	$\nu(2^0)=1$
$n=1$	*2* {prime}	$\sigma(2^1)=1$	$\nu(2^1)=1$
$n>1$	2^n {non-simple}	$\sigma(2^n)=0$	$\nu(2^n)=1$

A summary of the situation for finite FA's is:

$n\in\mathbf{N}$	2^n {Boolean}	$0\leq\sigma(2^n)\leq\nu(2^n)=1$

For the infinite FA's, the situation is entirely different:

$\kappa\geq\aleph_0$	{simple}	$\aleph_0\leq\kappa\leq\sigma(\kappa)\leq\nu(\kappa)$
$\kappa=2^\alpha$ with $\alpha\geq\aleph_0$		$\aleph_0<\kappa\leq\sigma(\kappa)$ and $\nu(\kappa)\geq 2^\kappa>2^{\aleph_0}$

Let us now compare the RA reducts of some finite and infinite FA's, in terms of their expandability indices, introduced in 5.1. We have:

$n\in\mathbf{N}$	$(2^n)_\lambda$ {Boolean}	$\theta[(2^n)_\lambda]=0$	$\varphi[(2^n)_\lambda]=1$
$\kappa\geq\aleph_0$	$\mathcal{P}_\kappa$ {non-Boolean}	$\theta(\mathcal{P}_\kappa)=\kappa$	$\varphi(\mathcal{P}_\kappa)=\kappa$

These infinite elastic RA's demonstrate quite clearly the diversity of possible non-Boolean fork operations. And there many such elastic RA's with each infinite cardinality:

($\kappa\geq\aleph_0$) there are at least $\kappa\geq\aleph_0$ (simple) infinite FA's of each infinite cardinality $\kappa\geq\aleph_0$;

($\alpha^+>\aleph_0$) there are at least $2^{\alpha^+}>2^{\aleph_0}$ (non-simple) FA's with cardinality $\alpha^+>\aleph_0$ for each infinite successor cardinal $\alpha^+>\aleph_0$.

Also, finite and infinite FA's are quite distinct from three viewpoints:

(ι) the finite FA's are all Boolean, whereas
there are non-Boolean infinite FA's of each infinite cardinality;

(σ) there are only two simple finite FA's (*1* and *2*), whereas
there are at least $\kappa\geq\aleph_0$ simple infinite FA's with cardinality $\kappa\geq\aleph_0$;

(ν) the finite FA's are characterised by their cardinalities 2^n, whereas
there are at least $\kappa\geq\aleph_0$ (simple) FA's with cardinality $\kappa\geq\aleph_0$
(for successor cardinal $\alpha^+>\aleph_0$ there are at least $2^{\alpha^+}>2^{\aleph_0}$
(non-simple) FA's with cardinality $\alpha^+>\aleph_0$).

The RA reducts of finite and infinite FA's are also distinct in two ways:

(φ) the RA reducts of the finite FA's have a single FA expansion, whereas
for each infinite cardinality $\kappa\geq\aleph_0$ there are κ (simple) FA's of cardinality κ with the same RA reduct;

(θ) the RA reducts of the finite FA's have no special expansion, whereas
there are (simple) infinite elastic FA's of each infinite cardinality
(whose RA reducts $\mathcal{P}_\kappa$ have $|\mathcal{P}_\kappa|=\kappa\geq\aleph_0$ special FA expansions).

We have employed the following main ideas. We control the size of RA's by controlling the sizes of their sets of generators. We guarantee that an RA has an FA expansion by putting the defined projections of the latter in its carrier. We force simple PFA's to be non-isomorphic by controlling the sizes of the set of fixpoints of their underlying codings. Finally, we construct non-isomorphic (non-simple) RA's by controlling the sizes of their sets of ideal elements or by controlling their prime (simple and non-trivial) direct-product factors.

We conclude by hinting at a possible refinement of these constructions, which may provide finer tools for the analysis and construction of infinite (simple, proper) FA's. In 6.1 we have presented a set-theoretical construction of coding with controlled sets of fixpoints: it yields an injective coding $*:U^2 \rightarrow U$ with given $fxpt(*):=\{u \in U / u*u=u\}$. Some refined constructions appear to be able to control entire families of sets of higher fixpoints; for instance given subsets $S \subseteq T \subseteq U$ with $|U-T|=|U|$, one would obtain an injective coding $*:U^2 \rightarrow U$ with $\{u \in U/u*u=u\}=S$ and $\{u \in U/u*(u*u)=u\}=T$. Similarly, one would control entire families of sets of higher fixpoints $S_0 \subseteq \ldots \subseteq S_n \subseteq \ldots \subseteq U$ with $|U-\cup_{n \in N} S_n|=|U|$.

APPENDIX: DETAILED PROOFS OF THE RESULTS

We present in this appendix detailed proofs of the results, except for those in sections 2 and 3 (whose proofs appear in [Veloso '96a]).

Lemma *Non-surjective codings for infinite sets*

For each infinite set U, there exists an injective, non-surjective coding function $*:U^2 \rightarrow U$.

Proof

Consider a set U with cardinality $|U|=\kappa \geq \aleph_0$ and select an element $u \in U \neq \varnothing$. Since $U^2=U \times U$ and $U-\{u\}$ have the same infinite cardinality κ, we have a bijection from U^2 onto $U-\{u\}$. This provides an injective, non-surjective coding function $*:U^2 \rightarrow U$.

QED

Proposition *Large simple non-Boolean FA's*

For each infinite cardinal $\kappa \geq \aleph_0$ there exists a simple non-Boolean FA with cardinality κ.

Proof

Consider the set Σ consisting of the FA equations together with Tarski's rule $\forall x(\neg x \approx 0 \rightarrow \infty;x;\infty \approx \infty)$ and the sentence $\exists x,y \neg x \nabla y = x \bullet y$.

Choose a set U with cardinality $|U|=\kappa \geq \aleph_0$ and, in view of the preceding lemma, an injective, non-surjective coding function $*:U^2 \rightarrow U$.

Now, consider the full PFA $\mathcal{P}^*(U^2)=<\wp(U^2),\varnothing,U^2,1_U,^{\sim},^{T},\cup,\cap,|,\angle^*>$ over U.

Notice that $\mathcal{P}^*(U^2)$ has the following properties.

1. $\mathcal{P}^*(U^2)$ is a PFA with cardinality $|\mathcal{P}^*(U^2)|=2^\kappa>\kappa$.

{ Indeed, $|\mathcal{P}^*(U^2)|=|2^{|U|}=2^\kappa>\kappa$. }

2. $\mathcal{P}^*(U^2)$ satisfies Tarski's rule $\forall x(\neg x \approx 0 \rightarrow \infty;x;\infty \approx \infty)$.

{ Indeed, since $U^2 \in \wp(U^2)$, it is simple and satisfies Tarski's rule. }

3. $\mathcal{P}^*(U^2)$ satisfies $\exists x,y \neg x \nabla y = x \bullet y$.

{ Indeed, since $*:U^2 \rightarrow U$ is not surjective, $\mathcal{P}^*(U^2)$ is non-Boolean. }

Thus $\mathcal{P}^*(U^2)$ is a model, with cardinality $2^\kappa > \kappa$, of set Σ, which is consistent. The Downward Löwenheim-Skolem Theorem then guarantees that the consistent set Σ has a model $\mathcal{F}$ of infinite cardinality $\kappa < 2^\kappa$.

Notice that model $\mathcal{F}$ of set Σ has the following properties.

a. Model $\mathcal{F}$ is a fork algebra.

{ Indeed, $\mathcal{F}$ satisfies the FA equations. }

b. Model $\mathcal{F}$ is a simple FA.

{ Indeed, $\mathcal{F}$ satisfies Tarski's rule $\forall x(\neg x \approx 0 \rightarrow \infty;x;\infty \approx \infty)$. }

c. Model $\mathcal{F}$ is a non-Boolean FA.

{ Indeed, $\mathcal{F}$ satisfies $\exists x,y \neg\, x\nabla y = x \bullet y$. }

Thus, $\mathcal{F}$ is a simple, non-Boolean FA with cardinality $|\mathcal{F}| = \kappa$.

QED

Proposition *Large simple non-Boolean PFA's*

For each infinite cardinal $\kappa \geq \aleph_0$ there exists a simple non-Boolean PFA $\mathcal{P}^*_\omega(U^2)$ with cardinality κ.

Proof

Choose a set U with cardinality $|U| = \kappa \geq \aleph_0$ and an injective, non-surjective coding function $*: U^2 \rightarrow U$.

Again, consider the full PFA $\mathcal{P}^*(U^2) = \langle \wp(U^2), \varnothing, U^2, 1_U, \tilde{}, ^T, \cup, \cap, |, \angle^* \rangle$ over U: it is a simple, non-Boolean (since $*: U^2 \rightarrow U$ is not surjective) PFA with cardinality $2^\kappa > \kappa$.

Notice the set $\wp_\omega(U^2) \subseteq \wp(U^2)$ of finite subsets of U^2 has cardinality $|\wp_\omega(U^2)| = \kappa.\aleph_0 = \kappa$.

Now, consider the subalgebra $\mathcal{P}^*_\omega(U^2) := \mathcal{P}^*(U^2)[\wp_\omega(U^2)]$ of the full PFA $\mathcal{P}^*(U^2)$ generated by $\wp_\omega(U^2)$. Notice that it has the following properties.

1. Subalgebra $\mathcal{P}^*_\omega(U^2)$ is a simple PFA.

{ Indeed, it is a subalgebra of the simple full PFA $\mathcal{P}^*(U^2)$. }

2. Subalgebra $\mathcal{P}^*_\omega(U^2)$ has cardinality $|\mathcal{P}^*_\omega(U^2)| = \kappa$.

{For $|\mathcal{P}^*_\omega(U^2)| = |\wp_\omega(U^2)|.\aleph_0 = \kappa.\aleph_0 = \kappa$ [Burris & Sankappanavar '81, p. 32]. }

3. Subalgebra $\mathcal{P}^*_\omega(U^2)$ is non-Boolean.

{ Since $*: U^2 \rightarrow U$ is not surjective, simple PFA $\mathcal{P}^*_\omega(U^2)$ is non-Boolean. }

Lemma *Expandability of subalgebras of reducts*

Consider an algebra $\mathcal{F}$ of FA-signature ϕ, with defined elements $\pi := (1\nabla\infty)^\dagger$ and $\rho := (\infty\nabla 1)^\dagger$, and a λ-subalgebra $\mathcal{R}$ of its λ-reduct $\mathcal{F}_\lambda$.

a) If $\mathcal{F}$ satisfies axiom (∇-def), then λ-subalgebra $\mathcal{R}$ of $\mathcal{F}_\lambda$ is expandable by $\nabla: F \times F \rightarrow F$ iff π and ρ are in R.

b) If π and ρ are in R and $\mathcal{F}$ is an FA, then so is the ∇-expansion $\mathcal{R}^\nabla$.

Proof

a) Since $\mathcal{F}$ satisfies (∇-def), we have $r\nabla s = (r;\pi^\dagger) \bullet (s;\rho^\dagger)$ for every $r,s \in R \subseteq F$. Thus, R is closed under ∇ iff $\pi = (1\nabla\infty)^\dagger$ and $\rho = (\infty\nabla 1)^\dagger$ are in R.

{ (⇒) If R is closed under ∇ then $\pi=(1\nabla\infty)^{\dagger}$ and $\rho=(\infty\nabla 1)^{\dagger}$ are in R.
(⇐) If $\pi,\rho\in R$ then $r\nabla s=(r;\pi^{\dagger})\bullet(s;\rho^{\dagger})\in R$, for every $r,s\in R$. }

b) Since R will be closed under ∇, the ∇-expansion $\mathcal{R}^{\nabla}=(\mathcal{R},\nabla)$ of subalgebra $\mathcal{R}$ of $\mathcal{F}_{\lambda}$ will be a ϕ-subalgebra of FA $\mathcal{F}$, and thus $\mathcal{R}^{\nabla}\in$ **FA**.

QED

Proposition *Upper bound on fork indices of RA's*

An RA $\mathcal{R}$ with cardinality $|R|=\kappa$ has fork index $\varphi(\mathcal{R})\leq\kappa^2$.

Proof

Consider an FA-expansion $\mathcal{F}$ of RA $\mathcal{R}$ with defined projections $\pi_{\mathcal{F}}:=(1\nabla_{\mathcal{F}}\infty)^{\dagger}$ and $\rho_{\mathcal{F}}:=(\infty\nabla_{\mathcal{F}}1)^{\dagger}$. Then $\mathcal{R}=\mathcal{F}_{\lambda}$, so by the preceding lemma, π and ρ are in R. This defines a function $P:\mathbf{FRK}(\mathcal{R})\rightarrow R^2$ from the set $\mathbf{FRK}(\mathcal{R})$ of FA-expansions of RA $\mathcal{R}$ into $R^2=R\times R$.

This function $P:\mathbf{FRK}(\mathcal{R})\rightarrow R^2$ is injective, in view of axiom (∇-def).

{ Indeed, given $\mathcal{F},\mathcal{G}\in\mathbf{FRK}(\mathcal{R})$ with $P(\mathcal{F})=P(\mathcal{G})$, we have $\nabla_{\mathcal{F}}=\nabla_{\mathcal{G}}$ and $\mathcal{F}=\mathcal{G}$, since $r\nabla_{\mathcal{F}}s=[r;(\pi_{\mathcal{F}}^{\dagger})]\bullet[s;(\rho_{\mathcal{F}}^{\dagger})]=[r;(\pi_{\mathcal{G}}^{\dagger})]\bullet[s;(\rho_{\mathcal{G}}^{\dagger})]=r\nabla_{\mathcal{G}}s$. }

Thus, $|\mathbf{FRK}(\mathcal{R})|\leq|R^2|=|R|^2=\kappa^2$, and $\varphi(\mathcal{R})\leq|\mathbf{FRK}(\mathcal{R})|$; so $\varphi(\mathcal{R})\leq\kappa^2$.

QED

Corollary *Upper bound on fork indices of infinite RA's*

An infinite RA $\mathcal{R}$, with cardinality $|R|=\kappa\geq\aleph_0$ has fork index $\varphi(\mathcal{R})\leq\kappa$.

Proof

By the preceding proposition, $\varphi(\mathcal{R})\leq\kappa^2$, and, since $\kappa\geq\aleph_0$, $\kappa^2=\kappa$.

QED

Lemma *Relativisation of FA to ideal elements*

Consider an FA $\mathcal{F}=(\mathcal{R},\nabla)$ with carrier R. Given an ideal element $i\in J(\mathcal{F})=J(\mathcal{R})$, consider the set of elements below i: $F^{(i)}:=\{f\in F/f\leq i\}$..

a) Set $F^{(i)}=\{f\in F/f\leq i\}$ is closed under fork $\nabla:F\times F\rightarrow F$.

b) Relativisation $\_^{(i)}:F\rightarrow F^{(i)}$, defined by $\_^{(i)}(f):=f\bullet i$, is a ϕ-homomorphism from FA $\mathcal{F}=(\mathcal{R},\nabla)$ onto the ∇-expansion $\mathcal{F}^{(i)}:=(\mathcal{R}^{(i)},\nabla)$, with kernel $\ker(\_^{(i)})=F^{(j)}$ where $j=i^{-}$. Thus $\mathcal{F}^{(i)}\cong\mathcal{F}/F^{(j)}$ with $j=i^{-}\in J(\mathcal{F})$.

c) $\mathcal{F}^{(i)}:=(\mathcal{R}^{(i)},\nabla)$ is an FA with $J(\mathcal{F}^{(i)})=J(\mathcal{F})\cap F^{(i)}=\{j\bullet i/j\in J(\mathcal{F})\}$.

d) FA $\mathcal{F}^{(i)}$ is prime iff i is an atom of BA $\mathcal{J}[\mathcal{F}]=\mathcal{J}[\mathcal{R}]$.

Proof

First, notice that, for $j,k\in J(\mathcal{F})=J(\mathcal{R})$, $j\nabla k=(j;\pi^{\dagger})\bullet(k;\rho^{\dagger})\leq(j;\infty)\bullet(k;\infty)\leq j\bullet k$.

a) Thus, for $r,s\in F^{(i)}$, we have by fork monotonicity [Frias et al. '95, '96], $r\nabla s\leq i\nabla i\leq i\bullet i=i$; and so $r\nabla s\in F^{(i)}$.

b) By part (a), $\mathcal{R}^{(i)}$ is expandable by fork ∇ to ϕ-algebra $\mathcal{F}=(\mathcal{R},\nabla)$.

To see that $\_^{(i)}:F\rightarrow F^{(i)}$ preserves fork, consider elements $r=r\bullet i+r\bullet i^{-}$ and $s=s\bullet i+s\bullet i^{-}$ of F, and notice the following calculations.

1. $r\nabla s=(r\bullet i)\nabla(s\bullet i)+(r\bullet i)\nabla(s\bullet i^{-})+(r\bullet i^{-})\nabla(s\bullet i)+(r\bullet i^{-})\nabla(s\bullet i^{-})$

{ Since fork ∇ distributes over + [Frias et al. '95, '96]. }

2. $(r\bullet i)\nabla(s\bullet i^-)=0$, $(r\bullet i^-)\nabla(s\bullet i)=0$, and $[(r\bullet i^-)\nabla(s\bullet i^-)]\bullet i=0$.

{ Fork monotonicity [Frias et al. '95, '96] and the above remark yield: $(r\bullet i)\nabla(s\bullet i^-)\leq i\nabla i^-\leq i\bullet i^-=0$, $(r\bullet i^-)\nabla(s\bullet i)\leq i^-\nabla i\leq i^-\bullet i=0$, and $[(r\bullet i^-)\nabla(s\bullet i^-)]\bullet i\leq(i^-\nabla i^-)\bullet i\leq(i^-\bullet i^-)\bullet i=i^-\bullet i=0$. }

3. $r\nabla s=(r\bullet i)\nabla(s\bullet i)+(r\bullet i^-)\nabla(s\bullet i^-)$.

{ By 1 and 2: $r\nabla s=(r\bullet i)\nabla(s\bullet i)+(r\bullet i)\nabla(s\bullet i^-)+(r\bullet i^-)\nabla(s\bullet i)+(r\bullet i^-)\nabla(s\bullet i^-)=$ $=(r\bullet i)\nabla(s\bullet i)+0+0+(r\bullet i^-)\nabla(s\bullet i^-)=(r\bullet i)\nabla(s\bullet i)+(r\bullet i^-)\nabla(s\bullet i^-)$. }

4. $(r\nabla s)\bullet i=(r\bullet i)\nabla(s\bullet i)$.

{ By 3 and 2, since $(r\bullet i)\nabla(s\bullet i)\leq i\bullet i=i$: $(r\nabla s)\bullet i=[(r\bullet i)\nabla(s\bullet i)+(r\bullet i^-)\nabla(s\bullet i^-)]\bullet i=$ $=[(r\bullet i)\nabla(s\bullet i)]\bullet i+[(r\bullet i^-)\nabla(s\bullet i^-)]\bullet i=[(r\bullet i)\nabla(s\bullet i)]\bullet i+0=(r\bullet i)\nabla(s\bullet i)$. }

Hence, $\_^{(i)}(r\nabla s)=(r\nabla s)\bullet i=(r\bullet i)\nabla(s\bullet i)=\_^{(i)}(r)\nabla\_^{(i)}(s)$.

By [Jónsson & Tarski '52, p. 132, Theorem 4.9], relativisation $\_^{(i)}:F\to F^{(i)}$, defined by $\_^{(i)}(f):=f\bullet i$, is an RA-homomorphism from RA $\mathcal{R}$ onto the relativisation $\mathcal{R}^{(i)}$ (with kernel $ker(\_^{(i)})=\{f\in F/f\bullet i=0\}=F^{(j)}$ where $j=i^-$).

Hence, relativisation $\_^{(i)}:F\to F^{(i)}$ is a ϕ-homomorphism from FA $\mathcal{F}=(\mathcal{R},\nabla)$ onto the ∇-expansion $\mathcal{F}^{(i)}:=(\mathcal{R}^{(i)},\nabla)$, with kernel $ker(\_^{(i)})=F^{(j)}$ where $j=i^-$.

Thus, $\mathcal{F}^{(i)}\cong\mathcal{F}/F^{(j)}$ with $j=i^-\in J(\mathcal{F}]$.

c) We show the inclusions $J(\mathcal{F}^{(i)})\subseteq J(\mathcal{F})\cap F^{(i)}\subseteq\{j\bullet i/j\in J(\mathcal{F})\}\subseteq J(\mathcal{F}^{(i)})$.

1. If $k\in J(\mathcal{F}^{(i)})\subseteq F^{(i)}$ then $k\in F^{(i)}$ and $k\in J(\mathcal{F})$.

{ If $k\in J(\mathcal{F}^{(i)})\subseteq F^{(i)}$ then $k\in F^{(i)}$ and $k=i;k;i$ so $\infty;k;\infty=\infty;i;k;i;\infty=i;k;i=k$. }

2. If $k\in J(\mathcal{F})\cap F^{(i)}$ then $k=i\bullet j$ for some $j\in J(\mathcal{F})$.

{ If $k\in F^{(i)}$ and $k\in J(\mathcal{F}]$ then $k=i\bullet k$ and $k\in J(\mathcal{F}]$. }

3. If $k=i\bullet j$ for some $j\in J(\mathcal{F})$ then $J(\mathcal{F}^{(i)}]$.

{ If $k=i\bullet j$ for some $j\in J(\mathcal{F})$ then $k=i\bullet j\leq i$ and $k=i\bullet j\in J(\mathcal{F})$, so $i;k;i=i\bullet k\bullet i=k$. }

d) Since the universal element of $\mathcal{F}^{(i)}$ is i, by the remark on simple FA's, FA $\mathcal{F}^{(i)}$ is simple iff $J(\mathcal{F}^{(i)})\subseteq\{0,i\}$. By part (c), $J(\mathcal{F}^{(i)})=\{j\bullet i/j\in J(\mathcal{F})\}$.
So, $J(\mathcal{F}^{(i)})\subseteq\{0,i\}$ iff $\{j\bullet i/j\in J(\mathcal{F})\}\subseteq\{0,i\}$ iff for every $j\in J(\mathcal{F})$: $j\bullet i\in\{0,i\}$.
Hence, FA $\mathcal{F}^{(i)}$ is prime iff $i\neq 0$ and for all $j\in J(\mathcal{F})$, $j\bullet i\in\{0,i\}$
iff i is an atom of Boolean algebra $\mathcal{J}[\mathcal{F}]=\mathcal{J}[\mathcal{R}]$.

QED

Proposition *Factors of FA and ideal elements*

Consider an FA $\mathcal{F}$ with set of ideal elements $J(\mathcal{F})$.

a) $\mathcal{F}\cong\mathcal{G}\times\mathcal{H}$ iff $\mathcal{G}\cong\mathcal{F}^{(g)}$ and $\mathcal{H}\cong\mathcal{F}^{(h)}$ for some ideal elements $g=h^-\in J(\mathcal{F})$.

b) FA $\mathcal{G}$ is a factor of FA $\mathcal{F}$ iff $\mathcal{G}\cong\mathcal{F}^{(g)}$ for some element $g\in(J(\mathcal{F})-\{\infty\})$.

Proof

Similar to [Jónsson & Tarski '52, p. 134, 135, Theorem 4.13].

a) The preceding lemma characterises relativisation $\mathcal{F}^{(i)}$ to ideal element $i\in J(\mathcal{F})$ as a homomorphic image of $\mathcal{F}$ under relativisation homomorphism $\_^{(i)}:F\to F^{(i)}$, defined by $\_^{(i)}(r):=r\bullet i$, so $\mathcal{F}^{(i)}\cong\mathcal{F}/F^{(j)}$ with $j=i^-\in J(\mathcal{F}]$.

($\Leftarrow$) With $g=h^{-}\in J(\mathcal{F})$, we have homomorphic images $\mathcal{F}^{(g)}$ and $\mathcal{F}^{(h)}$ of $\mathcal{F}$, and an isomorphism m from $\mathcal{F}$ onto $\mathcal{F}^{(g)}\times\mathcal{F}^{(h)}$, defined by $m(f):=<f\bullet g,f\bullet h>$.

{ By the preceding lemma, function $m:F\to F^{(g)}\times F^{(h)}$, defined by $m(f):=<f\bullet g,f\bullet h>$ is a homomorphism from $\mathcal{F}$ onto $\mathcal{F}^{(g)}\times\mathcal{F}^{(h)}$ with kernel $ker(m)=ker(\_^{(g)})\cap ker(\_^{(h)})=F^{(h)}\cap F^{(g)}=\{0\}$. }

($\Rightarrow$) Given FA's $\mathcal{G}$ and $\mathcal{H}$, call $\mathcal{P}:=\mathcal{G}\times\mathcal{H}$, and set $g:=<\infty_{\mathcal{G}},0_{\mathcal{H}}>$ and $h:=<0_{\mathcal{G}},\infty_{\mathcal{H}}>$.

Notice that g and h are ideal elements of $\mathcal{P}=\mathcal{G}\times\mathcal{H}$ such that $h=g^{-}$.

{ Clearly $g^{-}=<\infty_{\mathcal{G}},0_{\mathcal{H}}>^{-}=<\infty_{\mathcal{G}}^{-},0_{\mathcal{H}}^{-}>=<0_{\mathcal{G}},\infty_{\mathcal{H}}>=h$. Also, $g=<\infty_{\mathcal{G}},0_{\mathcal{H}}>\in J(\mathcal{G}\times\mathcal{H})$, because $\infty_{\mathcal{G}}\in J(\mathcal{G})$ and $0_{\mathcal{H}}\in J(\mathcal{H})$ (see [Veloso '96b; Appendix]). }

Now, consider the product projections $p:G\times H\to G$ and $q:G\times H\to H$.

Notice that projection $p:P\to G$ has kernel $ker(p)=P^{(h)}$.

{ Since $r=<g,h>\in ker(p)$ iff $g=0$ iff $r\le<0_{\mathcal{G}},\infty_{\mathcal{H}}>$ iff $r\in P^{(h)}$. }

Thus, $\mathcal{P}/ker(p)$ and $\mathcal{P}/P^{(h)}$ are homomorphic images of $\mathcal{P}$ with the same kernel $ker(p)=P^{(h)}$. Hence, $\mathcal{G}\cong\mathcal{P}/ker(p)\cong\mathcal{P}/P^{(h)}\cong\mathcal{P}^{(g)}$.

Similarly, projection $q:G\times H\to H$ has kernel $ker(q)=P^{(g)}$; so $\mathcal{P}/ker(q)$ and $\mathcal{P}/P^{(g)}$ are homomorphic images of $\mathcal{P}$ with the same kernel $ker(q)=P^{(g)}$. Hence, $\mathcal{H}\cong\mathcal{P}/ker(q)\cong\mathcal{P}/P^{(g)}\cong\mathcal{P}^{(h)}$.

Therefore, $\mathcal{P}=\mathcal{G}\times\mathcal{H}\cong\mathcal{P}^{(g)}\times\mathcal{P}^{(h)}$ with $g=h^{-}\in J(\mathcal{P}]$.

Now, if function $m:G\times H\to F$ is an isomorphism between $\mathcal{P}=\mathcal{G}\times\mathcal{H}$ and $\mathcal{F}$, then we have $\mathcal{F}\cong\mathcal{F}^{(m(g))}\times\mathcal{F}^{(m(h))}$ with $m(g)=m(h)^{-}\in J(\mathcal{F})$.

{ Because, then $m:P\to F$ gives an isomorphism between BA's $\mathcal{J}[\mathcal{P}]$ and $\mathcal{J}[\mathcal{F}]$. Thus, $\mathcal{G}\cong\mathcal{P}^{(g)}\cong\mathcal{F}^{(m(g))}$ and $\mathcal{H}\cong\mathcal{P}^{(h)}\cong\mathcal{F}^{(m(h))}$; so $\mathcal{P}=\mathcal{G}\times\mathcal{H}\cong\mathcal{F}^{(m(g))}\times\mathcal{F}^{(m(h))}$. }

b) By part (a), $\mathcal{F}\cong\mathcal{G}\times\mathcal{H}$ iff $\mathcal{G}\cong\mathcal{F}^{(g)}$ and $\mathcal{H}\cong\mathcal{F}^{(h)}$ for some $g=h^{-}\in J(\mathcal{F})$.

Now, relativisation $\mathcal{F}^{(h)}$ is non-trivial iff $h\neq 0$ iff $g=h^{-}\neq\infty$.

Thus, FA $\mathcal{G}$ is a factor of FA $\mathcal{F}$ iff $\mathcal{G}\cong\mathcal{F}^{(g)}$ for some $g\in(J(\mathcal{F})-\{\infty\})$.

QED

Corollary *Prime factors of direct product of prime FA's*

Consider a direct product $\mathcal{F}:=\times_{i\in I}\mathcal{F}_i$ of prime FA's $\mathcal{F}_i$, $i\in I$. Any prime factor $\mathcal{G}$ of FA $\mathcal{F}$ is isomorphic to some factor $\mathcal{F}$: $\mathcal{G}\cong\mathcal{F}_i$, for some $i\in I$.

Proof

Recall that $J(\times_{i\in I}\mathcal{F}_i)=\{k\in F/\forall i\in I: k(i)\in\{0_i,\infty_i\}\}$.

{ Since, for prime FA $\mathcal{F}_i$ $J(\mathcal{F}_i)=\{0_i,\infty_i\}$, $i\in I$, see [Veloso '96b; Appendix.] }

Notice that an ideal element $a\in J(\mathcal{F})$ is an atom of the BA $\mathcal{J}[\mathcal{F}]$ iff $a(i)\neq 0_i$ for a unique $i\in I$.

{ If $a(i)=0_i$ for a every $i\in I$ then $a=<\ldots,a_i,\ldots>=<\ldots,0_i,\ldots>=0$.

If $a(i)\neq 0_i$ and $a(j)\neq 0_j$ for distinct $i\neq j\in I$, then $a=<\ldots,a_i,\ldots,a_j,\ldots>$ has non-zero elements strictly below it (as $0=<\ldots,0_i,\ldots,0_j,\ldots><<\ldots,0_i,\ldots,a_j,\ldots><<\ldots,a_i,\ldots,a_j,\ldots>$). }

Now, consider the elements $\underline{i}:=<\ldots 0_j,\ldots,\infty_i,\ldots,0_k,\ldots>$, $i\in I$, of FA $\mathcal{F}$ (i. e. $\underline{i}(i):=\infty_i$ and $\underline{i}(j):=0_j$ for $j\in I-\{i\}$). Notice the following properties.

1. For each $i\in I$: $\underline{i}=<\ldots 0_j,\ldots,\infty_i,\ldots,0_k,\ldots>\in J(\mathcal{F})$ (see [Veloso '96b; Appendix]).

{ Because $\infty;\underline{i};\infty=<\ldots\infty_j;0_j;\infty_j,\ldots,\infty_i;\infty_i;\infty_i,\ldots,\infty_k;0_k;\infty_k,\ldots>=<\ldots 0_j,\ldots,\infty_i,\ldots,0_k,\ldots>=\underline{i}$. }

2. For each $i\in I$: $\mathcal{F}_i\cong\mathcal{F}^{(\underline{i})}$

{ As in the preceding proposition, projection $p_i:F\rightarrow F_i$ has kernel $\ker(p_i)=F^{(h)}$ where $h=\underline{i}^-\in J(\mathcal{F})$; so $\mathcal{F}_i\cong\mathcal{F}/\ker(p_i)\cong\mathcal{F}/F^{(h)}\cong\mathcal{F}^{(\underline{i})}$. }

3. For each $i\in I$: $\underline{i}=<\ldots 0_j,\ldots,\infty_i,\ldots,0_k,\ldots>$ is an atom of the BA $\mathcal{J}[\mathcal{F}]$.

{ For every $i,j\in I$: $\underline{i}(j):\neq 0_j$ iff $j=i$. }

4. The set of atoms of the BA $\mathcal{J}[\mathcal{F}]$ is $\{\underline{i}\in F/i\in I\}$.

{ Given an atom a of the BA $\mathcal{J}[\mathcal{F}]$, we have a unique $i\in I$ such that $a(i)\neq 0_i$ (so, since $k(i)\in\{0_i,\infty_i\}$, $a(i)=\infty_i$). Thus, $a=<\ldots 0_j,\ldots,\infty_i,\ldots>=\underline{i}$. }

Now, since $\mathcal{G}$ is a prime factor of FA $\mathcal{F}$, then, by the preceding proposition, $\mathcal{G}\cong\mathcal{F}^{(g)}$ with $g\in(J(\mathcal{F})-\{\infty\})$, and by the previous lemma on relativisation of FA to ideal elements, g is an atom of Boolean algebra $\mathcal{J}[\mathcal{F}]$.

Thus, for some $i\in I$, $g=\underline{i}$, whence $\mathcal{G}\cong\mathcal{F}^{(g)}=\mathcal{F}^{(\underline{i})}\cong\mathcal{F}_i$.

QED

Lemma *Lower bound on the special index of direct power of prime RA's*

For each prime RA $\mathcal{R}$, if $I\neq\varnothing$ then $\theta(\mathcal{R}^I)\geq\theta(\mathcal{R})$.

Proof

For each special expansion $\mathcal{F}\in\mathbf{SPC}(\mathcal{R})$ of RA $\mathcal{R}$, the direct power $\mathcal{F}^I$ is a special FA expansion of RA $\mathcal{R}^I$ with single prime factor $\mathcal{F}$.

{ Indeed, since $\mathcal{F}$ is an FA expansion of $\mathcal{R}$, so is $\mathcal{F}^I$ an FA expansion of $\mathcal{R}^I$; also, $\mathcal{F}^I$ is a special FA as it is a non-trivial direct power of special FA $\mathcal{F}$; its single prime factor is $\mathcal{F}$, by the preceding corollary. }

This defines a function $\_^I:\mathbf{SPC}(\mathcal{R})\rightarrow\mathbf{SPC}(\mathcal{R}^I)$ from the set of FA-expansions of RA $\mathcal{R}$ into the set of FA-expansions of the direct power $\mathcal{R}^I$.

Now, prime $\mathcal{G},\mathcal{H}\in\mathbf{SPC}(\mathcal{R})$ are isomorphic FA's iff the direct powers $\mathcal{G}^I,\mathcal{H}^I\in\mathbf{SPC}(\mathcal{R}^I)$ are isomorphic FA's

{ ($\Rightarrow$) If $\mathcal{G}\cong\mathcal{H}$ then clearly $\mathcal{G}^I\cong\mathcal{H}^I$.

($\Leftarrow$) If $\mathcal{G}^I\cong\mathcal{H}^I$ then so are their single prime factors $\mathcal{G}\cong\mathcal{H}$. }

So, we have a well-defined injective function from $\mathbf{SPC}(\mathcal{R})/\cong$ into $\mathbf{SPC}(\mathcal{R}^I)/\cong$.

Hence, $\theta(\mathcal{R})=|\mathbf{SPC}(\mathcal{R})/\cong|\leq|\mathbf{SPC}(\mathcal{R}^I)/\cong|=\theta(\mathcal{R}^I)$.

QED

Lemma *Special index of direct product of prime, special RA's*

Let $\mathcal{R}$ be a prime RA of cardinality $|\mathcal{R}|=\kappa$. Given a set of special prime FA's Q_i of cardinality $|Q_i|=\zeta_i$ with RA reducts $\mathcal{P}_i$, $i\in I$, consider the direct product $\mathcal{R}^*:=(\times_{i\in I}\mathcal{P}_i)\times\mathcal{R}$. If $\kappa\notin\{\zeta_i/i\in I\}$ then $\theta(\mathcal{R}^*)\geq\theta(\mathcal{R})$.

Proof

For each special expansion $\mathcal{F}\in\mathbf{SPC}(\mathcal{R})$ of RA $\mathcal{R}$, consider the direct product $\mathcal{F}^*:=(\times_{i\in I}Q_i)\times\mathcal{F}$. Notice that it has the following properties.

1. The direct product $\mathcal{F}^*=(\times_{i\in I}Q_i)\times\mathcal{F}$ is an FA expansion of RA $\mathcal{R}^*$.

{ Since each Q_i is an FA expansion of $\mathcal{P}_i$, $i\in I$, and $\mathcal{F}$ is an FA expansion of $\mathcal{R}$, so is $\mathcal{F}^*=(\times_{i\in I}Q_i)\times\mathcal{F}$ an FA expansion of $\mathcal{R}^*=(\times_{i\in I}\mathcal{P}_i)\times\mathcal{R}$. }

2. The direct product $\mathcal{F}^*=(\times_{i\in I}Q_i)\times\mathcal{F}$ is a special FA.

{ Since it is a non-trivial direct product of special FA's Q_i, $i\in I$, and $\mathcal{F}$. }

3. $\mathcal{F}^*=(\times_{i\in I}Q_i)\times\mathcal{F}$ has set of prime factors $\{Q_i/i\in I\}\cup\{\mathcal{F}\}$,
(with set of cardinalities $\{\zeta_i/i\in I\}\cup\{\kappa\}$).

{ By the corollary on prime factors of direct product of prime FA's, the prime factors of the direct product $\mathcal{F}^*=(\times_{i\in I}Q_i)\times\mathcal{F}$ are Q_i, $i\in I$, and $\mathcal{F}$. }

This defines a function $\_^*:\mathbf{SPC}(\mathcal{R})\rightarrow\mathbf{SPC}(\mathcal{R}^*)$ from the set of FA-expansions of RA $\mathcal{R}$ into the set of FA-expansions of the direct product $\mathcal{R}^*=(\times_{i\in I}\mathcal{P}_i)\times\mathcal{R}$.

Now, prime $\mathcal{G},\mathcal{H}\in\mathbf{SPC}(\mathcal{R})$ are isomorphic FA's iff the direct products $\mathcal{G}^*=(\times_{i\in I}Q_i)\times\mathcal{G}$ and $\mathcal{H}^*=(\times_{i\in I}Q_i)\times\mathcal{H}$ are isomorphic FA's.

($\Rightarrow$) If $\mathcal{G}\cong\mathcal{H}$ then clearly $\mathcal{G}^*=(\times_{i\in I}Q_i)\times\mathcal{G}\cong(\times_{i\in I}Q_i)\times\mathcal{H}=\mathcal{H}^*$.

($\Leftarrow$) If $\mathcal{G}^*\cong\mathcal{H}^*$ then so are their prime factors $\mathcal{G}\cong\mathcal{H}$.

{ If $\mathcal{G}^*\cong\mathcal{H}^*$, then prime factor $\mathcal{G}$ of $\mathcal{G}^*$ must be isomorphic to some prime factor (in the set $\{Q_i/i\in I\}\cup\{\mathcal{H}\}$) of $\mathcal{H}^*$.

But, $\mathcal{G}^*$ is not isomorphic to any Q_i, $i\in I$, since $|\mathcal{R}|=\kappa\notin\{\gamma_i/i\in I\}$; thus $\mathcal{G}\cong\mathcal{H}$. }

So, we have a well-defined injection from $\mathbf{SPC}(\mathcal{R})/\cong$ into $\mathbf{SPC}(\mathcal{R}^*)/\cong$.

Hence, $\theta(\mathcal{R})=|\mathbf{SPC}(\mathcal{R})/\cong|\leq|\mathbf{SPC}(\mathcal{R}^*)/\cong|=\theta(\mathcal{R}^*)$.

QED

Proposition *Simple PFA connection*: SI2 vs. fxpt(*)

Consider a simple PFA $Q=<Q,\varnothing,U^2,1_U,^{\sim},^{T},\cup,\cap,|,\angle^*>$ with fork $\angle^*$ induced by coding $*:U^2\rightarrow U$. Then $2\cap 1_U=1_{fxpt(*)}$; so $SI2(Q)=\wp[1_{fxpt(*)}]\cap Q$.

Proof

First, we see that $2\cap 1_U=1_{fxpt(*)}$, since $<u,v>\in 2\cap 1_U$ iff $v=u\in fxpt(*)$.

{ $<u,v>\in 2\cap 1_U$ iff $<u,v>\in 1_U$ and $<u,v>\in 1_U\nabla 1_U$ iff $u=v$ and $<u,v>=<u,u*u>$ iff $v=u=u*u$ iff $v=u\in fxpt(*)$. }

Hence, since $Q\subseteq\wp(U^2)$, $r\in SI2(Q)$ iff $r\in Q$ and $r\subseteq 1_{fxpt(*)}$ iff $r\in Q\cap\wp[1_{fxpt(*)}]$.

QED

Lemma *Set of fixpoints of coding and special simple PFA's*

Consider a simple PFA $Q=<Q,\varnothing,U^2,1_U,^{\sim},^{T},\cup,\cap,|,\angle^*>$ with fork $\angle^*$ induced by underlying coding $*:U^2\rightarrow U$ with set of fixpoints $fxpt(*)=\{u\in U/u*u=u\}$.

Then, simple PFA Q is special iff $*:U^2\rightarrow U$ is surjective and $fxpt(*)\neq U$.

Proof

We have $2=1_U\angle^*1_U$, and, by the preceding proposition, $2\cap 1_U=1_{fxpt(*)}$.

So $1_U\subseteq 1_U\angle^*1_U$ iff $1_U=2\cap 1_U$ iff $1_U=1_{fxpt(*)}$ iff $U=fxpt(*)$.

Also, $1_U\angle^*1_U\subseteq 1_U$ iff $2\subseteq 1_U$ iff $fxpt(*)=U$.

{ Because, $2\subseteq 1_U$ iff $<u,u*u>\in 1_U$ whenever $<u,u*u>\in 2$
iff for all $u\in U$: $u=u*u$ iff $fxpt(*)=U$. }

Hence, $1_U\angle^*1_U=1_U$ iff $fxpt(*)=U$, i. e. $1_U\angle^*1_U\neq 1_U$ iff $fxpt(*)\neq U$.

By the remark in 2.3, $U^2\angle^*U^2=U^2$ iff coding $*:U^2\rightarrow U$ is surjective.

QED

Lemma *Infinite simple PRA's and sets of generators*

Given an infinite cardinal $\kappa \geq \aleph$, algebra $\mathcal{P}$ is a simple PRA with cardinality $|P|=\kappa$ over set U iff $\mathcal{P}$ is the subalgebra $\mathcal{P}(U^2)[G]$ of the full PRA $\mathcal{P}(U^2)$ generated by some subset $G \subseteq \wp(U^2)$ with cardinality $|G|=\kappa$.

Proof

The assertion follows from the remark, in section 4, on subalgebras of $\mathcal{P}(U^2)$ generated by infinite subsets.

{ ($\Rightarrow$) If $\mathcal{P}$ is a simple PRA with cardinality $|P|=\kappa$ over set U, then $\mathcal{P}=\mathcal{P}(U^2)[G]$ with $G:=P \subseteq \wp(U^2)$ (so $|G|=|P|=\kappa$).

($\Leftarrow$) If $\mathcal{P}=\mathcal{P}(U^2)[G]$ with $G \subseteq \wp(U^2)$ and $|G|=\kappa$, then $\mathcal{P}$ has cardinality $|P|=|G|.\aleph_0=\kappa.\aleph_0=\kappa$ [Burris & Sankappanavar '81, p. 32]. }

QED

Proposition *Simple PRA's and PFA expansions*

Let $\mathcal{P}$ be a simple PRA over set U. Consider any injective function $*:U^2 \rightarrow U$ inducing fork operation $\angle^*$ and projections p^* and q^*.

a) PRA $\mathcal{P}$ is expandable by the induced fork $\angle^*$ iff induced projections p^* and q^* are in its carrier P.

b) If $\{p^*,q^*\} \subseteq P$ and $fxpt(*) \in 1^P(U)$ {where $1^P(U):=\{S \subseteq U / \wp(1_S) \subseteq P\}$, then PRA $\mathcal{P}$ has fork expansion $\mathcal{P}^*:=(\mathcal{P},\angle^*)$ size-controlled by cardinal $\gamma:=|fxpt(*)|$: $\mathcal{P}^*$ has set of sub-identities of 2 $SI2(\mathcal{P}^*)=\wp[1_{fxpt(*)}]$.

Proof

a) The RA reduct of the full PFA $\mathcal{P}^*(U^2)$ is the full PRA $\mathcal{P}(U^2)$.

So, $\{p^*,q^*\} \subseteq P$ iff $\mathcal{P}$ is expandable by the induced fork $\angle^*$, by the lemma on expandability of subalgebras of reducts in 5.1.

b) Consider such a fork expansion $\mathcal{P}^*:=(\mathcal{P},\angle^*)$ of $\mathcal{P}$. It is a simple PFA.

By the proposition on simple PFA connection between SI2 and fixpoint in 5.3, we have $SI2(\mathcal{P}^*)=\wp[1_{fxpt(*)}] \cap P^*=\wp[1_{fxpt(*)}] \cap P$.

But, since $fxpt(*) \in 1^P(U)$, we have $\wp[1_{fxpt(*)}] \subseteq P$.

Thus $SI2(\mathcal{P}^*)=\wp[1_{fxpt(*)}] \cap P=\wp[1_{fxpt(*)}]$; so $|SI2(\mathcal{P}^*)|=2^\gamma$, since $\gamma=|1_{fxpt(*)}|$.

QED

Proposition *Special coding with controlled set of fixpoints*

Consider an infinite set U of cardinality $\kappa \geq \aleph_0$. For each subset $T \subseteq U$ with $|T|=\kappa$, there exists a bijection $*^S:U^2 \rightarrow U$ with $fxpt(*^S)=S$, where $S:=U-T$.

Proof

First, since infinite set U has cardinality $|U|=\kappa \geq \aleph_0$, the following sets have the same infinite cardinality $\kappa \geq \aleph_0$:

its square $U^2=U \times U$, its identity (diagonal) relation $1_U:=\{<u,v> \in U^2 / u=v\}$ and its complement $1_U{}^{\sim}=\{<u,v> \in U^2 / u \neq v\}$.

{ Indeed, $|U^2|=|U|^2=\kappa.\kappa=\kappa$, $|1_U|=|U|=\kappa$ and $\kappa=|U| \leq |1_U{}^{\sim}| \leq |U|=\kappa$. }

Since $|T|=\kappa=|U|$, we can partition T into disjoint subsets A and B of U, both with cardinality κ. Since $|1_U{}^{\sim}|=\kappa=|A|$, we have a bijection $f:1_U{}^{\sim}\to A$.

Also, since $|T|=\kappa=|B|$, we have a bijection $g:T\to B$ without fixpoints.

{ We can partition B into $\aleph_0$ subsets B_n, $n\in\mathbf{N}$, all with cardinality $\kappa=|B|$. So, we have bijections $g_0:A\to B_0$, and $g_{n+1}:B_n\to B_{n+1}$, $n\in\mathbf{N}$, with pairwise disjoint domains and images. Their disjoint union gives a bijection g from $T=A\cup B$ onto $B=\cup_{n\in N}B_n$ without fixpoints, as required.

(Indeed, for each $t\in T=A\cup(\cup_{n\in N}B_n)$, $g(t)\neq t$, since, for $t\in A$ $g(t)=g_0(t)\in(B_0-A)$, and for $t\in B_n$ $g(t)=g_{n+1}(t)\in(B_{n+1}-B_n)$. }

We now define $*^S:U^2\to U$ as follows:

for $u\in S$ we set $u*^Su:=u$ (notice that $u\notin A\cup B$);

for $u\in T$ we set $u*^Su:=g(u)$ (notice that $g(u)\in B$);

for $<v,w>\in 1_U{}^{\sim}$ we set $v*^Sw:=f(v,w)$ (notice that $f(v,w)\in A$).

Hence, $*^S:U^2\to U$ is a bijection, from $U^2=1_U\cup 1_U{}^{\sim}$ onto $U=S\cup T$, because it is the disjoint union of bijections (1_S, g and f) with pairwise disjoint domains (1_S, 1_T and $1_U{}^{\sim}$) and images (S, B and A).

Also, $u*^Su=u$ iff $u\in S$ (because for $u\notin S$ $u*^Su=g(u)\neq u$). Thus $fxpt(*^S)=S$.

QED

Corollary *Simple PFA and special coding with smaller set of fixpoints*

Consider a simple PFA $\mathcal{Q}=<Q,\varnothing,U^2,1_U,{}^{\sim},{}^{T},\cup,\cap,|,\angle^*>$ over set U where $\angle^*$ is the fork induced by some special coding $*:U^2\to U$ with $|fxpt(*)|<|U|$. Then $\mathcal{Q}$ is a special FA, and hence non-Boolean.

Proof

Underlying coding $*:U^2\to U$ is surjective with $fxpt(*)\neq U$ (since $|fxpt(*)|<|U|$). The lemma on set of fixpoints of coding and special simple PFA's in 5.3 guarantees that simple PFA $\mathcal{Q}$ is special.

QED

Corollary *Many special codings with smaller sets of fixpoints*

Consider an infinite set U. For each subset $S\subseteq U$ with cardinality $|S|<|U|$, there exists a special coding $*^S:U^2\to U$ with $fxpt(*^S)=S$.

Proof

Given subset $S\subseteq U$, consider its complement: $T:=U-S=\{u\in U/u\notin S\}$.

Since $|S|<|U|$, we have $|T|=|U|$, and the previous proposition on special coding with controlled set of fixpoints gives a bijection $*^S:U^2\to U$ with $fxpt(*^S)=U-T=S$.

QED

Proposition *Many large prime, special PFA's*

For each infinite cardinal $\kappa\geq\aleph_0$, there exist at least κ, pairwise non-isomorphic, prime special PFA's.

Proof

Select a set U with cardinality $|U|=\kappa\geq\aleph_0$. For each subset $S\subseteq U$ with cardinality $|S|=\gamma<\kappa$, we exhibit a simple special PFA Q_S that is size-controlled by γ: $|SI2(Q_S)|=2^\gamma$.

For this purpose, we choose, by the corollary on many special codings with smaller sets of fixpoints in 6.1, a special coding $*^S:U^2\to U$ with $\mathrm{fxpt}(*^S)=S$, inducing fork $\angle^S$ and projections p^S and q^S. Set $G_S:=\wp(1_S)\cup\{p^S,q^S\}\cup\wp_\omega(U^2)$ and consider the subalgebra $\mathcal{P}_S:=\mathcal{P}(U^2)[G_S]$ of the full PRA $\mathcal{P}(U^2)$ generated by G_S. Notice the following properties.

1. Set $G_S=\wp(1_S)\cup\{p^S,q^S\}\cup\wp_\omega(U^2)$ has cardinality $|G_S|=\kappa$.
{For $\kappa=|\wp_\omega(U^2)|\leq|G_S|$ and $|G_S|\leq|\wp(1_S)|+|\{p^S,q^S\}|+|\wp_\omega(U^2)|=2^\gamma+2+\kappa=\kappa$ (as $2^\gamma\leq\kappa$.}
2. Algebra $\mathcal{P}_S=\mathcal{P}(U^2)[G_S]$ is a simple PRA with cardinality $|\mathcal{P}_S|=\kappa$, so prime.
{By the lemma on infinite simple PRA's and sets of generators in 5.4, since it is the subalgebra of the full PRA $\mathcal{P}(U^2)$ generated by G_S with $|G_S|=\kappa$.}
3. Simple PRA $\mathcal{P}_S=\mathcal{P}(U^2)[G_S]$ has a fork expansion $Q_S:=(\mathcal{P}_S,\angle^S)$ whose set of sub-identities of 2 has cardinality $|SI2(Q_S)|=2^\gamma$.
{By the proposition on simple PRA's and PFA expansions in 5.4, since $\{p^S,q^S\}\subseteq G_S\subseteq P_S$ and $\wp(1_S)\subseteq G_S\subseteq P_S$ with $\mathrm{fxpt}(*^S)=S$.}
4. Simple fork expansion $Q_S=(\mathcal{P}_S,\angle^S)$ is a special, so non-Boolean, PFA.
{By the corollary on simple PFA and special coding with smaller set of fixpoints in 6.1, since special coding $*^S:U^2\to U$ has $|\mathrm{fxpt}(*^S)|=\gamma<\kappa=|U|$.}

Therefore, there are at least κ, pairwise non-isomorphic, prime special PFA's Q_S with cardinality $|Q_S|=\kappa$, for $|S|<\kappa$.

QED

Theorem *Large prime, non-Boolean, elastic PRA's*

For each infinite cardinal $\kappa\geq\aleph_0$, there exists a prime, non-Boolean, elastic PRA $\mathcal{P}_\kappa$ of cardinality $|\mathcal{P}_\kappa|=\kappa$: PRA $\mathcal{P}_\kappa$ has κ, pairwise non-isomorphic, special FA expansions Q_γ for each smaller cardinal $\gamma<\kappa$.

Proof

Given infinite cardinal $\kappa\geq\aleph_0$, consider set $U:=\kappa$.

For each smaller cardinal $\xi<\kappa$, ξ is a subset $\xi\subseteq U$ of cardinality $|\xi|=\xi$.

In particular, $U:=\kappa$ is a set with cardinality $|U|=|\kappa|=\kappa\geq\aleph_0$.

So, by the corollary on many special codings with smaller set of fixpoints in 6.1, for each smaller cardinal $\gamma<\kappa$, we have a special coding $*^\gamma:U^2\to U$, with $\mathrm{fxpt}(*^\gamma)=\gamma$, inducing fork $\angle^\gamma$ and projections p^γ and q^γ.

(Note that $|1_{\mathrm{fxpt}(*)}|=|1_\gamma|=\gamma$, so $|\wp[1_{\mathrm{fxpt}(*)}]|=2^\gamma\leq\kappa$.)

Set $H:=\wp_\omega(U^2)\cup\bigcup_{\gamma<\kappa}[\{p^\gamma,q^\gamma\}\cup\wp(1_\gamma)]$ and consider the subalgebra $\mathcal{P}_\kappa:=\mathcal{P}(U^2)[H]$ of the full PRA $\mathcal{P}(U^2)$ generated by H. Notice the following properties.

1. Set $H=\wp_\omega(U^2)\cup\bigcup_{\gamma<\kappa}[\{p^\gamma,q^\gamma\}\cup\wp(1_\gamma)]$ has cardinality $|H|=\kappa$.
{For $\kappa=|\wp_\omega(U^2)|\leq|H|$ and $|H|\leq|\wp_\omega(U^2)|+|\bigcup_{\gamma<\kappa}[\{p^\gamma,q^\gamma\}\cup\wp(1_\gamma)]|\leq\kappa+\kappa.(2+\kappa)=\kappa$.}
2. Algebra $\mathcal{P}_\kappa=\mathcal{P}(U^2)[H]$ is a simple PRA with cardinality $\kappa\geq\aleph_0$, so prime.

{By the lemma on infinite simple PRA's and sets of generators in 5.4, since $\mathcal{P}_\kappa$ is the subalgebra of the full PRA $\mathcal{P}(U^2)$ generated by H with $|H|=\kappa$.}

3. For each cardinal $\gamma<\kappa$, simple PRA $\mathcal{P}_\kappa$ has a fork expansion $Q_\gamma:=(\mathcal{P}_\kappa,\angle^\gamma)$ whose set of sub-identities of 2 has cardinality $|SI2(Q_\gamma)|=2^\gamma$.

{By the proposition on simple PRA's and PFA expansions in 5.4, since $\{p^\gamma,q^\gamma\}\subseteq H\subseteq P$ and $fxpt(*^\gamma)=\gamma\in 1^P(U)$ (for $\wp[1_{fxpt(*)}]=\wp(1_\gamma)\subseteq H\subseteq P$).}

4. Simple fork expansion $Q_\gamma=(\mathcal{P}_\kappa,\angle^\gamma)$ is a special, so non-Boolean, PFA.

{By the corollary on simple PFA and special coding with smaller set of fixpoints in 6.1, since special coding $*^\gamma:U^2\to U$ has $|fxpt(*^\gamma)|=\gamma<\kappa=|U|$.}

5. PRA $\mathcal{P}_\kappa$ is elastic, so non-Boolean.

{By the remark in 5.1 and the fact that PRA $\mathcal{P}_\kappa$, of cardinality $|\mathcal{P}_\kappa|=\kappa$, has at least κ, pairwise non-isomorphic, special PFA expansions Q_γ, for $\gamma<\kappa$.}

Therefore, $\mathcal{P}_\kappa$ is a prime, non-Boolean, elastic PRA of cardinality $|\mathcal{P}_\kappa|=\kappa$.

QED

Theorem *Infinitely many large non-Boolean elastic RA's*

For each infinite cardinal $\kappa\geq\aleph_0$, there exist infinitely many pairwise non-isomorphic non-Boolean elastic RA's of cardinality κ.

Proof

By the preceding theorem on large prime, non-Boolean, elastic PRA's, we have a prime, non-Boolean, elastic PRA $\mathcal{P}$ of cardinality $|\mathcal{P}|=\kappa$.
For each $n\in\mathbf{N}$, consider the direct power $\mathcal{P}^{n+1}$ and notice that it has the following properties.

1. The direct power $\mathcal{P}^{n+1}$ has cardinality $|\mathcal{P}^{n+1}|=\kappa$.

{ Indeed, $|\mathcal{P}^{n+1}|=|\mathcal{P}|^{n+1}=\kappa^n.\kappa=\kappa$. }

2. The direct power $\mathcal{P}^{n+1}$ is a non-Boolean RA.

{ Indeed, $\mathcal{P}^{n+1}$ is a direct power of a non-Boolean RA $\mathcal{P}$. }

3. The direct power $\mathcal{P}^{n+1}$ is elastic, by the lower bound on direct powers.

{ Indeed, $\theta(\mathcal{P}^{n+1})\geq\theta(\mathcal{P})=\kappa$, by the lemma on direct powers in 5.2. }

4. The direct power $\mathcal{P}^{n+1}$ has exactly 2^{n+1} ideal elements.

{ Since simple RA $\mathcal{P}$ has 2 ideal elements, see [Veloso '96b; Appendix]. }

Thus, each RA $\mathcal{P}^{n+1}$ is elastic, so non-Boolean, with cardinality $|\mathcal{P}^{n+1}|=\kappa$.
For distinct $k\neq l\in\mathbf{N}$, the direct powers $\mathcal{P}^{k+1}$ and $\mathcal{P}^{l+1}$ are not isomorphic.
{ $\mathcal{P}^{k+1}$ and $\mathcal{P}^{l+1}$ have, respectively, 2^{k+1} and 2^{l+1} ideal elements. }
Hence, there are at least $\aleph_0$ pairwise non-isomorphic elastic RA's $\mathcal{P}^{n+1}$ of cardinality κ, for $n\in\mathbf{N}$.

QED

Theorem *Many large non-Boolean elastic RA's*

For each infinite cardinal $\kappa\geq\aleph_0$, there exist at least κ pairwise non-isomorphic non-Boolean elastic RA's $\mathcal{R}[\gamma]$ of cardinality κ (each one of them with special index $\theta(\mathcal{R}[\gamma])=\kappa$).

Proof

By the theorem on large prime, non-Boolean, elastic PRA's, we have
a prime, non-Boolean, elastic PRA $\mathcal{Q}$ of cardinality $|\mathcal{Q}|=\aleph_0$;
a prime, non-Boolean, elastic PRA $\mathcal{P}$ of cardinality $|\mathcal{P}|=\kappa$.
For each cardinal $\gamma<\kappa$, form the direct product $\mathcal{R}[\gamma]:=\mathcal{Q}^{\gamma}\times\mathcal{P}$
and notice that it has the following properties.

1. The direct product $\mathcal{R}[\gamma]$ has cardinality $|\mathcal{R}[\gamma]|=\kappa$.
{ Indeed, $\kappa=|\mathcal{P}|\leq|\mathcal{R}[\gamma]|$ and $|\mathcal{R}[\gamma]|=|\mathcal{Q}|^{\gamma}.|\mathcal{P}|\leq(\gamma^{\gamma}).\kappa\leq(2^{\gamma\gamma}).\kappa=\kappa$, since $2^{\gamma}\leq\kappa$. }
2. The direct product $\mathcal{R}[\gamma]$ is a non-Boolean RA.
{ Indeed, $\mathcal{R}[\gamma]$ is a direct product of non-Boolean RA's $\mathcal{Q}$ and $\mathcal{P}$. }
3. The direct product $\mathcal{R}[\gamma]$ is elastic, by the lower bound on direct products.
{ Since $|\mathcal{P}|=\kappa\neq\aleph_0=|\mathcal{Q}|$, $\theta(\mathcal{Q}^{\gamma}\times\mathcal{P})\geq\theta(\mathcal{P})=\kappa$, by the lemma in 5.2. }
4. The direct product $\mathcal{R}[\gamma]$ has exactly $2^{\gamma+1}$ ideal elements.
{ It is a direct product of $\gamma+1$ simple RA's, see [Veloso '96b; Appendix]. }

Thus, each RA $\mathcal{R}[\gamma]$ is elastic, so non-Boolean, with cardinality $|\mathcal{R}[\gamma]|=\kappa$.
For distinct $\delta\neq\eta<\kappa$, the direct products $\mathcal{R}[\delta]$ and $\mathcal{R}[\eta]$ are not isomorphic.
{ $\mathcal{R}[\delta]$ and $\mathcal{R}[\eta]$ have, respectively, $2^{\delta+1}$ and $2^{\nu+1}$ ideal elements. }
Therefore, there are at least κ pairwise non-isomorphic elastic RA's
$\mathcal{R}[\gamma]=\mathcal{Q}^{\gamma}\times\mathcal{P}$ of cardinality κ, for $\gamma<\kappa$.

QED

Proposition *Collections of large non-Boolean elastic RA's*

For each infinite cardinal $\kappa>\aleph_0$ such that $\kappa?\leq\kappa$, there exist at least 2^{κ} pairwise non-isomorphic non-Boolean elastic RA's $\mathcal{R}[I]$ with cardinality κ.

Proof

By the theorem on large prime, non-Boolean, elastic PRA's,
for each infinite cardinal $\aleph_0\leq\xi\leq\kappa$, we have:
a prime, non-Boolean, elastic PRA $\mathcal{P}_{\xi}$ of cardinality $|\mathcal{P}_{\xi}|=\xi$.
Consider a set $I\subseteq\kappa-\aleph_0$ of infinite cardinals strictly below κ,
and notice that $|I|\leq|\kappa-\aleph_0|=\kappa$ (since $\kappa>\aleph_0$) and for each $\gamma\in I$, $\gamma<\kappa$ so $2^{\gamma}\leq\kappa$.
Now, form the direct product $\mathcal{R}[I]:=(\times_{\gamma\in I}\mathcal{P}_{\gamma})\times\mathcal{P}_{\kappa}$
and notice that it has the following properties.

1. The direct product $\mathcal{R}[I]$ has cardinality $|\mathcal{R}[I]|=\kappa$.
{ Indeed, $\kappa=|\mathcal{P}_{\kappa}|\leq|\mathcal{R}[I]|$ and $|\mathcal{R}[I]|=|(\times_{\gamma\in I}\mathcal{P}_{\gamma})\times\mathcal{P}_{\kappa}|\leq|\times_{\gamma\in\kappa}\mathcal{P}_{\gamma}|\leq\Pi_{\gamma\in\kappa}2^{\gamma}=\kappa?\leq\kappa$. }
2. The direct product $\mathcal{R}[I]$ is a non-Boolean RA.
{ Indeed, $\mathcal{R}[I]$ is a direct product of non-Boolean RA's $\mathcal{P}_{\gamma}$, for $\gamma\in I$, and $\mathcal{P}_{\kappa}$. }
3. The direct product $\mathcal{R}[I]$ has set of simple factors $\{\mathcal{P}_{\gamma}/\gamma\in I\}\cup\{\mathcal{P}_{\kappa}\}$,
(with set of cardinalities $\{\gamma/\gamma\in I\}\cup\{\kappa\}$).
{ By the remark on prime factors of direct product of prime RA's in 5.2,
the prime factors of the direct product $(\times_{\gamma\in I}\mathcal{P}_{\gamma})\times\mathcal{P}_{\kappa}$ are $\mathcal{P}_{\gamma}$, $\gamma\in I$, and $\mathcal{P}_{\kappa}$. }
4. The direct product $\mathcal{R}[I]$ is elastic, by the lower bound on direct products.

{ Since $|\mathcal{P}_\kappa|=\kappa\notin I=\{|\mathcal{P}_\gamma|/\gamma\in I\}$, $\theta[(\times_{\gamma\in I}\mathcal{P}_\gamma)\times\mathcal{P}_\kappa]\geq\theta(\mathcal{P}_\kappa)=\kappa$, by the lemma in 5.2. }

Thus, each RA $\mathcal{R}[I]$ is elastic, so non-Boolean, with cardinality $|\mathcal{R}[I]|=\kappa$.

For distinct $M\neq N\in\wp(\kappa-\{\kappa,\aleph_0\})$, FA's $\mathcal{R}[M]$ and $\mathcal{R}[N]$ are not isomorphic.

{ For definiteness, say $M\subset N\subseteq\kappa-\{\kappa,\aleph_0\}$ and consider an element $\nu\in(N-M)\neq\varnothing$.

Then, $\mathcal{P}_\nu$ is a prime factor of $\mathcal{R}[N]$ with $|\mathcal{P}_\nu|=\nu$.

The prime factors $\mathcal{P}$ of $\mathcal{R}[M]$ have $|\mathcal{P}|\in M\cup\{\kappa\}$.

Thus, since $\nu\notin M\cup\{\kappa\}$, $\mathcal{P}_\nu$ cannot be a prime factor of $\mathcal{R}[M]$. }

Therefore, there are at least 2^κ pairwise non-isomorphic elastic RA's $\mathcal{R}[I]=(\times_{\gamma\in I}\mathcal{P}_\gamma)\times\mathcal{P}_\kappa$ of cardinality κ, for $I\in\wp(\kappa-\aleph_0)$.

QED

Theorem *Very large collections of large non-Boolean elastic RA's*

For each infinite successor cardinal $\kappa=2^\alpha$ with $\alpha\geq\aleph_0$, there exist at least 2^κ pairwise non-isomorphic non-Boolean elastic RA's $\mathcal{R}[I]$ with cardinality κ.

Proof

Since $\alpha\geq\aleph_0$, we have $\kappa=2^\alpha>\alpha\geq\aleph_0$.

So, by the preceding proposition it suffices to show that $\kappa?\leq\kappa$.

Indeed, for each $\gamma\in\kappa=2^\alpha$, we have $\gamma<2^\alpha$, so $\gamma\leq\alpha$, whence $2^\gamma\leq 2^\alpha=\kappa$.

Thus, $\kappa?=\Pi_{\gamma\in\kappa}2^\gamma\leq\kappa.\kappa=\kappa$.

QED

Corollary *Large collections of large special FA's*

Consider an infinite cardinal $\kappa\geq\aleph_0$.

a) There exist at least κ pairwise non-isomorphic special (simple, proper) FA's with cardinality κ.

b) If κ is a successor cardinal $\kappa=2^\alpha$ with $\alpha\geq\aleph_0$, then there exist at least 2^κ pairwise non-isomorphic special FA's with cardinality κ.

Proof

a) By the proposition on many large prime, special PFA's.

b) For each infinite successor cardinal $\kappa=2^\alpha$ with $\alpha\geq\aleph_0$, the preceding theorem, gives at least 2^κ pairwise non-isomorphic non-Boolean elastic RA's $\mathcal{R}[I]$ with cardinality κ, each one of them has an FA expansion $\mathcal{F}[I]$.

We thus have a collection with at least 2^κ pairwise non-isomorphic special FA's $\mathcal{F}[I]$ with cardinality κ

QED

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