PUC

ISSN 0103-9741

Monografias em Ciência da Computação
nº 23/96

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PONTIFÍCIA UNIVERSIDADE CATÓLICA DO RIO DE JANEIRO
RUA MARQUÊS DE SÃO VICENTE, 225 - CEP 22453-900
RIO DE JANEIRO - BRASIL

PUC RIO - DEPARTAMENTO DE INFORMÁTICA ISSN 0103-9741

Monografias em Ciência da Computação, Nº 23/96

Editor: Carlos J. P. Lucena July, 1996

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* This work has been sponsored by the Ministério de Ciência e Tecnologia da Presidência da República Federativa do Brasil.

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PUCRioInf MCC 21/96

Abstract. A fork algebra is a relational algebra enriched with a new binary operation. This class of algebras was introduced because its equational calculus has applications in program construction. It also has some interesting connections with algebraic logic. In this report we examine the set-theoretical nature of fork algebras, namely to what extent fork algebras of relations are concrete. We investigate whether every fork algebra of relations can be represented as a cartesian one (whose underlying coding is true cartesian-product pair forming). We argue that the answer is affirmative, but with a proviso. The main idea is using the room provided by the neutral element for relational composition to accommodate the cartesian behaviour. We first show that these widened fork algebras of relations are still fork algebras, in that they satisfy the fork equations, and then that they can be represented by proper fork algebra (with real identity). We also establish that the fork algebras of relations can be represented as cartesian ones, and exhibit a simple proper cartesian fork algebra of each given infinite cardinality. We finally show that the enlargement of the identity is essential, by exhibiting large collections of (simple) proper fork algebras of relations that cannot be represented as cartesian ones.

Key words: Fork algebras, relational algebras, representability, fork algebras of relations, proper fork algebras, cartesian coding, cartesian algebras of relations, non-cartesian fork algebras.

Resumo. Uma álgebra de fork é uma álgebra relacional enriquecida com uma nova operação binária. Tal classe de álgebras foi introduzida porque seu cálculo equacional tem aplicações em construção de programas, tendo também interessantes conexões com lógica algébrica. Neste trabalho examina-se a natureza conjuntista das álgebras de fork de relações, em que medida álgebras de fork de relações são concretas. Investigamos se toda álgebra de fork de relações pode ser representada como uma a cartesiana (cuja codificação subjacente é realmente formação de pares cartesianos). Argumentamos que a resposta é afirmativa, mas com uma ressalva. A idéia central se baseia em usar o espaço fornecido pelo elemento neutro da composição relacional para acomodar o comportamento cartesiano. Inicialmente mostra-se que tais versões liberais de álgebras de fork de relações ainda são álgebras de fork, uma vez que satisfazem as equações de fork, e que elas podem ser representadas pelas álgebras de fork próprias (com a real identidade). Também se estabelece que as álgebras de fork de relações podem ser representadas pelas cartesianas, e exibe-se uma álgebra cartesiana de fork simples própria de cada cardinalidade infinita. Finalmente mostra-se que tal liberalização da identidade é essencial, exibindo-se vastas coleções de álgebras de fork (simples) que não representadas pelas cartesianas.

Palavras chave: Álgebras de fork, álgebras relacionais, representabilidade, álgebras de fork de relações, álgebras de fork próprias, codificação cartesiana, álgebras cartesianas de relações, álgebras de fork não cartesianas.

NOTE AND ACKNOWLEDGEMENTS

A preliminary version of the research reported herein was orally presented at an informal workshop on Relation Algebras and Applications to Computing, held at PUC-Rio in August 1994. Discussions with participants in this workshop were instrumental in sharpening many ideas. In particular, Chris Brink, Roger Maddux, and Don Pigozzi are to be thanked.

Rather than delaying the appearance of the written version of this report, it was thought advisable to present the ideas basically in their original form, taking care to correct only the most obvious mistakes. This may serve as an explanation, and perhaps excuse, for the relatively few references to more recent literature.

CONTENTS

1. INTRODUCTION

In this report we examine the set-theoretical nature of fork algebras, namely to what extent fork algebras of relations are really concrete.

A fork algebra (FA, for short) is a relational algebra (RA, for short) enriched with a new binary operation, called fork. This class of algebras was introduced because its equational calculus has applications in program construction. It also has some interesting connections with algebraic logic.

Algebras of relations involve relations on a set (of points), whereas fork algebras of relations (FAR's, for short) deal with relations on a set of objects (trees) structured by an underlying pair-packing coding. The original intuition behind this structuring operation is the true cartesian pair forming. Subsequently this idea was widened to an injective coding, which is enough for many purposes. But this widening was criticised on the grounds that such FAR's have a hidden underlying coding, and thus fail to be concrete (see, e. g. [Sain '94]). Indeed, while the operations of a (simple, proper) algebra of relations are natural set-theoretical operations, which are determined once the underlying set is given, this does not occur even with the simple, proper fork algebra of relations (due to the arbitrary nature of the hidden underlying coding).

So, we examine the question: can every FAR be represented as a cartesian FAR (one whose underlying pair-packing coding is true cartesian-product pair forming)? We will argue that the answer is yes, but with a proviso.

Both the positive answer and the constraining proviso come from a simple, though crucial, idea of using the room provided by the neutral element for relational composition to accommodate the required behaviour.

We show that every FA can indeed be represented as a cartesian FAR by making use of the room provided by the neutral element I for relational composition |. The drawback of this representation is the choice for I, which renders them not so concrete. The constraining proviso is that if one insists in taking I as the concrete identity (diagonal) relation, then such representation is not always possible: we show that there are many non-trivial FAR's that cannot be represented as proper cartesian FAR's.

As mentioned, the main device for getting cartesian representability amounts to considering this wider class of FAR's, where I is not required to be the identity on the underlying set. We begin in section 2 with some preliminaries on algebras of relations. Then we examine in section 3 how a fork operation on relations is induced by a coding on its underlying set.

In section 4 we introduce this class of FAR's, which is wider than the class of proper fork algebras (those where $I=1_U$). We also argue, along the lines of [Frias et al. '96], that every PFA (short for proper FAR) is a fork algebra in that it satisfies the three fork equations. We then establish that the subclass of proper FAR's is sufficiently wide to represent all FAR's. So, even though the relaxed FAR's are less restricted than the PFA's, they are still AFA's.

In section 5 we examine the class of cartesian fork algebras of relations, whose underlying coding is true pair forming. By showing that every coding algebra is a homomorphic image of some cartesian coding algebra, we establish that the subclass of the CAR's (short for cartesian fork algebras of relations) is sufficiently wide to represent all FAR's. We also exhibit a simple PCA of each given infinite cardinality.

Sections 4 and 5 together establish that two distinct subclasses of FAR's - the PFA's and the CAR's - are sufficiently wide to represent all FAR's.

The question of representability by proper CAR's is taken up in section 6. We first introduce the subclass of weird algebra of relations (WAR's, for short) and show they cannot be represented as PCA's. We then establish that this subclass is populated by exhibiting first some admittedly uninteresting (essentially Boolean algebras) PWA's (short for proper WAR's) of each nonzero cardinality. With the aim of showing some more interesting WAR's, we introduce the merge algebras of relations, whose underlying coding is a merge-like operation on infinite sequences, and observed that most MAR's are non-Boolean WAR's. With these results we can exhibit, for each infinite cardinality, a simple proper MAR as well as large collections of pairwise non-isomorphic PWA's. Thus, at each infinite cardinality, the class of PCA's, though populated, fails to represent many PFA's.

These results establish the positive answer as well as the constraining proviso concerning the question of presentation of FAR's in set-theoretical cartesian terms. Indeed, every FAR can be represented as a cartesian FAR (one whose underlying pair-packing coding is true cartesian-product pair forming), and in this sense one may say that the nature of FAR's is set theoretical. But, this representation is achieved at the price of enlarging the true identity to an equivalence relation that is a neutral element for relational composition, in this choice residing the only source of non-concrete nature. This widening of the identity is unavoidable if one wishes to encompass all the (proper) FAR's: at each infinite cardinality, the class of proper cartesian FAR's, though populated, fails to represent many PFA's.

2. PRELIMINARIES: ALGEBRAS OF RELATIONS

We begin by recalling the concept of algebra of relations (or AR, for short) [Jónsson & Tarski '52; Veloso '74].

An *algebra of relations* (on set U) is an algebra $\mathcal{Q}=\langle Q,\cup,\cap,\varnothing,V,\tilde{},|,^{T},I\rangle$, where

- its reduct $\langle Q,\cup,\cap,\varnothing,V\rangle$ is a field of subsets of $V\subseteq U^2$ (with $U^2:=U\times U$);
- operation $^{T}:Q\to Q$ is relation transposition, i. e. $r^{T}=\{\langle v,u\rangle\in U^2/\langle u,v\rangle\in r\}$;
- operation $|:Q\times Q\to Q$ is relation composition, i. e. $r|s=\{\langle u,w\rangle\in U^2/\exists v\in U\,[\langle u,v\rangle\in r\,\&\,\langle v,w\rangle\in s]\}$;
- $I\in R$ is a neutral element for operation $|$: $r|I=r=I|r$.

We shall call an AR $\mathcal{P}=\langle P,\cup,\cap,\varnothing,V,\tilde{},|,^{T},I\rangle$ on set U *proper* (a PAR, for short) iff I is the identity (diagonal) relation on U: $I=1_U:=\{\langle u,v\rangle\in U^2/u=v\}$.

It is well known that every AR $\mathcal{Q}$ is isomorphic to a proper AR $\mathcal{P}$. We recall [Jónsson & Tarski '52, Theorem 4.24, p. 140; Veloso '74, p. 7] that, in an AR $\mathcal{Q}$ on set U, both elements I and V are equivalence relations on U. Also, the quotient projection $n:U \rightarrow \underline{U}$ with $\underline{U}:=U/I$ induces an image mapping $N:\wp(U^2) \rightarrow \wp(\underline{U}^2)$, defined by $N(r):=\{<n(x),n(y)> \in \underline{U}^2 / <x,y> \in r\}$. Thus, every AR $\mathcal{Q}=<Q,\cup,\cap,\emptyset,V,^{\sim},|,^{T},I>$ (on set U) is isomorphic to a proper AR $\mathcal{P}:=N(\mathcal{Q})=<N(Q),\cup,\cap,\emptyset,N(V),^{\sim},|,^{T},1_{\underline{U}}>$ (on set $\underline{U}=U/I$) [Jónsson & Tarski '52, Theorem 4.27, p. 142; Veloso '74, p. 7].

Given $V \subseteq U^2$, the *powerset PAR of* V is the PAR $\mathcal{P}(V):=<\wp(V),\cup,\cap,\emptyset,V,^{\sim},|,^{T},1_U>$. The *full PAR on* set U is the PAR $\mathcal{P}(U^2):=<\wp(U^2),\cup,\cap,\emptyset,V,^{\sim},|,^{T},1_U>$.

Recall that an AR $\mathcal{Q}=<Q,\cup,\cap,\emptyset,V,^{\sim},|,^{T},I>$ on U with $V=U^2$ is simple [Jónsson & Tarski '52, Theorem 4.28, p. 142; Veloso '74, p. 7, 12]. So, the full PAR $\mathcal{P}(U^2)$ and its subalgebras are simple.

We shall also have occasion to use the following construction.

Lemma *Change of underlying set of AR's*

Given a surjective function $h:W \rightarrow U$, every AR $\mathcal{Q}=<Q,\cup,\cap,\emptyset,V,^{\sim},|,^{T},I>$ on U is isomorphic to an AR $H(\mathcal{Q})$ on W.

Proof outline

Function $h:W \rightarrow U$ induces a preimage mapping $H:\wp(U^2) \rightarrow \wp(W^2)$, by $H(r):=\{<w',w''> \in W^2 / <h(w'),h(w'')> \in r\}$, which preserves Boolean structure.

If $h:W \rightarrow U$ is surjective, then $H:\wp(U^2) \rightarrow \wp(W^2)$ is injective and preserves Peircean structure.

QED

3. Pair Coding and Induced Fork

Algebras of relations involve relations on a set (of points), whereas fork algebras of relations deal with relations involving structured objects. Such structured objects present a behaviour akin to that of pairs. We first examine how a fork operation on relations is induced by a coding on its underlying set.

If set U happens to be closed under cartesian product $(U \times U \subseteq U)$, then we already have in U ordered pairs (v,w) of elements $v,w \in U$. True pair forming gives an insertion of $U^2=U \times U$ into U, assigning $(v,w) \in U$ to $<v,w> \in U^2$.

In general, we may resort to a function $*:U^2 \rightarrow U$. Such a function $*:U^2 \rightarrow U$ induces a binary operation $_*\angle$ on relations on U. This operation, called *fork induced by* $*:U^2 \rightarrow U$, is defined by

$$r\,_*\angle\, s := \{<u,v> \in U^2 / \exists v',v'' \in U\ [v' * v'' = v \,\&\, <u,v'> \in r \,\&\, <u,v''> \in s]\}.$$

So, $r\,_*\angle\, s = \{<u,v*w> \in U^2 / <u,v> \in r \,\&\, <u,w> \in s\}$.

A simple property of the induced fork is its monotonicity with respect to inclusion: if $r \subseteq p$ and $s \subseteq q$ then $r\,_*\angle\, s \subseteq p\,_*\angle\, q$.

Now, given a (universal) relation $V \subseteq U^2$, we wish to have the fork of any two relations included in V. In view of monotonicity, it suffices to guarantee that $V_* \angle V \subseteq V$. We say that $*: U^2 \to U$ *packs* relation $V \subseteq U^2$ iff $<v, v*w> \in V$ whenever $<v,w> \in V$. { For a transitive relation $V \subseteq U^2$, packing V is a necessary and sufficient condition for $V_* \angle V \subseteq V$. }

Notice that, if $*: U^2 \to U$ is injective, one can recover v and w from v*w. But, one can also relax this requirement to injectivity over universal $V \subseteq U^2$. We shall say that $*: U^2 \to U$ is a *coding* for relation $V \subseteq U^2$ iff $*: U^2 \to U$ packs relation V and the restriction ${}_{V|}*: V \to U$ of $*: U^2 \to U$ to $V \subseteq U^2$ is injective.

For the case where I is not necessarily the identity, it is natural to consider relaxed versions of these ideas.

A function $\circ: U^2 \to U$ induces another binary operation ${}_\circ\angle^I$ on relations on U: the *(relaxed) fork induced by* $\circ: U^2 \to U$ *under* I as follows

$r{}_\circ\angle^I s := \{<u,v> \in U^2 / \exists v',v'' \in U\ [<u,v'> \in r \& <u,v''> \in s \& <v' \circ v'', v> \in I]\}$.

Note that $r{}_\circ\angle^I s = (r{}_\circ\angle s)|I$. So, ${}_\circ\angle^I$ is still monotonic with respect to inclusion.

By a *coding algebra* on set U we mean an algebra of the form $<U, \circ>$ with $\circ: U^2 \to U$. We shall say that coding algebra $<U, \circ>$ is a *pair-packing for* relation $V \subseteq U^2$ *under* relation $I \subseteq V$ iff $<v,z> \in V$ whenever $<v,w> \in V$ and $<v \circ w, z> \in I$ and whenever $<u',v'>, <u'',v''> \in V$: $<u' \circ v', u'' \circ v''> \in I$ iff $<u',u''> \in I$ and $<v',v''> \in I$.

Notice that these concepts reduce to the previous ones in the case $I = 1_U$.

4. Fork Algebras of Relations

Algebras of relations involve relations on a set (of points), whereas fork algebras of relations (FAR's, for short) deal with relations on a set of objects (trees) structured by an underlying pair-packing coding.

Consider an AR $\mathcal{Q} = <Q, \cup, \cap, \emptyset, V, \sim, |, ^T, I>$ on set U. A function $\circ: U^2 \to U$ induces a relaxed fork operation ${}_\circ\angle^I$ on $\wp(U^2)$. If Q is closed under the relaxed fork operation ${}_\circ\angle^I$ we can expand AR $\mathcal{Q}$ by ${}_\circ\angle^I$.

A *fork algebra of relations* (on set U) is an expansion $(\mathcal{Q}, {}_\circ\angle^I)$ of an AR $\mathcal{Q} = <Q, \cup, \cap, \emptyset, V, \sim, |, ^T, I>$ on U by the binary operation relaxed fork ${}_\circ\angle^I$ induced by some underlying pair-packing for relation $V \subseteq U^2$ under relation $I \subseteq V$.

In other words, an FAR (on set U) is an algebra $\mathcal{G} = <G, \cup, \cap, \emptyset, V, \sim, |, ^T, I, \angle>$, where

- its reduct $\mathcal{G}_\angle = <G, \cup, \cap, \emptyset, V, \sim, |, ^T, I>$ is an AR on set U;
- for some underlying pair-packing $<U, \circ>$ for $V \subseteq U^2$ under $I \subseteq V$: operation $\angle: G \times G \to G$ is relaxed fork induced by $\circ$ under I, i. e. $r \angle s = r{}_\circ\angle^I s = (r{}_\circ\angle s)|I = \{<u,w> \in U^2 / \exists v \in U\ [<u,v> \in r{}_\circ\angle s \& <v,w> \in I]\}$.

It is not difficult to see that if the reduct $\mathcal{G}_\angle$ is simple then so is $\mathcal{G}$.

We call FAR $\mathcal{H}$ *proper* (a PFA, for short) iff its reduct $\mathcal{H}_\angle$ is a proper AR.

We shall use **FAR** for the class of FAR's and **PFA** for the class of PFA's.

For PFA's the relaxed fork reduces to the induced fork. It is not difficult to see that in a PFA $\mathcal{H}=<H,\cup,\cap,\varnothing,V,^{\sim},|,^T,I,_*\angle>$ the induced fork $_*\angle$ depends only on the restriction $_{V|}*:V\rightarrow U$ of $*:U^2\rightarrow U$ to the universal relation V.

Notice that the restriction $_{V|}*:V\rightarrow U$ of $*:U^2\rightarrow U$ to $V\subseteq U^2$ is injective iff $<U,*>$ is a pair-packing for V under the identity 1_U. Thus, each injective $*:V\rightarrow U$ gives rise to a *powerset PFA* $\mathcal{P}^*(V)$ *of* V as the $_*\angle$-expansion $(\mathcal{P}(V),_*\angle)$ of the powerset PAR $\mathcal{P}(V)=<\wp(V),\cup,\cap,\varnothing,V,^{\sim},|,^T,1_U>$ of V. In particular, for injective $*:U^2\rightarrow U$, we have a *full PFA* $\mathcal{P}^*(U^2):=(\mathcal{P}(U^2),_*\angle)$. The full PFA's and their subalgebras are simple.

The abstract versions of the algebras of relations are the relational algebras: algebras $\mathcal{R}=<R,+,\bullet,0,\infty,^-,;,^\dagger,1>$ satisfying familiar equations [Jónsson & Tarski '52, Definition 4.1, p. 127, 128; Veloso '74, p. 7, 8]. Similarly, the abstract versions of the FAR's form the class **AFA** of (abstract) fork algebras, which are expansions of relational algebras by a fork operation satisfying three equations. A *fork algebra* (AFA, for short) is an expansion $\mathcal{R}^\nabla=(\mathcal{R},\nabla)$ of a relational algebra $\mathcal{R}$ by a binary operation $\nabla:R\times R\rightarrow R$ satisfying the following three fork equations: $(r\nabla s);(p\nabla q)^\dagger=(r;p^\dagger)\bullet(s;q^\dagger)$, $r\nabla s=(r;\pi^\dagger)\bullet(s;\rho^\dagger)$ and $\pi\nabla\rho\leq 1$ (with $\pi:=(1\nabla\infty)^\dagger$ and $\rho:=(\infty\nabla 1)^\dagger$).

It is not difficult to see that these three fork equations are satisfied in every proper FA [Frias et al. '96].

Proposition *Properties of PFA's*: **PFA**$\subseteq$**AFA**

Given PFA $\mathcal{H}=<H,\cup,\cap,\varnothing,V,^{\sim},^T,|,1_U,\angle>$ define $\pi:=(1_U\angle V)^T$ and $\rho:=(V\angle 1_U)^T$.

a) $(r\angle s)|(p\angle q)^T=(r|p^T)\cap(s|q^T)$.

b) $\pi\angle\rho\subseteq 1_U$.

c) $r\angle s=(r|\pi^T)\cap(s|\rho^T)$.

Proof outline

These equations follow from properties of fork induced by coding.

{ First, inclusions $(r|p^T)\cap(s|q^T)\subseteq(r\angle s)|(p\angle q)^T$ and $r\angle s\subseteq(r|\pi^T)\cap(s|\rho^T)$ are easy. Now,$(r\angle s)|(p\angle q)^T\subseteq(r|p^T)\cap(s|q^T)$ and $\pi\angle\rho\subseteq 1_U$ use the injectivity of $_{V|}*:V\rightarrow U$. Finally, inclusion $(r|\pi^T)\cap(s|\rho^T)\subseteq r\angle s$ follows form (a) and (b). }

QED

The next result shows that the subclass of proper FAR's is sufficiently wide to represent all FAR's.

Theorem *Representation of FAR's as PFA's*: **FAR**$\subseteq$**I[PFA]**

Every FAR $\mathcal{G}$ is isomorphic to some proper PFA $\mathcal{H}$.

Proof outline

Consider FAR $\mathcal{G}$ with underlying pair-packing coding $\circ:U^2\rightarrow U$.

We have a surjective mapping $n:U\rightarrow \underline{U}$ with $\underline{U}:=U/I$, inducing a relational isomorphism $N:\mathcal{G}_\angle\rightarrow\mathcal{P}$ onto PAR $\mathcal{P}$ with universal relation $\underline{V}\subseteq\underline{U}^2$.

We can define a coding $*:\underline{U}^2\rightarrow\underline{U}$ so that $n(u)*n(v):=n(u\circ v)$ for $<n(u),n(v)>\in\underline{V}$.
Then $*:\underline{U}^2\rightarrow\underline{U}$ is well-defined pair-packing for $\underline{V}$ under $1_{\underline{U}}$.
Also, N preserves forks: $N(r\angle s)=N(r)_*\angle N(s)$.

Hence, N(G) is closed under $_*\angle$; so we can expand PAR $\mathcal{P}$ to PFA $\mathcal{H}=(\mathcal{P},_*\angle)$, and $N:G\rightarrow N(G)$ is an FA-isomorphism between FAR $\mathcal{G}=(\mathcal{G}_\angle,\angle)$ and PFA $\mathcal{H}=(\mathcal{P},_*\angle)$.

QED

As a consequence of this representation, we see that FAR's are AFA's: widening the diagonal relation 1_U to a larger neutral element I for | does not preclude satisfaction of the three fork equations.

Corollary *Properties of FAR's*: **FAR⊆AFA**

Every FAR $\mathcal{G}=<G,\cup,\cap,\varnothing,V,\tilde{},|,^T,I,\angle>$ satisfies the three fork equations:

a) $(r\angle s)|(p\angle q)^T=(r|p^T)\cap(s|q^T)$,

b) $\pi\angle\rho\subseteq I$,

c) $r\angle s=(r|\pi^T)\cap(s|\rho^T)$,

with the defined 'projections' $\pi:=(I\angle V)^T$ and $\rho:=(V\angle I)^T$.

We thus have the following inclusions **PFA⊆FAR⊆AFA** as well as the representation **FAR⊆I[PFA]⊆AFA**, whence **I[PFA]=I[FAR]⊆AFA**.

We should also mention the Representability Theorem [Frias et al. '95,'96].

Theorem *Representation of AFA's as PFA's*: **AFA⊆I[PFA]**

Every AFA $\mathcal{F}$ is isomorphic to some proper PFA $\mathcal{H}$.

Summing up, we have the following inclusions **PFA⊆FAR⊆AFA** as well as the representations **I[PFA]=AFA=I[FAR]**.

5. Cartesian Fork Algebras (of Relations)

We now examine cartesian fork algebras of relations, where the underlying pair-packing coding is true pair forming.
We shall show that every FAR is isomorphic to some cartesian FAR.

Consider a set U closed under cartesian product ($U\times U\subseteq U$). Then, true pair formation gives an insertion $(,):U^2\rightarrow U$, assigning $(v,w)\in U$ to $<v,w>\in U^2$. Since $(,):U^2\rightarrow U$ is injective, it is a coding for any $V\subseteq U^2$.

The fork induced by $(,):U^2\rightarrow U$ on relations on U is

$r\ {}^{(,)}\!\angle\ s = \{<u,(v,w)>\in U^2/<u,v>\in r\&<u,w>\in s\}$.

We call FAR $\mathcal{C}=<C,\cup,\cap,\varnothing,V,\tilde{},|,^T,I,\angle>$ on set U *cartesian* (a CAR, for short) iff

- U is closed under cartesian product: $U\times U\subseteq U$;
- insertion $(,):U^2\rightarrow U$ is a coding for $V\subseteq U^2$ under $I\subseteq V$;
- operation $\angle:C\times C\rightarrow C$ is the relaxed fork induced by $(,):U^2\rightarrow U$ under I, i. e.
  $r\angle s = r_{(,)}\angle^I s = \{<u,z>\in U^2/<u,v>\in r\&<u,w>\in s\&<(v,w),z>\in I\}$.

We shall use **CAR** for the class of cartesian FAR's. Clearly **CAR**$\subseteq$**FAR**.

By the *cartesian closure* of a set U we mean the union $U^{\times}:=\cup_{n\in N}U_n$ where $U_0:=U$ and $U_{n+1}:=U_n\cup(U_n\times U_n)$. Clearly $U^{\times}\times U^{\times}\subseteq U^{\times}$, so $U^{\times}$ is closed under cartesian product. Also, for an infinite set U with cardinality $|U|=\kappa\geq\aleph_0$, its cartesian closure has cardinality $\kappa\leq|U^{\times}|\leq\aleph_0.\kappa^2=\kappa$.

The next result shows that the cartesian coding algebras generate by homomorphic images all coding algebras.

Proposition *Cartesian coding*

Every coding algebra $<U,*>$ is a homomorphic image of some cartesian coding algebra.

Proof outline

Consider the term algebra $<T,\circ>$ freely generated by the elements of U.
Evaluation is a homomorphism from $<T,\circ>$ onto $<U,*>$.

Now, consider the closure $U^{\times}$ of U under cartesian product.
Since we have a single operation, $<U^{\times},(,)>$ is a homomorphic image of $<T,\circ>$.

We thus have a surjective homomorphism k from $<U^{\times},<(,)>$ onto $<U,*>$.

QED

We can now show that the subclass of cartesian FAR's is sufficiently wide to represent all FAR's. We will then have the inclusion **CAR**$\subseteq$**FAR** together with the representation **FAR**$\subseteq$I**[CAR]**, whence I**[CAR]**=I**[FAR]**=I**[PFA]**.

Theorem *Representation of FAR's as CAR's*: **FAR**$\subseteq$I**[CAR]**

Every FAR $\mathcal{G}$ is isomorphic to some cartesian algebra of relations $\mathcal{C}$.

Proof outline

Consider FAR $\mathcal{G}$, which may be assumed proper, with coding $*:U^2\to U$.

By the preceding proposition, we have a surjective homomorphism k from cartesian coding algebra $<W,(,)>$ onto $<U,*>$; call $\underline{I}:=\ker(k)$ its kernel.

Now, surjective $k:W\to U$ induces, by preimage, injective $k:\wp(U^2)\to\wp(W^2)$.
This preimage mapping gives a relational isomorphism $K:G\to Q$ from proper AR $\mathcal{G}_{\angle}=<G,\cup,\cap,\emptyset,V,^{\sim},|,^{T},1_U>$ onto AR $\mathcal{Q}=<K(G),\cup,\cap,\emptyset,\underline{V},^{\sim},|,^{T},\underline{I}>$ with $\underline{V}:=K(V)\subseteq W^2$.

Also, $<W,(,)>$ is a pair-packing for relation $\underline{V}=K(V)$ under relation $\underline{I}\subseteq\underline{V}$.

Since k is an epimorphism of coding algebras, K preserves induced forks.

{ For $r,s\subseteq V$, we have $K(r)_{(,)}\angle K(s)\subseteq K(r\angle s)$ and $K(r\angle s)\subseteq[K(r)_{(,)}\angle K(s)]|\underline{I}$;
whence $K(r)_{(,)}\angle^{\underline{I}}K(s)=[K(r)_{(,)}\angle K(s)]|\underline{I}=K(r\angle s)|\underline{I}$.
Now, for $r,s\in G$, $K(r\angle s)\in K(G)$ and so $K(r)_{(,)}\angle^{\underline{I}}K(s)=K(r\angle s)|\underline{I}=K(r\angle s)$. }

Thus, K(G) is closed under ${}_{(,)}\angle^{\underline{I}}$; so we can expand AR $\mathcal{Q}$ to CAR $(\mathcal{Q},{}_{(,)}\angle^{\underline{I}})$, and $K:G\to K(G)$ is an FA-isomorphism form FAR $\mathcal{G}=(\mathcal{G}_{\angle},\angle)$ onto CAR $\mathcal{C}=(\mathcal{Q},{}_{(,)}\angle^{\underline{I}})$.

QED

Notice that for a proper FAR $\mathcal{H}$, the above cartesian FAR $\mathcal{C}=K(\mathcal{H})$ has I=ker(k) and cannot be guaranteed to be proper. So, natural question concerns the representability of FAR's as proper CAR's.

We first examine the proper cartesian algebra of relations.

A proper cartesian algebra of relations (a PCA, for short) on set U is a CAR $\mathcal{D}=<D,\cup,\cap,\varnothing,V,\bar{~},|,^T,I,\angle>$ with $I=1_U$, i. e. an FAR $\mathcal{D}=<D,\cup,\cap,\varnothing,V,\bar{~},|,^T,1_U,_{(,)}\angle>$. We shall use **PCA** for the class **PFA**$\cap$**CAR** of proper cartesian FAR's

We now exhibit a simple PCA of each given infinite cardinality.

Proposition *Large simple proper cartesian algebras of relations*
For each infinite cardinal $\kappa\geq\aleph_0$ there exists a simple proper cartesian algebra of relations $\mathcal{D}$ with cardinality $|\mathcal{D}|=\kappa$.

Proof outline

Select a set U with $|U|=\kappa\geq\aleph_0$, and consider its cartesian closure $W:=U^\times$.

Since set $W=U^\times$ is closed under cartesian product, we can expand the full PAR $\mathcal{P}(W^2)$ to $\mathcal{P}^{(,)}(W^2)=(\mathcal{P}(W^2),_{(,)}\angle)$, which is a simple, proper CAR.

Then, the subalgebra $\mathcal{D}$ of the full PFA $\mathcal{P}^{(,)}(W^2)$ generated by $\wp_\omega(W^2)$ is a simple PCA, on set $U^\times$, with cardinality $|\mathcal{D}|=|\wp_\omega(W^2)|=|W^2|=\kappa$.

QED

6. Non-Cartesian Coding

As mentioned, every FAR can be represented as a cartesian FAR, which is in general not proper. We now examine the next natural question: the representability of FAR's as proper cartesian FAR's. We shall exhibit large classes of FAR's that cannot be isomorphic to proper cartesian FAR's.

We begin by noticing that the usual idea of ordered pair leads to $u\neq(u,u)$.
[If $u=(u,u)$ then elements $(u,u)\in U\times U$, $(u,(u,u))\in U\times(U\times U)$, and so forth, would be equal, even though one would like to view them as ordered pairs, triples, etc. In the usual set-theoretical constructions of ordered pairs as sets, say $(v,w)=\{\{v\},\{v,w\}\}$, $u=(u,u)$ would lead to the cycle $u=(u,u)\in\{u\}\in u$, and thus to an infinite descending $\in$-chain $\ldots(u,(u,u))\in\{(u,u)\}\in(u,u)\in\{u\}\in u$.]

Thus, in proper cartesian FAR C on set U one must have the equation $(1_U\angle 1_U)\cap 1_U=\varnothing$.

Now, by a *weird algebra of relations* (WAR, for short) we mean an FAR $\mathcal{W}=<W,\cup,\cap,\varnothing,V,\bar{~},|,^T,I,\angle>$ where $(I\angle I)\cap I\neq\varnothing$. We use PWA as short for proper WAR, and denote their respective classes by **PWA** and **WAR**.

We thus immediately have the next non-representability result: the class **PCA**=**PFA**$\cap$**CAR** of proper cartesian FAR's cannot represent any WAR.

Proposition *Non-cartesian coding*: **WAR**$\cap$**I[PCA]**$=\varnothing$

A WAR cannot be isomorphic to a proper cartesian FAR.

It remains to exhibit such WAR's. Some Boolean FAR's (with $\angle=\cap$) provide somewhat uninteresting examples of WAR's.

Lemma *Many proper weird algebras of identities*
For each nonempty set $U\neq\varnothing$, there exists a proper WAR, on set U, with cardinality $|U|$.

Proof outline

Consider the powerset PAR of 1_U: the PAR $\mathcal{P}(1_U)=<\wp(1_U),\cup,\cap,\emptyset,1_U,^{\sim},|,^{T},1_U>$.
We have the diagonal bijection $d:1_U\to U$ with $d(u,u)=u$, and d induces diagonal fork $_d\angle$ on $\wp(1_U)$ which behaves as intersection: $r\ _d\angle\ s = r\cap s$.
Since $\wp(1_U)$ is closed under diagonal fork $_d\angle$, we can expand PAR $\mathcal{P}(1_U)$ to $\mathcal{P}^d(1_U)=(\mathcal{P}(1_U),_d\angle)$. Clearly $(1_U\ _d\angle\ 1_U)\cap 1_U = 1_U \neq \emptyset$.
Then, the subalgebra I of the powerset PFA $\mathcal{P}^d(1_U)$ generated by $\wp_\omega(1_U)$ is a WAR, on set U, with cardinality $|I|=|\wp_\omega(1_U)|=|U|$.

QED

The WAR's of identities exhibited in this lemma are essentially Boolean algebras. It would be of interest to have non-Boolean WAR's. For this purpose, we first define merge coding.

Given a set A, consider the set $U:=A^\omega$ of all ω-sequences $<a_0,a_1,\ldots,a_n,\ldots>$ of elements of A. We now define the merge operation $\int:S^2\to S$ as follows

$<a_0,a_1,\ldots,a_n,\ldots>\int<b_0,b_1,\ldots,a_n,\ldots>:=<a_0,b_0,a_1,b_1,\ldots,a_n,b_n,\ldots>$

Notice that one can recover both arguments a and b from the result $a\int b$ (since, for each $n\in\mathbf{N}$: $a_n=(a\int b)_{2n}$ and $b_n=(a\int b)_{2n+1}$). Thus $\int:U^2\to U$ is injective.

Moreover, for a constant sequence $a^\omega=<a,a,\ldots,a,\ldots>\in A^\omega$, we have $a^\omega\int a^\omega=a^\omega$.

We also have similar merge operations on A^I, for every infinite set I.

The fork induced by $\int:U^2\to U$ on relations on $U=A^\omega$ is

$r\ _\int\angle\ s = \{<u,v\int w>\in S^2/<u,v>\in r\&<u,w>\in s\}$.

Now, by a *merge algebra of relations* <u>over</u> set A (MAR, for short) we mean a FAR $\mathcal{M}=<M,\cup,\cap,\emptyset,V,^{\sim},|,^{T},I,\angle>$ <u>on</u> set $U:=A^\omega$ such that

- operation $\angle:M\times M\to M$ is relaxed fork induced by the merge operation $\int:S^2\to S$ under I (i. e. $r\angle s = r\ _\int\angle^I\ s = (r\ _\int\angle\ s)\ |\ I$).

We then clearly have most MAR's as examples of non-Boolean WAR's.

Lemma *Merge algebras of relations as (non Boolean) WAR's*:
Consider a merge algebra of relations $\mathcal{M}$ over set A (so, on set $U:=A^\omega$).
a) If set A is nonempty ($A\neq\emptyset$) then $\mathcal{M}$ is a WAR.
b) If $I^{\sim}\neq\emptyset$, then $\mathcal{M}$ is a non-Boolean WAR.

Proof outline

a) We have some constant sequence $a^\omega=<a,a,\ldots,a,\ldots>\in A^\omega$.
So $<a^\omega,a^\omega>=<a^\omega,a^\omega\int a^\omega>\in(1_U\,_\int\angle\,1_U)\cap 1_U\subseteq(I_\int\angle I)\cap I$. Thus $(I_\int\angle^I I)\cap I\neq\emptyset$.

b) Since $\alpha\neq\beta$, $|A|>1$. Hence $\mathcal{M}$ is a WAR, by (a).
Also $<\alpha,\alpha>\in 1_U\subseteq I$, so $<\alpha,\alpha\int\beta>\in I_\int\angle I^{\sim}\subseteq I_\int\angle^I I^{\sim}$; whereas $I\cap I^{\sim}=\emptyset$. So, $\mathcal{M}$ is non-Boolean.

QED

We now exhibit a simple MAR of each given infinite cardinality, so a WAR.

Proposition *Large simple proper non-Boolean MAR's*
For each infinite cardinal $\kappa \geq \aleph_0$ there exists a simple proper non-Boolean MAR $\mathcal{M}$ with cardinality $|\mathcal{M}|=\kappa$.

Proof outline

Select a set A with cardinality $|A|=\kappa$ and consider the merge operation $\int$ on $U=A^{\omega}$. Then, the full PFA $\mathcal{P}^{\int}(U^2)$ is a simple proper MAR.

Consider the subset $H=\{<a^{\omega},b^{\omega}>/a,b\in A\}\subseteq \wp(U^2)$ (note that $|H|=|A|^2=\kappa$).
Thus, the subalgebra $\mathcal{M}$ of the full PFA $\mathcal{P}^{\int}(U^2)$ generated by H is a simple proper non-Boolean MAR with cardinality $|\mathcal{M}|=|H|=\kappa$.

QED

We now use direct products to exhibit large collections of pairwise non-isomorphic non-Boolean PWA's of each given infinite cardinality. For this purpose, we notice that a direct product of proper non-Boolean WAR's is isomorphic to a proper non-Boolean WAR.

Theorem *Many large proper non Boolean WAR's*
Consider an infinite cardinal $\kappa \geq \aleph_0$.

a) There exists a simple non-Boolean PWA $\mathcal{M}_\kappa$ with cardinality $|\mathcal{M}_\kappa|=\kappa$.

b) There exist at least κ, pairwise non-isomorphic, non-Boolean PWA's with cardinality κ.

c) If κ is a successor cardinal ($\kappa=2^{\alpha}$ with $\alpha\geq\aleph_0$), then there exist at least 2^{κ}, pairwise non-isomorphic, non-Boolean PWA's with cardinality κ.

Proof outline

a) Follows immediately from the preceding results.

b) For each cardinal $\gamma<\kappa$, form the direct product $\mathcal{G}[\gamma]:=\mathcal{M}^{\gamma}\times\mathcal{M}_\kappa$, where $\mathcal{M}$ is a simple non-Boolean PWA with cardinality $|\mathcal{M}|=\aleph_0$.

Then, $\mathcal{G}[\gamma]$ is (isomorphic to) a non-Boolean PWA, of cardinality $\kappa\leq|\mathcal{G}[\gamma]|\leq\kappa.\kappa$, which has exactly $2^{\gamma+1}$ ideal elements (see [Veloso '96b; Appendix]).
Hence, there are at least κ pairwise non-isomorphic non-Boolean PWA's $\mathcal{G}[\gamma]=\mathcal{M}^{\gamma}\times\mathcal{M}_\kappa$ of cardinality κ, for $\gamma<\kappa$.

c) For each set $I\subseteq\kappa-\aleph_0$, form the direct product $\mathcal{H}[I]:=(\times_{\gamma\in I}\mathcal{M}_\gamma)\times\mathcal{M}_\kappa$.
Then, $\mathcal{H}[I]$ is (isomorphic to) a non-Boolean PWA, and $\mathcal{H}[I]=(\times_{\gamma\in I}\mathcal{M}_\gamma)\times\mathcal{M}_\kappa$ has cardinality $\kappa\leq|\mathcal{H}[I]|\leq|\mathcal{H}[\kappa]|\leq 2^{\alpha\alpha}.\alpha.\kappa=\kappa$ (since $\kappa=\alpha^{+}=2^{\alpha}$).
The non-trivial simple factors of $\mathcal{H}[I]$ are $\mathcal{M}_\gamma,\gamma\in I$, and $\mathcal{M}_\kappa$ (see [Veloso '96c]).
Therefore, there are at least 2^{κ} pairwise non-isomorphic non-Boolean PWA's $\mathcal{H}[I]=(\times_{\gamma\in I}\mathcal{M}_\gamma)\times\mathcal{M}_\kappa$ of cardinality κ, for $I\in\wp(\kappa-\aleph_0)$.

QED

7. CONCLUSION

We have examined the set-theoretical nature of fork algebras, namely to what extent fork algebras of relations are really concrete.

Algebras of relations involve relations on a set (of points), whereas fork algebras of relations (FAR, for short) deal with relations on a set of objects (trees) structured by an underlying pair-packing coding. The original intuition behind this structuring operation is the true pair forming of the cartesian product. So, we have examined the question: can every FAR be represented as a cartesian FAR (one whose underlying pair-packing coding is true cartesian-product pair forming)?

The answer, in a nutshell, is yes, with a proviso. One can indeed represent each FAR as a cartesian FAR by making use of the room provided by the neutral element I for relational composition |. The drawback of this representation is the choice for I, which renders them not so concrete. The constraining proviso is that if one insists in taking I as the concrete identity (diagonal) relation, then there are many non-trivial FAR's that cannot be represented in this manner, as proper cartesian FAR's.

The main device for getting cartesian representability amounts to considering this wider class of FAR's, where I is not required to be the identity on the underlying set. In section 3 we have examined how a fork operation on relations is induced by a coding on its underlying set.

In section 4 we have introduced this class of FAR's, which is wider than the class **PFA** of proper fork algebras (those where $I=1_U$): **PFA**$\subset$**FAR**. We also argued, along the lines of [Frias et al. '96], that every PFA is an AFA (**PFA**$\subseteq$**AFA**) in that it satisfies the three fork equations. We have then established that the subclass of proper FAR's is sufficiently wide to represent all FAR's: **FAR**$\subseteq$**I[PFA]**. So, even though the relaxed FAR's are less restricted than the PFA's, they are still AFA's. We thus have the inclusions **PFA**$\subseteq$**FAR**$\subseteq$**AFA** and the representations **I[PFA]**=**I[FAR]**. The latter, with the Representability Theorem **AFA**$\subseteq$**I[PFA]** of [Frias et al. '95,'96], yields **I[PFA]**=**I[FAR]**=**AFA**.

In section 5 we have examined the class of cartesian fork algebras of relations, where the underlying pair-packing coding is true pair forming. By showing that every coding algebra is a homomorphic image of some cartesian coding algebra, we have established that the subclass **CAR**$\subseteq$**FAR** is sufficiently wide to represent all FAR's: **FAR**$\subseteq$**I[CAR]**. We have also exhibited a simple PCA of each given infinite cardinality.

The question of representability by proper CAR's is taken up in section 6. We have first introduced the subclasses **WAR** of weird algebra of relations and **PWA** of the proper WAR's (**PWA**$\subseteq$**WAR**$\subseteq$**FAR**) and argued they cannot be represented as PCA's: **WAR**$\cap$**I[PCA]**$=\varnothing$. We then have established that these classes are populated by exhibiting first some admittedly uninteresting (in that they are essentially Boolean algebras) PWA's of each nonzero cardinality. With the aim of showing some more interesting WAR's, we have introduced the merge algebras of relations, whose underlying coding is a merge-like operation on infinite sequences, and observed that most MAR's are non-Boolean WAR's. With these results we have been able to exhibit, for each infinite cardinality, a simple proper MAR as well as large collections of pairwise non-isomorphic PWA's.

Hence, two subclasses, **PFA** and **CAR**, of **FAR** have the following properties.

- They can represent all the FAR's: $\mathbf{FAR} \subseteq \mathrm{I}[\mathbf{PFA}]$ and $\mathbf{FAR} \subseteq \mathrm{I}[\mathbf{CAR}]$ (so $\mathrm{I}[\mathbf{PFA}] = \mathrm{I}[\mathbf{FAR}] = \mathrm{I}[\mathbf{CAR}]$).
- They can represent all the AFA's: $\mathrm{I}[\mathbf{PFA}] = \mathrm{I}[\mathbf{CAR}] = \mathrm{I}[\mathbf{FAR}] = \mathbf{AFA}$ (in view of the Representability Theorem [Frias et al. '95,'96]).

But their intersection $\mathbf{PCA} = \mathbf{PFA} \cap \mathbf{CAR}$ is not wide enough: we have a large class $\mathbf{WAR} \subseteq \mathbf{FAR}$ such that $\mathbf{WAR} \cap \mathrm{I}[\mathbf{PCA}] = \varnothing$. Indeed, **PCA** fails to represent many (simple) proper fork algebras at each infinite cardinality.

To summarise the non-representability results, let us introduce some notation. Given a class **C** of algebras, we use the notations $\mathbf{C}\{\kappa\}$ for the class of algebras in **C** with cardinality κ and $|\mathbf{C}/{\equiv}|$ for the cardinal number of pairwise non-isomorphic algebras in class **C** (the cardinality $|\mathbf{C}/{\equiv}|$ where $\equiv$ is the equivalence relation of being isomorphic between algebras in class **C**). We also use $\mathbf{C}\langle\kappa\rangle$ to abbreviate the cardinal number $|\mathbf{C}\{\kappa\}/{\equiv}|$ of pairwise non-isomorphic algebras in class **C** with cardinality κ, and **SMPL** to denote the class of simple fork algebras.

For each infinite cardinal $\kappa \geq \aleph_0$, even though $\mathbf{SMPL} \cap \mathbf{PCA}\{\kappa\} \neq \varnothing$, we have:

- $\mathcal{M}_\kappa \in \mathbf{SMPL} \cap \mathbf{PFA}\{\kappa\} - \mathrm{I}[\mathbf{PCA}] \neq \varnothing$
  (simple MAR $\mathcal{M}_\kappa$, with cardinality κ, not representable as a PCA);
- $(\mathbf{PFA} - \mathrm{I}[\mathbf{PCA}])\langle\kappa\rangle \geq \kappa \geq \aleph_0$
  (κ pairwise non-isomorphic PWA's $(\mathcal{M}_\omega)^\gamma \times \mathcal{M}_\kappa \in \mathbf{PFA}\{\kappa\} - \mathrm{I}[\mathbf{PCA}]$, for $\gamma < \kappa$);
- $(\mathbf{PFA} - \mathrm{I}[\mathbf{PCA}])\langle\kappa\rangle \geq 2^\kappa > \aleph_0$ if κ is a successor cardinal ($\kappa = 2^\alpha$ with $\alpha \geq \aleph_0$)
  (2^κ pairwise non-isomorphic $(\times_{\gamma \in I} \mathcal{M}_\gamma) \times \mathcal{M}_\kappa \in \mathbf{PFA}\{\kappa\} - \mathrm{I}[\mathbf{PCA}]$, for $I \subseteq \kappa - \aleph_0$).

Thus, at each infinite cardinality, the class $\mathbf{PCA} = \mathbf{PFA} \cap \mathbf{CAR}$ of proper cartesian FAR's, even though populated, fails to represent many PFA's.

Appendix: Detailed Proofs of the Results

We present in this appendix detailed proofs of the results.

Lemma *Change of underlying set of AR's*

Given a surjective function $h: W \rightarrow U$, every AR $\mathcal{Q} = \langle Q, \cup, \cap, \varnothing, V, {}^{\sim}, |, {}^{T}, I \rangle$ on U is isomorphic to an AR $H(\mathcal{Q})$ on W.

Proof

Function $h: W \rightarrow U$ induces a preimage mapping $H: \wp(U^2) \rightarrow \wp(W^2)$ by $H(r) := \{\langle w', w'' \rangle \in W^2 / \langle h(w'), h(w'') \rangle \in r\}$.

a) This preimage mapping $H: \wp(U^2) \rightarrow \wp(W^2)$ preserves Boolean structure: H is a Boolean homomorphism from $\langle Q, \cup, \cap, \varnothing, V, {}^{\sim} \rangle$ onto $\langle H(Q), \cup, \cap, \varnothing, H(V), {}^{\sim} \rangle$.

(i) H preserves inclusion: if $r \subseteq s$ then $H(r) \subseteq H(s)$

If $\langle w', w'' \rangle \in H(r)$ then $\langle h(w'), h(w'') \rangle \in r \subseteq s$, so $\langle w', w'' \rangle \in H(s)$.

(ii) $H(Q) \subseteq \wp[H(V)]$

If $s \in H(Q)$ then, for some $r \in Q$, $s=H(r)$, so $r \subseteq V$ and $s=H(r) \subseteq H(V)$.

(iii) $H(r \cap s)=H(r) \cap H(s)$

$\langle w',w''\rangle \in H(r \cap s)$ iff $\langle h(w'),h(w'')\rangle \in r \cap s$ iff $\langle h(w'),h(w'')\rangle \in r$ and $\langle h(w'),h(w'')\rangle \in s$ iff $\langle w',w''\rangle \in H(r)$ and $\langle w',w''\rangle \in H(s)$ iff $\langle w',w''\rangle \in H(r) \cap H(s)$.

(iv) $H(\varnothing)=\varnothing$

$\langle w',w''\rangle \in H(\varnothing)$ iff $\langle h(w'),h(w'')\rangle \in \varnothing$; so $H(\varnothing) \subseteq \varnothing$.

(v) $H(r^{\sim})=H(r)^{\sim}$

$\langle w',w''\rangle \in H(r^{\sim})$ iff $\langle h(w'),h(w'')\rangle \in V \cap r^{\sim}$ iff $\langle h(w'),h(w'')\rangle \in V$ and $\langle h(w'),h(w'')\rangle \notin r$ iff $\langle w',w''\rangle \in H(V)$ and $\langle w',w''\rangle \notin H(r)$ iff $\langle w',w''\rangle \in H(V) \cap H(r)^{\sim}=H(r)^{\sim}$.

(vi) $H(r \cup s)=H(r) \cup H(s)$

$H(r \cup s)=H[(r \cap s)^{\sim}]=[H(r) \cap H(s)]^{\sim}=H(r) \cup H(s)$.

b) If $h:W \to U$ is surjective then $H:\wp(U^2) \to \wp(W^2)$ is injective and preserves Peircean structure.

- $H:\wp(U^2) \to \wp(W^2)$ is injective

If $r \neq \varnothing$ then, for some $u',u'' \in U$, $\langle u',u''\rangle \in r$, so, for some $w',w'' \in W$, $\langle h(w'),h(w'')\rangle = \langle u',u''\rangle \in r$, thus $\langle w',w''\rangle \in H(r) \neq \varnothing$.

Now, consider $H(\mathcal{Q}):=\langle H(Q),\cup,\cap,\varnothing,H(V),^{\sim},^{T},|,H(I)\rangle$

(i) $H(r^T)=H(r)^T$

$\langle w',w''\rangle \in H(r^T)$ iff $\langle h(w'),h(w'')\rangle \in r^T$ iff $\langle h(w''),h(w')\rangle \in r$ iff $\langle w'',w'\rangle \in H(r)$ iff $\langle w',w''\rangle \in H(r)^T$.

(ii) $H(r)|H(s) \subseteq H(r|s)$

If $\langle w',w''\rangle \in H(r)|H(s)$ then, for some $w \in W$, $\langle w',w\rangle \in H(r)$ and $\langle w,w''\rangle \in H(s)$. So, with $h(w) \in U$, $\langle h(w'),h(w)\rangle \in r$ and $\langle h(w),h(w'')\rangle \in s$, whence $\langle h(w'),h(w'')\rangle \in r|s$ and $\langle w',w''\rangle \in H(r|s)$.

(iii) $H(r|s) \subseteq H(r)|H(s)$

If $\langle w',w''\rangle \in H(r|s)$ then $\langle h(w'),h(w'')\rangle \in r|s$, so, for some $u \in U$, $\langle h(w'),u\rangle \in r$ and $\langle u,h(w'')\rangle \in s$. Thus, since h is onto U, for some $w \in W$, $h(w)=u$, and so $\langle h(w'),h(w)\rangle \in r$ and $\langle h(w),h(w'')\rangle \in s$. Hence $\langle w',w\rangle \in H(r)$ and $\langle w,w''\rangle \in H(s)$, and thus $\langle w',w''\rangle \in H(r)|H(s)$.

Hence, H is an isomorphism of AR $\mathcal{Q}$ onto $H(\mathcal{Q})$, which is then an AR.

QED

Proposition *Properties of PFA's*: **PFA⊆AFA**

Given PFA $\mathcal{H}=\langle H,\cup,\cap,\varnothing,V,^{\sim},^{T},|,1_U,\angle\rangle$ define $\pi:=(1_U \angle V)^T$ and $\rho:=(V \angle 1_U)^T$.

a) $(r \angle s)|(p \angle q)^T=(r|p^T) \cap (s|q^T)$.

b) $\pi \angle \rho \subseteq 1_U$.

c) $r \angle s=(r|\pi^T) \cap (s|\rho^T)$.

Proof

First notice the following explicit descriptions of the defined projections: $\pi:=(1_{U*} \angle V)^T=\{\langle u*v,u\rangle \in V / \langle u,v\rangle \in V\}$ and $\rho:=(V_* \angle 1_U)^T=\{\langle u*v,v\rangle \in V / \langle u,v\rangle \in V\}$.

Indeed, $\pi^T:=1_{U*} \angle V=\{\langle u,u'*v\rangle \in U^2 / \langle u,u'\rangle \in 1_U$ and $\langle u,v\rangle \in V\}$.

Similarly, $\rho^T:=V_* \angle 1_U=\{\langle v,u*v''\rangle \in U^2 / \langle v,u\rangle \in V$ and $\langle v,v''\rangle \in 1_U\}$.

a) $(r|p^T)\cap(s|q^T)\subseteq(r_*\angle s)|(p_*\angle q)^T$ and $(r_*\angle s)|(p_*\angle q)^T\subseteq(r|p^T)\cap(s|q^T)$

(i) For $(r|p^T)\cap(s|q^T)\subseteq(r_*\angle s)|(p_*\angle q)^T$, consider $<u,z>\in(r|p^T)\cap(s|q^T)$.

Since $<u,z>\in r|p^T$, we have some $v\in U$ such that $<u,v>\in r$ and $<v,z>\in p^T$.
Since $<u,z>\in s|q^T$, we have some $w\in U$ such that $<u,w>\in s$ and $<w,z>\in q^T$.
From $<u,v>\in r$ and $<u,w>\in s$ we have $<u,v*w>\in r_*\angle s$.

From $<z,v>\in p$ and $<z,w>\in q$ we have $<z,v*w>\in p_*\angle q$.

So, we have $y=v*w\in U$ such that $<u,y>\in r_*\angle s$ and $<y,z>\in(p_*\angle q)^T$.

Hence $<u,z>\in(r_*\angle s)|(p_*\angle q)^T$.

(ii) For $(r_*\angle s)|(p_*\angle q)^T\subseteq(r|p^T)\cap(s|q^T)$, consider $<x,z>\in(r_*\angle s)|(p_*\angle q)^T$.

We then have some $y\in U$ such that $<x,y>\in r_*\angle s$ and $<y,z>\in(p_*\angle q)^T$.

From $<x,y>\in r_*\angle s$, we get $<u',u''>\in V$ so that $y=u'*u''$ with $<x,u'>\in r$ and $<x,u''>\in s$.

From $<z,y>\in p_*\angle q$, we get $<v',v''>\in V$ so that $y=v'*v''$ with $<z,v'>\in p$ and $<z,v''>\in q$.
Since $<u'*u'',v'*v''>=<y,y>\in 1_U\subseteq V$, we have $u'=v'$ and $u''=v''$.
From $<x,u'>\in r$, with $u'=v'$, and $<v',z>\in p^T$, we have $<x,z>\in r|p^T$.
From $<<x,u''>\in s$, with $u''=v''$, and $<v'',z>\in q^T$, we have $<x,z>\in s|q^T$.
Hence $<x,z>\in(r|p^T)\cap(s|q^T)$.

b) $\pi\angle\rho\subseteq 1_U$

Assume $<z,w>\in\pi_*\angle\rho$.

Then, for some $<u,v>\in V$, $w=u*v$ with $<z,u>\in\pi$ and $<z,v>\in\rho$.
From $<z,u>\in\pi$, we have $z=u*y$ for some $y\in U$.
From $<z,v>\in\rho$, we have $z=x*v$ for some $x\in U$.
Since $<u*y,x*v>=<z,z>\in 1_U\subseteq V$, we have $u=x$ and $y=v$.
Thus $<z,w>=<u*y,u*v>=<z,z>\in 1_U$.

c) $r_*\angle s\subseteq(r|\pi^T)\cap(s|\rho^T)$ and $(r|\pi^T)\cap(s|\rho^T)\subseteq r_*\angle s$

(i) For $r_*\angle s\subseteq(r|\pi^T)\cap(s|\rho^T)$, consider $<u,z>\in r_*\angle s$.

Then, for some $<v,w>\in V$, $z=v*w$ with $<u,v>\in r$ and $<u,w>\in s$.
From $z=v*w$ with $<v,w>\in V$ we get $<v,z>=<v,v*w>\in\pi^T$ and $<w,z>=<w,v*w>\in\rho^T$.
We thus have $v\in U$ such that $<u,v>\in r$ and $<v,z>\in\pi^T$, so $<u,z>\in r|\pi^T$.
We also have $w\in U$ such that $<u,w>\in s$ and $<w,z>\in\rho^T$, so $<u,z>\in s|\rho^T$.
Hence $<u,z>\in(r|\pi^T)\cap(s|\rho^T)$.

(i) For the other inclusion, item (a) gives $(r|\pi^T)\cap(s|\rho^T)=(r_*\angle s)|(\pi_*\angle\rho)^T$.

So, by part (b), we have $(r|\pi^T)\cap(s|\rho^T)=(r_*\angle s)|(\pi_*\angle\rho)^T\subseteq(r_*\angle s)|(1_U)^T=r_*\angle s$.

QED

Theorem *Representation of FAR's as PFA's*: **FAR⊆I[PFA]**

Every FAR $\mathcal{G}$ is isomorphic to some proper PFA $\mathcal{H}$.

Proof

Consider FAR $\mathcal{G}=<G,\cup,\cap,\emptyset,V,^-,|,^T,I,\angle>$ with fork induced by underlying pair-packing $<U,\circ>$ for universal $V\subseteq U^2$ under relation $I\subseteq V$

Since I is an equivalence relation on U, we have a surjective quotient projection $n:U\to\underline{U}$, mapping $u\in U$ to $u/I\in\underline{U}:=U/I$. This induces injective $N:\wp(U^2)\to\wp(\underline{U}^2)$ by $N(r):=\{\langle n(x),n(y)\rangle\in\underline{U}^2/\langle x,y\rangle\in r\}$. Since $I=\ker(n)$, $N(I)=1_{\underline{U}}$.

We thus have a relational isomorphism $N:G\to P$ from $\mathcal{G}_{\angle}=\langle G,\cup,\cap,\emptyset,V,\tilde{},|,^T,I\rangle$ onto proper AR $\mathcal{P}=\langle N(G),\cup,\cap,\emptyset,\underline{V},\tilde{},|,^T,1_{\underline{U}}\rangle$ with universal $\underline{V}:=N(V)\subseteq\underline{U}^2$.

Now, since $\langle U,\circ\rangle$ is pair-packing for V under I, we can define $*:\underline{U}^2\to\underline{U}$ by $(u/I)*(v/I):=(u\circ v)/I$ for $\langle(u/I),(v/I)\rangle\in\underline{V}$ and arbitrarily outside $\underline{V}$.
Then $*:\underline{U}^2\to\underline{U}$ is well-defined pair-packing for $\underline{V}$ under $1_{\underline{U}}$.

{ One can see that $*:\underline{U}^2\to\underline{U}$ is well-defined as follows.
Consider $\langle u,v\rangle\in V$. If $\langle u,x\rangle,\langle v,y\rangle\in I$ then $\langle x,y\rangle\in I^T|V|I\subseteq V$, whence $\langle u\circ v,x\circ y\rangle\in I$.
Thus, for $\langle(u/I),(v/I)\rangle\in\underline{V}$, whenever $\langle u,x\rangle,\langle v,y\rangle\in I$, we have $(u\circ v)/I=(x\circ y)/I$.
Now, $*:\underline{U}^2\to\underline{U}$ packs $\underline{V}$, for $\langle v,w\rangle\in V$ entails $\langle v,v\circ w\rangle\in V$ since $\langle v\circ w,v\circ w\rangle\in 1_U\subseteq I$.
Thus, $\langle(v/I),(v/I)*(w/I)\rangle=\langle(v/I),(v\circ w/I)\rangle\in\underline{V}$ whenever $\langle(v/I),(w/I)\rangle\in\underline{V}$.
Finally, one can see that $*:\underline{U}^2\to\underline{U}$ is injective over $\underline{V}$ as follows.
Given $\langle u,v\rangle$ and $\langle x,y\rangle$ in V, $\langle u*v,x*y\rangle\in I$ yields both $\langle u,x\rangle,\langle v,y\rangle$ in I
Thus, over $\underline{V}$, if $(u/I)*(v/I)=(x/I)*(y/I)$ then $(u/I)=(x/I)$ and $(v/I)=(y/I)$. }
Also, the RA-homomorphism N preserves induced forks: $N(r\angle s)=N(r)\,{}_*\!\angle N(s)$.

{ Indeed, $\langle x/I,z/I\rangle\in N(r\angle s)$ iff $\langle x,z\rangle\in r\angle s$ iff, for some $\langle v,w\rangle\in V$, $\langle x,v\rangle\in r$, $\langle x,w\rangle\in s$ and $\langle v\circ w,z\rangle\in I$ iff, for some $\langle\underline{v},\underline{w}\rangle\in\underline{V}$, $\langle x/I,\underline{v}\rangle\in N(r)$, $\langle x/I,\underline{w}\rangle\in N(s)$ and $\underline{v}*\underline{w}=z/I$ iff $\langle x/I,z/I\rangle\in N(r)\,{}_*\!\angle N(s)$. }

Hence, N(G) is closed under induced fork ${}_*\!\angle$, and we can expand PAR $\mathcal{P}$ to a PFA $\mathcal{H}=(\mathcal{P},{}_*\!\angle)$. Then $N:G\to N(G)$ gives an FA-isomorphism from FAR $\mathcal{G}=(\mathcal{G}_{\angle},\angle)$ onto proper FAR $\mathcal{H}=(\mathcal{P},{}_*\!\angle)$.

QED

Corollary *Properties of FAR's*: **FAR**$\subseteq$**AFA**

Every FAR $\mathcal{G}=\langle G,\cup,\cap,\emptyset,V,\tilde{},|,^T,I,\angle\rangle$ satisfies the three fork equations:

a) $(r\angle s)|(p\angle q)^T=(r|p^T)\cap(s|q^T)$,

b) $\pi\angle\rho\subseteq I$,

c) $r\angle s=(r|\pi^T)\cap(s|\rho^T)$,

with the defined 'projections' $\pi:=(I\angle V)^T$ and $\rho:=(V\angle I)^T$.

Proof

By the preceding theorem we have FAR $\mathcal{G}$ isomorphic to some PFA $\mathcal{H}$ and the latter satisfies these equations since **PFA**$\subseteq$**AFA**.

QED

Proposition *Cartesian coding*

Every coding algebra $\langle U,*\rangle$ is a homomorphic image of some cartesian coding algebra.

Proof

Consider the term algebra $\langle T,\circ\rangle$ freely generated by the elements of U.
[T is the disjoint union $\cup_{n\in N}T_n$ where $T_0:=U$ and $T_{n+1}:=T_n\cup\{t'\circ t''/t',t''\in T_n\}$.]

We have an evaluation mapping $e{:}T\rightarrow U$ defined by $e(u)=u$ for $u\in U$ and $e(t'\circ t'')=e(t')*e(t'')$ for $t'\circ t''\notin U$.

[e is defined by $e(u)=u$ for $u\in T_0=U$ and $e(t'\circ t'')=e(t')*e(t'')$ for $t'\circ t''\in(T_{n+1}-T_n)$.]

Evaluation $e{:}T\rightarrow U$ is a homomorphism of $\langle T,\circ\rangle$ onto $\langle U,*\rangle$.

[It is onto ,since $e(u)=u$ for $u\in U$, and $e(t'\circ t'')=e(t')*e(t'')$ for $t',t''\in T=\cup_{n\in\mathbf{N}}T_n$.]

Now, consider the closure $U^{\times}$ of U under cartesian product.

[$U^{\times}$ is the (disjoint) union $\cup_{n\in\mathbf{N}}U_n$ where $U_0=U$ and $U_{n+1}=U_n\cup(U_n\times U_n)$.]

Since we have a single operation, $\langle T,\circ\rangle$ is isomorphic to $\langle U^{\times},(,)\rangle$.

[Define mapping $i{:}T\rightarrow U^{\times}$ as the (disjoint) union $i:=\cup_{n\in\mathbf{N}}i_n$ of bijections, where $i_0:=1_U:T_0\rightarrow U_0$ and $i_{n+1}{:}(T_{n+1}-T_n)\rightarrow U_n\times U_n$ with $i_{n+1}[t'\circ t'']:=(i_n[t'],i_n[t''])$.]

We thus have a surjective homomorphism k from $\langle U^{\times},(,)\rangle$ onto $\langle U,*\rangle$.

[Mapping $k{:}U^{\times}\rightarrow U$ is the composite bijective $i^{-1}{:}U^{\times}\rightarrow T$ followed by surjective $e{:}T\rightarrow U$. So, $k(u)=u$ for $u\in U_0=U$ and $k[(v,w)]=k[v]*k[w]$ for $(v,w)\in U_n\times U_n$.]

QED

Theorem *Representation of FAR's as CAR's*: **FAR⊆I[CAR]**

Every FAR $\mathcal{G}$ is isomorphic to some cartesian algebra of relations $\mathcal{C}$.

Proof

In view of the representation of FAR's as PFA's, we may assume $\mathcal{G}$ to be a proper FAR $\langle G,\cup,\cap,\varnothing,V,{}^{\sim},{}^{T},|,1_U,\angle\rangle$, with underlying coding algebra $\langle U,*\rangle$.

By the preceding proposition, we have a surjective homomorphism k from cartesian coding algebra $\langle W,(,)\rangle$ onto $\langle U,*\rangle$; call $\underline{I}:=\ker(k)$ its kernel.

Now, surjective $k{:}W\rightarrow U$ induces, by preimage, injective $K{:}\wp(U^2)\rightarrow\wp(W^2)$.

This preimage mapping gives a relational isomorphism $K{:}G\rightarrow Q$ from proper AR $\mathcal{G}_{\angle}=\langle G,\cup,\cap,\varnothing,V,{}^{\sim},|,{}^{T},1_U\rangle$ onto AR $\mathcal{Q}=\langle K(G),\cup,\cap,\varnothing,\underline{V},{}^{\sim},|,{}^{T},\underline{I}\rangle$ with $\underline{V}:=K(V)\subseteq W^2$.

Since k is a homomorphism of coding algebras, $\langle W,(,)\rangle$ is a pair-packing for relation $\underline{V}:=K(V)$ under relation $\underline{I}\subseteq\underline{V}$.

{ First, insertion $(,){:}U^2\rightarrow U$ is injective.
Also, if $\langle\underline{v},\underline{w}\rangle\in\underline{V}=K(V)$ and $\langle(\underline{v},\underline{w}),\underline{z}\rangle\in\underline{I}=\ker(k)$, then $\langle k(\underline{v}),k(\underline{w})\rangle\in V$ and $k(\underline{v})*k(\underline{w})=k[(\underline{v},\underline{w})]=k(\underline{z})$; so $\langle k(\underline{v}),k(\underline{z})\rangle=\langle k(\underline{v}),k(\underline{v})*k(\underline{w})\rangle\in V$ and $\langle\underline{v},\underline{z}\rangle\in\underline{V}$.
Finally, $\langle(\underline{v},\underline{w}),(\underline{x},\underline{y})\rangle\in\underline{I}=\ker(k)$ iff $k(\underline{v})*k(\underline{w})=k[(\underline{v},\underline{w})]=k[(\underline{x},\underline{y})]=k(\underline{x})*k(\underline{y})$ iff $k(\underline{v})=k(\underline{x})$ and $k(\underline{w})=k(\underline{y})$ iff $\langle\underline{v},\underline{x}\rangle\in\underline{I}$ and $\langle\underline{w},\underline{y}\rangle\in\underline{I}$. }

We now show that relational isomorphism $K{:}G\rightarrow K(G)$ preserves induced forks and that $K(G)$ is closed under ${}_{(,)}\angle^{\underline{I}}$.

For this purpose, we establish $K(r)_{(,)}\angle^{\underline{I}}K(s)=[K(r)_{(,)}\angle K(s)]|\underline{I}=K(r\angle s)|\underline{I}$ for $r,s\subseteq V$.

1. First, we show that for $r,s\subseteq V$: $K(r)_{(,)}\angle K(s)\subseteq K(r\angle s)$.

{ To see that $K(r)_{(,)}\angle K(s)\subseteq K(r\angle s)$, consider $\langle\underline{u},\underline{z}\rangle\in K(r)_{(,)}\angle K(s)$.
Then, for some $\langle\underline{v},\underline{w}\rangle\in\underline{V}$, $\langle\underline{u},\underline{v}\rangle\in K(r)$ and $\langle\underline{u},\underline{w}\rangle\in K(s)$ with $(\underline{u},\underline{w})=\underline{z}$.
So, $\langle k(\underline{u}),k(\underline{v})\rangle\in r$ and $\langle k(\underline{u}),k(\underline{w})\rangle\in s$ and $\langle k(\underline{v}),k(\underline{w})\rangle\in V$ with $k[(\underline{u},\underline{w})]=k(\underline{z})$.
Thus, $\langle k(\underline{u}),k(\underline{v})*k(\underline{w})\rangle\in r\angle s$ with $k(\underline{v})*k(\underline{w})=k[(\underline{u},\underline{w})]=k(\underline{z})$.
Hence $\langle\underline{u},\underline{z}\rangle\in K(r\angle s)$. }

2. We also show that for $r,s\subseteq V$: $K(r\angle s)\subseteq[K(r)_{(,)}\angle K(s)]|\underline{I}$.

{ To see that $K(r\angle s)\subseteq[K(r)_{(,)}\angle K(s)]|\underline{I}$, consider $<\underline{u},\underline{z}>\in K(r\angle s)$; so $<k(\underline{u}),k(\underline{z})>\in r\angle s$.
Then, for some $<x,y>\in V$, $<k(\underline{u}),x>\in r$ and $<k(\underline{u}),y>\in s$ with $x*y=k(\underline{z})$.
So, for some $<\underline{x},\underline{y}>\in \underline{V}$, $<k(\underline{u}),k(\underline{x})>\in r$ and $<k(\underline{u}),k(\underline{y})>\in s$ with $k(\underline{x})*k(\underline{y})=k(\underline{z})$.
Thus, $<\underline{u},\underline{x}>\in K(r)$ and $<\underline{u},\underline{y}>\in K(s)$ with $k[(\underline{x},\underline{y})]=k(\underline{x})*k(\underline{y})=k(\underline{z})$.
Then $<\underline{u},(\underline{x},\underline{y})>\in K(r)_{(,)}\angle K(s)$ and $<(\underline{x},\underline{y}),\underline{z}>\in ker(k)=\underline{I}$; so $<\underline{u},\underline{z}>\in [K(r)_{(,)}\angle K(s)]|\underline{I}$. }

3. We now have for $r,s\subseteq V$: $K(r\angle s)|\underline{I}=[K(r)_{(,)}\angle K(s)]|\underline{I}=K(r)_{(,)}\angle^{\underline{I}}K(s)$.

{ Indeed, $K(r)_{(,)}\angle K(s)\subseteq K(r\angle s)$ yields $[K(r)_{(,)}\angle K(s)]|\underline{I}\subseteq K(r\angle s)|\underline{I}$.
Also, $K(r\angle s)\subseteq[K(r)_{(,)}\angle K(s)]|\underline{I}$ yields $K(r\angle s)|\underline{I}\subseteq[K(r)_{(,)}\angle K(s)]|\underline{I}|\underline{I}=[K(r)_{(,)}\angle K(s)]|\underline{I}$.
Hence $K(r\angle s)|\underline{I}\subseteq[K(r)_{(,)}\angle K(s)]|\underline{I}\subseteq K(r\angle s)|\underline{I}$, i. e. $K(r\angle s)|\underline{I}=K(r\angle s)|\underline{I}$. }

4. In particular, for $r,s\in G$: $K(r)_{(,)}\angle^{\underline{I}}K(s)=[K(r)_{(,)}\angle K(s)]|\underline{I}=K(r\angle s)|\underline{I}\in K(G)$.

{ Indeed, we have $r\angle s\in G$, so $K(r\angle s)\in K(G)$; thus, since $\mathcal{Q}$ is an AR,
$K(r\angle s)|\underline{I}=K(r\angle s)\in K(G)$, whence $K(r)_{(,)}\angle^{\underline{I}}K(s)=K(r\angle s)|\underline{I}\in K(G)$. }

Therefore, K(G) is closed under relaxed fork $_{*(,)}\angle^{\underline{I}}$. So we can expand AR $\mathcal{Q}$ to CAR $\mathcal{C}=(\mathcal{Q},_{(,)}\angle^{\underline{I}})$. Then $K:G\to K(G)$ gives an FA-isomorphism from FAR $\mathcal{G}=(\mathcal{G}_{\angle},\angle)$ onto CAR $\mathcal{C}=(\mathcal{Q},_{(,)}\angle^{\underline{I}})$.

QED

Proposition *Large simple proper cartesian algebras of relations*

For each infinite cardinal $\kappa\geq\aleph_0$ there exists a simple proper cartesian algebra of relations $\mathcal{D}$ with cardinality $|\mathcal{D}|=\kappa$.

Proof

Select a set U with $|U|=\kappa\geq\aleph_0$, and consider its cartesian closure $W:=U^{\times}$.

Since set $W=U^{\times}$ is closed under cartesian product, we can expand the full PAR $\mathcal{P}(W^2)$ to $\mathcal{P}^{(,)}(W^2)=(\mathcal{P}(W^2),_{(,)}\angle)$, which is a simple, proper CAR.

Set $G:=\wp_{\omega}(W^2)$ (note that $\kappa=|W|\leq|G|\leq\aleph_0.|W|=\kappa$) and let $\mathcal{D}$ be the subalgebra of the full PFA $\mathcal{P}^{(,)}(W^2)$ generated by G (note that $\kappa=|G|\leq|\mathcal{D}|=\aleph_0.|G|=\kappa$).

Therefore, $\mathcal{D}$ is a simple PCA, on set $U^{\times}$, with cardinality $|\mathcal{D}|=\kappa$.

QED

Proposition *Non-cartesian coding*: **WAR**$\cap$I**[PCA]**$=\varnothing$

A WAR cannot be isomorphic to a proper cartesian FAR.

Proof

In WAR $\mathcal{W}$ we have $(I\angle I)\cap I\neq\varnothing$, so $\mathcal{W}$ does not satisfy the equation $(I\angle I)\cap I=\varnothing$.

A proper cartesian FAR C on set U has $I=1_U$, so C satisfies $(I\angle I)\cap I=\varnothing$.

QED

Lemma *Many proper weird algebras of identities*

For each nonempty set $U\neq\varnothing$, there exists a proper WAR, on set U, with cardinality $|U|$.

Proof

Consider the powerset PAR of 1_U: the PAR $\mathcal{P}(1_U)=<\wp(1_U),\cup,\cap,\varnothing,1_U,^{\sim},|,^{T},1_U>$.

{ Notice that for relations $r,s\subseteq 1_U$, $r|s=r\cap s\subseteq 1_U$ and $r^T=r$. Thus, $\mathcal{P}(1_U)=<\wp(1_U),\cup,\cap,\emptyset,1_U,^{\sim},\cap,id,1_U>$ is essentially a Boolean algebra. }

Also, we have the diagonal bijection $d:1_U\to U$ with $d(u,u)=u$.

Now, d induces fork ${}_d\angle$ on $\wp(1_U)$, which behaves as intersection: $r\ {}_d\angle\ s = r\cap s$.

{ First $r\cap s\subseteq r\,{}_d\angle s$ (because if $<u,v>\in r\cap s\subseteq 1_U$ then $u=v$ and $d(u,u)=u$, so $<u,v>=<u,d(u,u)>$ with $<u,u>\in r$ and $<u,u>\in s$, and thus $<u,v>\in r\,{}_d\angle s$).
Also $r\,{}_d\angle s\subseteq r\cap s$ (because, if $<u,v>\in r\,{}_d\angle s$ then, for some $<v',v''>\in 1_U$, $<u,v'>\in r\subseteq 1_U$ and $<u,v''>\in s\subseteq 1_U$ with $d(v',v'')=v$; so $v'=u=v''$ and $v=d(v',v'')=d(u,u)=u$, whence $<u,v>=<u,v'>\in r$ and $<u,v>=<u,v''>\in s$. }

Now, since $\wp(1_U)$ is closed under diagonal fork ${}_d\angle$, we can expand PAR $\mathcal{P}(1_U)$ to $\mathcal{P}^d(1_U)=(\mathcal{P}(1_U),{}_d\angle)$. Clearly $(1_U\,{}_d\angle\,1_U)\cap 1_U=(1_U\cap 1_U)\cap 1_U=1_U\neq\emptyset$.

Set $K:=\wp_\omega(1_U)$ (note that $|U|\leq|K|\leq\aleph_0.|1_U|=|U|$) and let I be the subalgebra of the powerset PFA $\mathcal{P}^d(1_U)$ generated by K (note that $|U|=|K|\leq|I|=\aleph_0.|K|=|U|$).

Therefore, I is a WAR, on set U, with cardinality $|I|=|U|$.

QED

Lemma *Merge algebras of relations as (non Boolean) WAR's*:

Consider a merge algebra of relations $\mathcal{M}$ over set A (so, on set $U:=A^\omega$).

a) If set A is nonempty ($A\neq\emptyset$) then $\mathcal{M}$ is a WAR.

b) If $I^{\sim}\neq\emptyset$, then $\mathcal{M}$ is a non-Boolean WAR.

Proof

a) We have some $a\in A\neq\emptyset$ and constant sequence $a^\omega=<a,a,\ldots,a,\ldots>\in A^\omega$.

Since $a^\omega\int a^\omega=a^\omega$, we have $<a^\omega,a^\omega\int a^\omega>=<a^\omega,a^\omega>\in(1_{U\int}\angle 1_U)\cap 1_U$ and $(1_{U\int}\angle 1_U)\cap 1_U\neq\emptyset$; whence $\emptyset\neq(1_{U\int}\angle 1_U)\cap 1_U\subseteq(I_\int\angle I)\cap I\subseteq[(I_\int\angle I)|I]\cap I=(I_\int\angle^I I)\cap I$.

b) We have some $<\alpha,\beta>\in I^{\sim}\neq\emptyset$.

First $\alpha\neq\beta$ (since $<\alpha,\beta>\in I^{\sim}\subseteq(1_U)^{\sim}$). Thus $|A|>1$, and $A\neq\emptyset$ [so $\mathcal{M}$ is a WAR, by (a)].

Now, consider $<\alpha,\beta>\in I^{\sim}$ and $<\alpha,\alpha>\in 1_U\subseteq I$. Notice that $<\alpha,\alpha\int\beta>\in I_\int\angle I^{\sim}$, so $I_\int\angle I^{\sim}\neq\emptyset$ and $\emptyset\neq I_\int\angle I^{\sim}\subseteq I_\int\angle^I I^{\sim}$; whereas $I\cap I^{\sim}=\emptyset$. Hence $I_\int\angle^I I^{\sim}\neq\emptyset=I\cap I^{\sim}$, and $\mathcal{M}$ is non-Boolean.

QED

Proposition *Large simple proper non-Boolean MAR's*

For each infinite cardinal $\kappa\geq\aleph_0$ there exists a simple proper non-Boolean MAR $\mathcal{M}$ with cardinality $|\mathcal{M}|=\kappa$.

Proof

Select a set A with cardinality $|A|=\kappa$ and consider the merge operation $\int$ on $U=A^\omega$. Then, the full PFA $\mathcal{P}^\int(U^2)$ is a simple proper MAR.

Set $H:=\{<a^\omega,b^\omega>/a,b\in A\}$ (note that $\kappa=|A|\leq|A|^2=\kappa$) and let $\mathcal{M}$ be the subalgebra of the full PFA $\mathcal{P}^\int(U^2)$ generated by H (note that $\kappa=|H|\leq|\mathcal{M}|=\aleph_0.|H|=\kappa$).

Since A is infinite, we have distinct $a\neq b\in A$. So $<a^\omega,b^\omega>\in(1_U)^{\sim}\neq\emptyset$.

Thus, by the lemma, $\mathcal{M}$ is a non-Boolean MAR

Therefore, $\mathcal{M}$ is a simple proper non-Boolean MAR of cardinality κ.

QED

Theorem *Many large proper non Boolean WAR's*

Consider an infinite cardinal $\kappa \geq \aleph_0$.

a) There exists a simple non-Boolean PWA $\mathcal{M}_\kappa$ with cardinality $|\mathcal{M}_\kappa| = \kappa$.

b) There exist at least κ, pairwise non-isomorphic, non-Boolean PWA's with cardinality κ.

c) If κ is a successor cardinal ($\kappa = 2^\alpha$ with $\alpha \geq \aleph_0$), then there exist at least 2^κ, pairwise non-isomorphic, non-Boolean PWA's with cardinality κ.

Proof outline

a) By the preceding results, we have a simple proper non-Boolean MAR $\mathcal{M}_\kappa$ with cardinality $|\mathcal{M}_\kappa| = \kappa$; and $\mathcal{M}_\kappa$ is a PWA.

b) For each cardinal $\gamma < \kappa$, form the direct product $\mathcal{G}[\gamma] := \mathcal{M}^\gamma \times \mathcal{M}_\kappa$, where $\mathcal{M}$ is a simple non-Boolean PWA with cardinality $|\mathcal{M}| = \aleph_0$.

Then, $\mathcal{G}[\gamma]$ is (isomorphic to) a non-Boolean PWA, and $\mathcal{G}[\gamma] := \mathcal{M}^\gamma \times \mathcal{M}_\kappa$ has cardinality $\kappa = |\mathcal{M}_\kappa| \leq |\mathcal{G}[\gamma]| \leq \aleph_0{}^\gamma . \kappa \leq \kappa . \kappa = \kappa$.

Also, $\mathcal{G}[\gamma]$ has exactly $2^{\gamma+1}$ ideal elements (see [Veloso '96b; Appendix]).

Hence, there are at least κ pairwise non-isomorphic non-Boolean PWA's $\mathcal{G}[\gamma] = \mathcal{M}^\gamma \times \mathcal{M}_\kappa$ of cardinality κ, for $\gamma < \kappa$.

c) For each set $I \subseteq \kappa - \aleph_0$, form the direct product $\mathcal{H}[I] := (\times_{\gamma \in I} \mathcal{M}_\gamma) \times \mathcal{M}_\kappa$.

Then, $\mathcal{H}[I]$ is (isomorphic to) a non-Boolean PWA.

Notice that $\mathcal{H}[I]$ has cardinality $\kappa = |\mathcal{M}_\kappa| \leq |\mathcal{H}[I]| \leq |\mathcal{H}[\kappa]|$ and $\mathcal{H}[\kappa]$ has cardinality $|\mathcal{H}[\kappa]| = |(\times_{\gamma \in \kappa} \mathcal{M}_\gamma)| . |\mathcal{M}_\kappa| = |(\times_{\gamma \in \alpha} \mathcal{M}_\gamma)| . |\mathcal{M}_\alpha| . |\mathcal{M}_\kappa| \leq \alpha^\alpha . \alpha . \kappa \leq 2^{\alpha\alpha} . \alpha . \kappa = \kappa$ (since $\kappa = \alpha^+ = 2^\alpha$). Also, for any non-trivial factorisation $\mathcal{H}[I] \cong \mathcal{F} \times \mathcal{G}$, if $\mathcal{F}$ is simple non-trivial then $\mathcal{F}$ must be isomorphic to $\mathcal{M}_\kappa$ or to some $\mathcal{M}_\gamma$, with $\gamma \in I$. Thus, the set of non-trivial simple factors of $\mathcal{H}[I]$ is $\{\mathcal{M}_\kappa\} \cup \{\mathcal{M}_\gamma / \gamma \in I\}$ (see [Veloso '96c]).

Therefore, there are at least 2^κ pairwise non-isomorphic non-Boolean PWA's $\mathcal{H}[I] = (\times_{\gamma \in I} \mathcal{M}_\gamma) \times \mathcal{M}_\kappa$ of cardinality κ, for $I \in \wp(\kappa - \aleph_0)$.

QED

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